

Polinomi di Taylor

Dato $f: I \rightarrow \mathbb{R}$, I aperto, $x_0 \in I$,

f derivabile almeno n volte in x_0 ,

si dice Polinomio di Taylor di grado n centrato in x_0 il polinomio

$$\begin{aligned} & f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \\ & \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = \\ & = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k \end{aligned}$$

Teorema (Resto di Peano)

$f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ x_0 interno ad I

f n volte derivabile in x_0 . Allora

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k + \boxed{R_n(x_0)}$$

$$\boxed{R_n(x_0) = o((x-x_0)^n)}$$

Polinomi di Taylor delle funzioni elementari.

$$1) e^x = \sum_{k=0}^n \frac{e^{x_0}}{k!} (x - x_0)^k + o((x - x_0)^n)$$

in $x_0 = 0$

$$e^x = \sum_{k=0}^n \frac{x^k}{k!} + o(x^n)$$

$$2) \sin x = \sum_{k=0}^n (-1)^k \frac{x^{2k+1}}{(2k+1)!} + o(x^{2n+2}) \quad x_0 = 0$$

$$3) \cos x = \sum_{k=0}^n (-1)^k \frac{x^{2k}}{(2k)!} + o(x^{2n+1}) \quad x_0 = 0$$

$$4) \log(1+x) = \sum_{k=1}^n (-1)^{k-1} \frac{x^k}{k} \quad x_0 = 0$$

$$5) \arctan(x) = \sum_{k=0}^n (-1)^k \frac{x^{2k+1}}{(2k+1)!} + o(x^{2n+2}) \quad \text{in } x_0 = 0$$

$$6) (1+x)^2 = \sum_{k=0}^n \binom{2}{k} x^k + o(x^n)$$

$$\text{dove } \binom{2}{k} = \frac{2(2-1) \dots (2-k+1)}{k!}$$

Esempi 1) $\lim_{x \rightarrow 0} \frac{x + 2x^3 - \sin x}{x^3} = \textcircled{*}$

$$\sin x = x + o(x^2)$$

$$= \lim_{x \rightarrow 0} \frac{\cancel{x} + 2x^3 - \cancel{x} + o(x^2)}{x^3} = \frac{0}{0} \quad \text{0/0}$$

$$\sin x = x - \frac{x^3}{6!} + o(x^4)$$

$$\begin{aligned}
 (*) &= \lim_{x \rightarrow 0} \frac{\cancel{x} + 2x^3 - \cancel{x} + \frac{x^3}{6} + o(x^4)}{x^3} = \\
 &= \lim_{x \rightarrow 0} \frac{\frac{13}{6}x^3 + o(x^4)}{x^3} = \lim_{x \rightarrow 0} \frac{13}{6} + \frac{o(x^4)}{x^3} = \\
 &= \frac{13}{6}
 \end{aligned}$$

$$2) \lim_{x \rightarrow 0} \frac{\log(1+x) - \sin x}{1 - \cos x} = (*)$$

$$= \lim_{x \rightarrow 0} \frac{x + o(x) - (x + o(x^2))}{1 - (1 - \frac{x^2}{2} + o(x^3))} =$$

$$= \lim_{x \rightarrow 0} \frac{o(x) + o(x^2)}{\frac{x^2}{2} + o(x^3)} \quad ? \quad ?$$

$$(*) \lim_{x \rightarrow 0} \frac{\cancel{x} - \frac{x^2}{2} + \boxed{o(x^2)} - \cancel{x} + \boxed{o(x^2)}}{\frac{x^2}{2} + o(x^3)} =$$

$$\lim_{x \rightarrow 0} \frac{-\frac{1}{2} + \frac{o(x^2)}{x^2}}{\frac{1}{2} + \frac{o(x^3)}{x^2}} = -1$$

Osservazione

$$o(x^m) = o(x^n) \quad \forall m > n$$

$$\frac{o(x^m)}{x^m} = x^{m-n} \frac{o(x^n)}{x^n}$$

$$g(x) = \begin{cases} \frac{e^x - 1}{x^a} & \text{se } x > 0 \\ 1 + bx + x^2 & \text{se } x \leq 0 \end{cases} \quad \left(\text{Compito } 13/12/12 \right)$$

1) $a > 0, b \in \mathbb{R}$

continua in $x_0 = 0$

2) $a > 0, b \in \mathbb{R}$

derivabile in $x_0 = 0$

$$\lim_{x \rightarrow 0^-} g(x) = \lim_{x \rightarrow 0^-} 1 + bx + x^2 = 1$$

$$\begin{aligned} \lim_{x \rightarrow 0^+} g(x) &= \lim_{x \rightarrow 0^+} \frac{e^x - 1}{x^a} = \lim_{x \rightarrow 0^+} \frac{e^x - 1}{x} \cdot \frac{x}{x^a} \\ &= \lim_{x \rightarrow 0^+} \left[\frac{e^x - 1}{x} \right] \cdot \left[x^{1-a} \right] = \begin{cases} 0 & \text{se } a < 1 \\ 1 & \text{se } a = 1 \\ +\infty & \text{se } a > 1 \end{cases} \end{aligned}$$

$\lim_{x \rightarrow 0^+} x^{1-a} = \begin{cases} 0 & 1-a > 0 \\ 1 & 1-a = 0 \\ +\infty & 1-a < 0 \end{cases}$

g è continua in $x_0 = 0$ se $a = 1$

2) se $x \leq 0 \quad g(x) = 1 + bx + x^2$

$$g'_-(0) = b$$

$$g'_+(0) = \lim_{x \rightarrow 0^+} \frac{g(x) - g(0)}{x} =$$

$$\lim_{x \rightarrow 0^+} \frac{\frac{e^x - 1}{x} - 1}{x} = \lim_{x \rightarrow 0^+} \frac{e^x - 1 - x}{x^2} = \textcircled{x}$$

$$= \lim_{x \rightarrow 0^+} \frac{1 + \cancel{x} + o(x) - 1 - \cancel{x}}{x^2} \quad 99$$

$$\textcircled{a=1}$$

$$(*) = \lim_{x \rightarrow 0^+} \frac{\cancel{1} + \cancel{x} + \cancel{\frac{x^1}{2}} + \frac{\sigma(x^2)}{\cancel{x^2}} - \cancel{1} - \cancel{x}}{\cancel{x^2}} = \frac{1}{2} = f'_+(0)$$

g è derivabile in $x_0 = 0$ se soltanto se
 $a = 1$ e $b = \frac{1}{2}$.

Compito del 8/01/13

$$1) \quad f(x) = \begin{cases} (1 + \sin x)^{\frac{3}{x}} & \text{se } x > 0 \\ x^2 + ax + b & \text{se } x \leq 0 \end{cases}$$

1) continuità in $x_0 = 0$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} x^2 + ax + b = b$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \left[(1 + \sin x)^{\frac{1}{\sin x}} \right]^{\frac{\sin x}{x} \cdot 3}$$

$$\downarrow$$

$$t = \sin x$$

$$\lim_{t \rightarrow 0} (1+t)^{\frac{1}{t}} = e$$

$$= e^3$$

$$b = e^3$$

2) derivabilità: supponiamo $b = e^3$

$$f'_-(0) = \frac{d}{dx} (x^2 + ax + b) \Big|_{x=0} = 2x + a \Big|_{x=0} = a$$

$$f'_+(0) = \lim_{x \rightarrow 0^+} \frac{(1 + \sin x)^{\frac{3}{x}} - e^3}{x} =$$

$$= \lim_{x \rightarrow 0^+} \frac{e^{\frac{3}{x} \log(1 + \sin x)} - e^3}{x} =$$

$$= \lim_{x \rightarrow 0^+} \frac{e^3 \left[e^{\left(\frac{3}{x} \log(1 + \sin x) - 3\right)} - 1 \right]}{x} =$$

$$= \lim_{x \rightarrow 0^+} e^3 \cdot \frac{e^{\left[\frac{3}{x} \log(1 + \sin x) - 3 \right]} - 1}{\left[\frac{3}{x} \log(1 + \sin x) - 3 \right] \frac{1}{x}}$$

||

$$\rightarrow \lim_{x \rightarrow 0^+} \left[\frac{3}{x} \log(1 + \sin x) \right] = 3 \text{ infatti}$$

$$\lim_{x \rightarrow 0^+} \frac{3 \sin x}{x} \left[\frac{\log(1 + \sin x)}{\sin x} \right] = 3$$

$\downarrow$
 1

$\sin x = t$
 $\lim_{t \rightarrow 0} \log \frac{(1+t)}{t} = 1$

$$\rightarrow t = \frac{3}{x} \log(1 + \sin x) - 3$$

$$\lim_{t \rightarrow 0} \frac{e^t - 1}{t} = 1$$

$$\lim_{x \rightarrow 0^+} \cdot \left[\frac{3}{x} \log(1 + \sin x) - 3 \right] \frac{1}{x} =$$

$$= 3 \lim_{x \rightarrow 0^+} \frac{\log(1 + \sin x) - x}{x^2} =$$

$$= \frac{3}{2}$$

$\left(\frac{0}{0} \right)$
 infatti:

$$\log(1 + \sin x)$$

$$\log(1 + t) = t + \frac{t^2}{2} + o(t^2)$$

$$\log(1 + \sin x) = \sin x + \frac{\sin^2 x}{2} + o(\sin^2 x)$$

$$\lim_{x \rightarrow 0^+} \frac{\log(1 + \sin x) - x}{x^2} =$$

$$= \lim_{x \rightarrow 0^+} \frac{\sin x + \frac{1}{2} \sin^2 x + o(\sin^2 x) - x}{x^2} =$$

$$= \lim_{x \rightarrow 0^+} \frac{\cancel{-\frac{x^3}{6}} + \cancel{o(x^4)} + \frac{1}{2} \sin^2 x + \cancel{o(\sin^2 x)} - \cancel{x}}{x^2}$$

$$= \lim_{x \rightarrow 0^+} \left[\underbrace{-\frac{x}{6}}_0 + \underbrace{\frac{o(x^2)}{x^2}}_0 + \frac{1}{2} \left(\underbrace{\frac{\sin x}{x}}_1 \right)^2 + \underbrace{\frac{o(x^2)}{x^2}}_0 \right] = \frac{1}{2}$$

$$\underbrace{\frac{o(\sin^2 x)}{x^2}}_0 = \underbrace{\frac{\sin^2 x}{x^2}}_1 \cdot \underbrace{\frac{o(\sin^2 x)}{\sin^2 x}}_0$$

$o(\sin^2 x) = o(x^2)$

f é derivabile se $b = e^3$ e $a = \frac{3}{2} e^3$

$$2) f(x) = (x+2) [3 - \log(x+2)]$$

$$1) \text{ dom } f = \{ x \in \mathbb{R} : x+2 > 0 \} \\ = (-2, +\infty)$$

$$2) A = \{ \text{punti di acc. di dom } f \} = \\ = [-2, +\infty) \cup \{+\infty\}$$

$$\lim_{x \rightarrow x_0} f(x) \quad \forall x_0 \in A$$

f è continua in dom $f \Rightarrow \forall x_0 \in \text{dom } f$

$$\lim_{x \rightarrow x_0} f(x) = f(x_0)$$

$$\lim_{x \rightarrow -2^+} (x+2) [3 - \log(x+2)] = \\ = \lim_{x \rightarrow -2^+} \underbrace{(x+2) \cdot 3}_{\rightarrow 0} - \lim_{x \rightarrow -2^+} \underbrace{(x+2) \log(x+2)}_{\substack{t = x+2 \\ \lim_{t \rightarrow 0^+} t \log t \\ = 0}} = 0$$

$$\lim_{x \rightarrow +\infty} (x+2) [3 - \log(x+2)] = -\infty$$

3) Asintoti:

non ci sono asintoti verticali

non ci sono asintoti orizzontali
obliqui?

$$\lim_{x \rightarrow +\infty} \frac{(x+2) [3 - \log(x+2)]}{x} = -\infty$$

$\Rightarrow$ non ci sono asintoti obliqui

$$4) f(x) = (x+2) [3 - \log(x+2)]$$

$$B = \text{dom } f$$

$$\underline{f'(x)} = [3 - \log(x+2)] + \cancel{(x+2)} \left(-\frac{1}{\cancel{(x+2)}} \right) =$$

$$= 3 - \log(x+2) - 1 =$$

$$= \underline{2 - \log(x+2)}$$

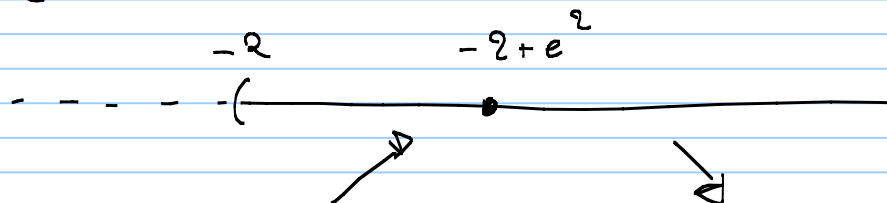
$$5) f'(x) > 0 \quad 2 - \log(x+2) > 0$$

$$\log(x+2) < 2$$

$$e^{\log(x+2)} < e^2$$

$$(x+2) < e^2$$

$$x < e^2 - 2$$



f è strett. crescente in $(-2, -2 + e^2)$

f è strett. decrescente in $(-2 + e^2, +\infty)$

$$6) \max / \min \quad \text{rel} / \text{ass.}$$

$$f'(x) = 0 \quad \Leftrightarrow \quad x = e^2 - 2 \quad \max \text{ rel e assoluto!}$$

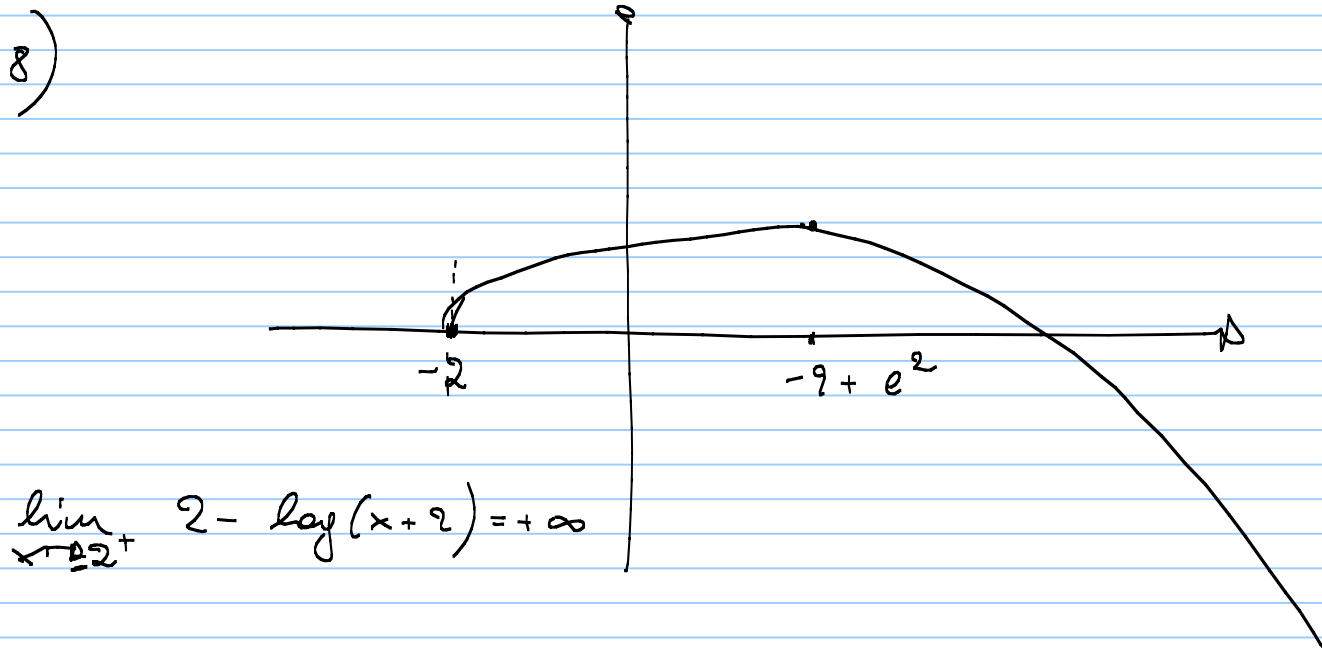
$$7) f''(x) = \frac{d}{dx} (2 - \log(x+2)) =$$

$$= -\frac{1}{x+2} > 0 \quad \Leftrightarrow \quad \frac{1}{x+2} < 0$$

$$\Leftrightarrow x < -2 \quad \boxed{\max}$$

$$\text{dom } f = (-2, +\infty)$$

$\Rightarrow f$ è concava in $(-2, +\infty)$



3) 1)

2)

3) $x^7 + x^6 + 1 = 0$

$$x^7 + x^6 + 1 \Big|_{x=-2} = (-2)^7 + (-2)^6 + 1 = -63 < 0$$

$$x^7 + x^6 + 1 \Big|_{x=1} = 3 > 0$$

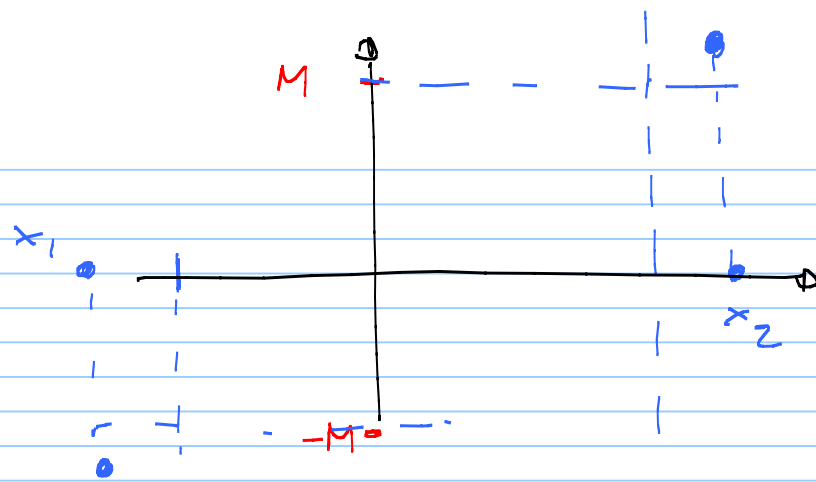
applicare il teo di Weier degli zeri

$\Rightarrow \exists$ una sol. in $(-2, 1)$

In alternativa:

$$\lim_{x \rightarrow -\infty} x^7 + x^6 + 1 = -\infty$$

$$\lim_{x \rightarrow +\infty} x^7 + x^6 + 1 = +\infty$$



per il teo di Weierstrass $\exists$ almeno una soluzione.

Compito del 28/06/13

$$1) \quad g(x) = \begin{cases} \frac{e^{-2x} - 1}{x\sqrt{a}} & \text{se } x > 0 \\ b + cx + x^2 & \text{se } x \leq 0 \end{cases}$$

$$a > 0, \quad b, c \in \mathbb{R}$$

2) g sia continua in $x_0 = 0$

$$\lim_{x \rightarrow 0^-} b + cx + x^2 = \boxed{b}$$

$$\lim_{x \rightarrow 0^+} \frac{e^{-2x} - 1}{x\sqrt{a}} = \lim_{x \rightarrow 0^+} \left[\frac{-2x}{x\sqrt{a}} \right] \left[\frac{e^{-2x} - 1}{-2x} \right] =$$

$\lim_{t \rightarrow 0} \frac{e^t - 1}{t} \quad t = -2x$

$$= (-2) \lim_{x \rightarrow 0^+} x^{1-\sqrt{a}} = \begin{cases} 0 & \text{se } 1-\sqrt{a} > 0 \\ -2 & \text{se } 1-\sqrt{a} = 0 \\ -\infty & \text{se } 1-\sqrt{a} < 0 \end{cases}$$

$$1 - \sqrt{a} > 0 \quad 1 > \sqrt{a} \quad 0 < a < 1$$

$$= \begin{cases} 0 & \text{se } 0 < a < 1 \\ -2 & \text{se } a = 1 \\ -\infty & \text{se } a > 1 \end{cases}$$

g è continua se $0 < a < 1$ e $b = 0$
oppure se $a = 1$ e $b = -2$

2) derivabilit 

i) $0 < a < 1 \quad b = 0$

ii) $a = 1 \text{ e. } b = -2$

i)

$$f(x) = \begin{cases} \frac{e^{-2x} - 1}{x^{\sqrt{a}}} & \text{e } x > 0 \\ cx + x^2 & \text{e } x \leq 0 \end{cases}$$

$$f'_-(0) = \left. \frac{d}{dx} (cx + x^2) \right|_{x=0} = c + 2x \Big|_{x=0} = c$$

$$\begin{aligned} f'_+(0) &= \lim_{x \rightarrow 0^+} \frac{\frac{e^{-2x} - 1}{x^{\sqrt{a}}}}{x} = \lim_{x \rightarrow 0^+} \frac{e^{-2x} - 1}{x^{1+\sqrt{a}}} \\ &= \lim_{x \rightarrow 0^+} \frac{e^{-2x} - 1}{-2x} \cdot \frac{-2x}{x^{1+\sqrt{a}}} = \\ &= \lim_{x \rightarrow 0^+} (-2) \underbrace{\left[\frac{e^{-2x} - 1}{-2x} \right]}_{\rightarrow 1} \cdot \underbrace{(x^{-\sqrt{a}})}_{\rightarrow +\infty} = +\infty \end{aligned}$$

f non   derivabile

ii) $a = 1 \quad b = -2$

$$f(x) = \begin{cases} \frac{e^{-2x} - 1}{x} & \text{e } x > 0 \\ -2 + cx + x^2 & \text{e } x \leq 0 \end{cases}$$

$$f'_-(0) = \left. \frac{d}{dx} (-2 + cx + x^2) \right|_{x=0} = c + 2x \Big|_{x=0} = c$$

$$f'_+(0) = \lim_{x \rightarrow 0^+} \frac{\frac{e^{-2x} - 1}{x} + 2}{x} =$$

$$= \lim_{x \rightarrow 0^+} \frac{e^{-2x} - 1 + 2x}{x^2} \quad (\infty)$$

$$e^t = 1 + t + \frac{t^2}{2} + o(t^2)$$

$$e^{-2x} = 1 - 2x + \frac{(-2x)^2}{2} + \underbrace{o(x^2)}_{o((-2x)^2)}$$

$$(\infty) \lim_{x \rightarrow 0^+} \frac{1 - \cancel{2x} + \frac{4x^2}{2} + o(x^2) - 1 + \cancel{2x}}{x^2} =$$

$$= 2$$

f è derivabile se e solo se
 $a=1 \quad b=-2 \quad c=2$

$$2) f(x) = \arcsin\left(\frac{|x+2|}{x^2+1}\right)$$

$$\text{dom } f = \left\{ x \in \mathbb{R} : -1 \leq \frac{|x+2|}{x^2+1} \leq 1 \right\}$$

$$\begin{cases} \frac{|x+2|}{x^2+1} \leq 1 \\ \frac{|x+2|}{x^2+1} \geq -1 \end{cases}$$

$$\bullet) x+2 \geq 0 \quad (x \geq -2)$$

$$\frac{x+2}{x^2+1} \leq 1$$

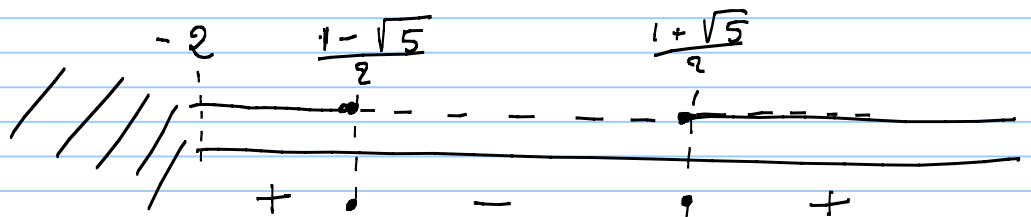
$$\frac{x+2-x^2-1}{x^2+1} \leq 0$$

$$\frac{x^2-x-1}{x^2+1} \geq 0$$

$$\Delta \geq 0 \quad \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$x < \frac{1-\sqrt{5}}{2} \text{ or } x > \frac{1+\sqrt{5}}{2}$$

$$\Delta > 0 \quad \forall x$$



$$-2 < \frac{1-\sqrt{5}}{2}$$

$$-4 < 1-\sqrt{5} \\ \sqrt{5} < 5 \text{ vero}$$

$$\frac{x+2}{x^2+1} \geq -1$$

$$\frac{x+2+x^2+1}{x^2+1} \geq 0$$

$$\frac{x^2+x+3}{x^2+1} \geq 0$$

$$\forall x \geq -2$$

$$\Delta \geq 0 \quad \forall x$$

$$\Delta > 0 \quad \forall x$$

Quindi se $x \geq -2$

$$-2 \leq x \leq \frac{1-\sqrt{5}}{2} \quad \text{or} \quad x > \frac{1+\sqrt{5}}{2}$$

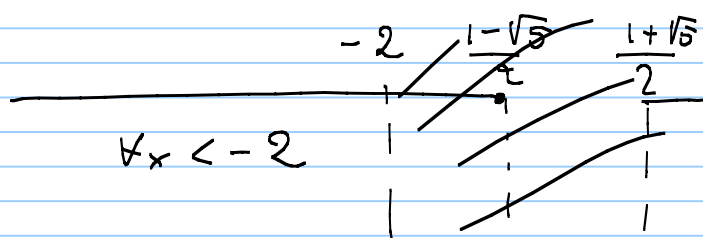
•) $x < -2$

$$\begin{cases} -\frac{x+2}{x^2+1} \leq 1 \\ -\frac{x+2}{x^2+1} \geq -1 \end{cases}$$

$$\frac{x+2}{x^2+1} \geq -1$$

$\forall x$ (vedi sopra!)

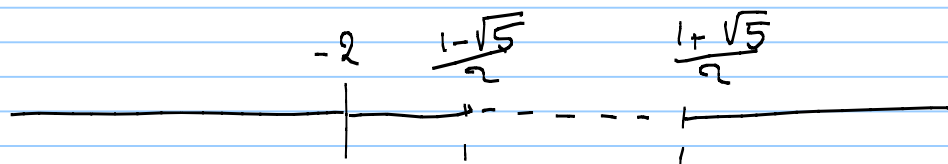
$$\frac{x+2}{x^2+1} \leq 1$$



$$\boxed{\forall x < -2}$$

dom f

"



$$(-\infty, \frac{1-\sqrt{5}}{2}] \cup [\frac{1+\sqrt{5}}{2}, +\infty)$$

$$A = \text{dom } f \cup \{+\infty\} \cup \{-\infty\}$$

$$\lim_{x \rightarrow x_0} f(x) = \quad \forall x_0 \in A$$

$$\forall x_0 \in \text{dom } f (\subset A) \Rightarrow \lim_{x \rightarrow x_0} f(x) = f(x_0)$$

perché f è continua

$$\lim_{x \rightarrow +\infty} \text{ess} \sin \frac{|x+2|}{\underbrace{x^2+1}_{\rightarrow 0}} = 0$$

$$\lim_{x \rightarrow -\infty} \arcsin \frac{|x+2|}{x^2+1} = 0$$

$\underbrace{\frac{|x+2|}{x^2+1}}_{\rightarrow 0}$

3) Asintoti:

- non ci sono asintoti verticali
- $y=0$ è un asintoto orizzontale sia $x \rightarrow +\infty$ che $x \rightarrow -\infty$
- non ci sono asintoti obliqui

4) $B = \{x \in \mathbb{R} : f \text{ è derivabile in } x\}$

$$f(x) = \arcsin \left(\frac{|x+2|}{x^2+1} \right)$$

• $\arcsin$ è derivabile in $(-1, 1)$

$$-1 < \frac{|x+2|}{x^2+1} < 1$$

• $|x|$ è derivabile in $\mathbb{R} \setminus \{0\}$

$$x+2 \neq 0$$

se $-1 < \frac{|x+2|}{x^2+1} < 1$ e $x \neq -2$

allora per i teoremi su algebra delle derivate e composizione di derivate f è derivabile.

se $\left(x < \frac{1-\sqrt{5}}{2} \text{ o } x > \frac{1+\sqrt{5}}{2} \right)$ e $x \neq -2$

$$f'(x) = \frac{1}{\sqrt{1 - \left(\frac{x+2}{x^2+1} \right)^2}} \cdot \frac{x^2+1 - (x+2)2x}{(x^2+1)^2} =$$

$$= \frac{1}{\sqrt{1 - \left(\frac{x+2}{x^2+1}\right)^2}} \cdot \frac{-x^2 - 4x + 1}{(x^2+1)^2}$$

$$x < -2$$

$$f(x) = \arcsin\left(-\frac{x+2}{x^2+1}\right) =$$

$$= -\arcsin\left(\frac{x+2}{x^2+1}\right)$$

$$f'(x) = -\frac{1}{\sqrt{1 - \left(\frac{x+2}{x^2+1}\right)^2}} \cdot \frac{-x^2 - 4x + 1}{(x^2+1)^2}$$

$$x = -2, \quad x = \frac{1-\sqrt{5}}{2}, \quad x = \frac{1+\sqrt{5}}{2}$$

↓
↓ $f'(\cdot)$
↓ $f'(\cdot)$

Osservazione:

Teorema

$f: I \rightarrow \mathbb{R}$

x_0 sia interno ad I

f sia continua in I , derivabile in $I \setminus \{x_0\}$.

Se $\lim_{x \rightarrow x_0} f'(x) = l \in \mathbb{R}$

$$\left(\begin{array}{l} \lim_{x \rightarrow x_0^+} = l \\ \lim_{x \rightarrow x_0^-} = l \end{array} \right)$$

Allora f è derivabile

Se $\lim_{x \rightarrow x_0^+} f'(x) = l_1$ e $\lim_{x \rightarrow x_0^-} f'(x) = l_2$

$$l_1 \neq l_2$$

$\Rightarrow$ allora f non è derivabile.

Esempio: $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

f è continua in $\mathbb{R}$

è derivabile in $\mathbb{R} \setminus \{0\}$

$$\lim_{x \rightarrow 0} f'(x) = \lim_{x \rightarrow 0} \underbrace{2x}_{\rightarrow 0} \underbrace{\sin \frac{1}{x}}_{\text{oscillates}} - \cancel{x^2} \underbrace{\left(\cos \frac{1}{x}\right)}_{\text{oscillates}} \underbrace{\left(\frac{1}{x}\right)}_{\rightarrow \infty}$$

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$$

derivabilità in $x = -2$

$$\lim_{x \rightarrow -2^+} \frac{1}{\sqrt{1 - \left(\frac{x+2}{x^3+1}\right)^2}} \cdot \frac{-x^3 - 4x + 1}{(x^3+1)^2} = \frac{1}{5}$$

$$\lim_{x \rightarrow -2^-} \left(- \frac{1}{\sqrt{1 - \left(\frac{x+2}{x^3+1}\right)^2}} \cdot \frac{-x^3 - 4x + 1}{(x^3+1)^2} \right) = -\frac{1}{5}$$

$\Rightarrow f$ non è derivabile in $x = -2$

$$B = (-\infty, -2) \cup \left(-2, \frac{1-\sqrt{5}}{2}\right) \cup \left(\frac{1+\sqrt{5}}{2}, +\infty\right)$$

5) monotonia

$$f'(x) > 0$$

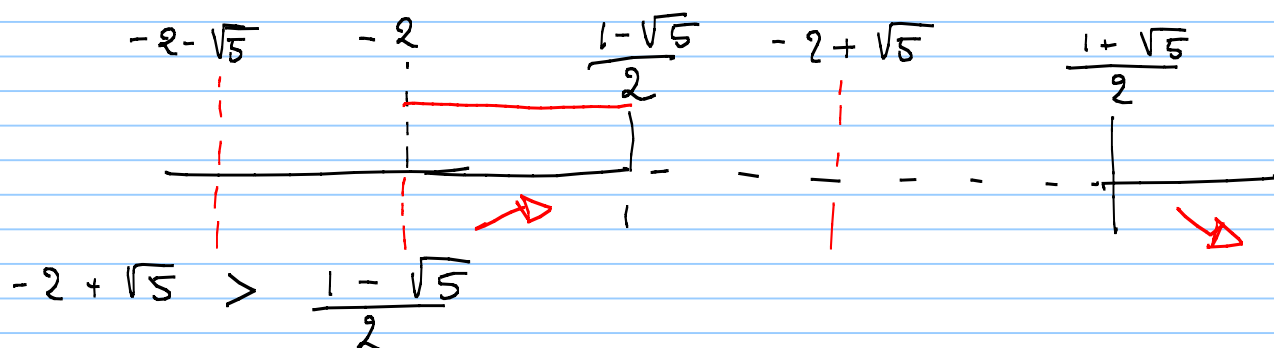
$$x > -2$$

$$\frac{1}{\sqrt{1 - \left(\frac{x+2}{x^3+1}\right)^2}} \cdot \frac{-x^3 - 4x + 1}{(x^3+1)^2} > 0$$

$$Q=D \quad -x^2 - 4x + 1 > 0$$

$$Q=D \quad \boxed{x^2 + 4x - 1 < 0}$$

$$x_{1,2} = -2 \pm \sqrt{4+1} = -2 \pm \sqrt{5}$$



$$-4 + 2\sqrt{5} > 1 - \sqrt{5}$$

$$3\sqrt{5} > 5$$

$$9 \cdot 5 > 25 \quad (S1)$$

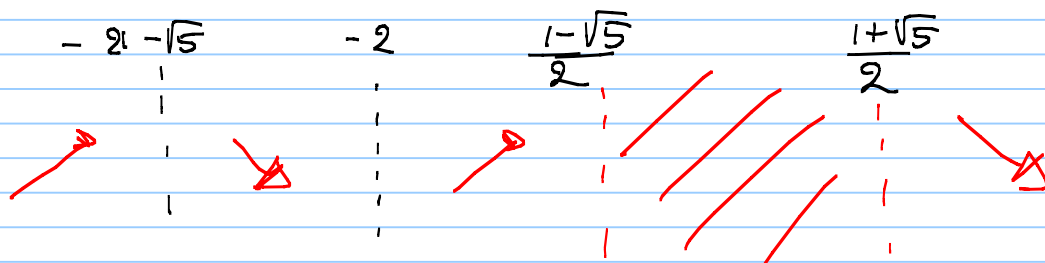
$$\bullet \quad x < -2$$

$$f'(x) = - \frac{1}{\sqrt{1 - \left(\frac{x+2}{x^2+1}\right)^2}} \cdot \frac{-x^2 - 4x + 1}{(x^2+1)^2} > 0$$

$$Q=D \quad x^2 + 4x - 1 > 0$$

$$x < -2 - \sqrt{5}$$

$$(x > -2 + \sqrt{5})$$

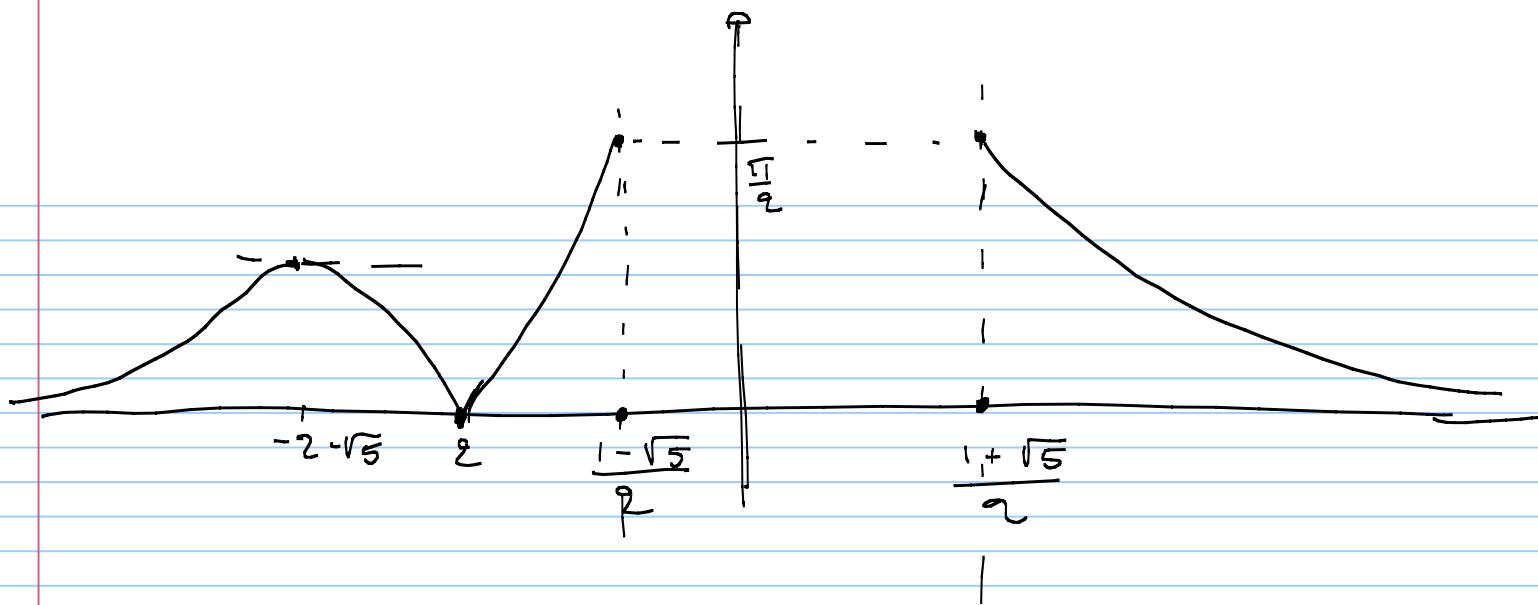


strettamente crescente in $(-\infty, -2 - \sqrt{5}]$

e in $[-2, \frac{1 - \sqrt{5}}{2}]$

strettamente decrescente in $[-2 - \sqrt{5}, -2]$

e in $[\frac{1 + \sqrt{5}}{2}, +\infty)$



3) $h(x) = x^7 + 6x^6 - x$ invertible in $(\frac{1}{2}, +\infty)$

$$7x^6 + 36x^5 - 1 > 0 \quad (\text{or } < 0)$$

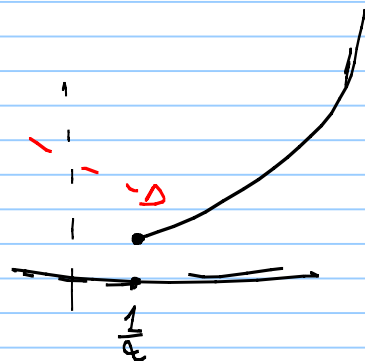
$$7 \cdot \frac{1}{2^6} + 36 \cdot \frac{1}{2^5} - 1 > 0$$

$$> 1$$

$$- - - - - > 0$$

$$42x^5 + 36 \cdot 5x^4 > 0$$

$$x > \frac{1}{2}$$



$\Rightarrow x^7 + 6x^6 - x$ is invertible in $(\frac{1}{2}, +\infty)$

• $h|_{(\frac{1}{2}, \infty)}$ $(h^{-1})'(6) = \frac{1}{h'(\boxed{h^{-1}(6)})}$

$$h^{-1}(6) = x$$

$\uparrow$

$$(\frac{1}{2}, +\infty)$$

$$6 = h(x)$$

$$6 = x^7 + 6x^6 - 1$$

$$\boxed{x = 1}$$

$$(h^{-1})'(6) = \frac{1}{7 + 36 - 1} = \frac{1}{42}$$

Compito del 6/08/13

$$1) g(x) = \begin{cases} \frac{1 - \cos x^a}{x^4} & \text{se } x > 0 \\ x^9 + b & \text{se } x \leq 0 \end{cases}$$

$$a > 0, b \in \mathbb{R}$$

1) continuità:

$$\lim_{x \rightarrow 0^-} x^9 + b = b = g(0)$$

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{1 - \cos x^a}{x^4} &= \lim_{x \rightarrow 0^+} \frac{1 - \cos x^a}{x^{2a}} \cdot \frac{x^{2a}}{x^4} \\ &= \frac{1}{2} \lim_{x \rightarrow 0^+} x^{2a-4} = \begin{cases} 0 & \text{se } a > 2 \\ \frac{1}{2} & \text{se } a = 2 \\ +\infty & \text{se } a < 2 \end{cases} \end{aligned}$$

$$g \text{ è continua in } x_0 = 0 \Leftrightarrow \begin{aligned} &a > 2 \text{ e } b = 0 \\ \text{o } &a = 2 \text{ e } b = \frac{1}{2} \end{aligned}$$

2) derivabilità:

$$g'_-(0) = 2x \Big|_{x=0} = 0$$

$$g'_+(0) = \lim_{x \rightarrow 0^+} \frac{g(x) - g(0)}{x - 0} =$$

i) $a > 2$

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{\frac{1 - \cos x^a}{x^4}}{x} &= \lim_{x \rightarrow 0^+} \frac{1 - \cos x^a}{x^5} = \\ &= \lim_{x \rightarrow 0^+} \underbrace{\frac{1 - \cos x^a}{x^{2a}}}_{\sim \frac{1}{2}} \cdot x^{2a-5} = \begin{cases} 0 & \text{se } a > \frac{5}{2} \\ \frac{1}{2} & \text{se } a = \frac{5}{2} \\ +\infty & \text{se } a < \frac{5}{2} \end{cases} \end{aligned}$$

ii) $a = 2$

$$\lim_{x \rightarrow 0^+} \frac{\frac{1 - \cos x^2}{x^4}}{x} = \frac{1}{2}$$

$$\begin{aligned}
 & \lim_{x \rightarrow 0^+} \frac{1 - \cos x^2 - \frac{1}{2}x^4}{x^5} = \\
 & = \lim_{x \rightarrow 0^+} \frac{\cancel{1} - \left(\cancel{1} - \cancel{\frac{x^4}{2}} + \frac{1}{4!}x^8 + o(x^{10}) \right) - \cancel{\frac{1}{2}x^4}}{x^5} \\
 & = 0
 \end{aligned}$$

Quindi g è derivabile se $a > \frac{5}{2}$ e $b = 0$

o $a = 2$ e $b = \frac{1}{2}$

$$2) f(x) = \frac{|x|}{(x-2)^2} e^{(x-2)}$$

$$1) \text{ dom } f = \{x \in \mathbb{R} : x \neq 2\} = (-\infty, 2) \cup (2, +\infty)$$

$$2) A = \{x \in \mathbb{R} : x \text{ è di accumulazione per dom } f\}$$

$$A = \mathbb{R}^*$$

$\forall x_0 \in \text{dom } f$, f è continua in x_0 (perché composizione, prodotto, ecc..., di funzioni continue).

Quindi $\lim_{x \rightarrow x_0} f(x) = f(x_0) \quad \forall x_0 \in \text{dom } f$.

$$\lim_{x \rightarrow +\infty} \frac{|x|}{(x-2)^2} e^{(x-2)} = \lim_{x \rightarrow +\infty} \underbrace{\left[\frac{|x|}{(x-2)} \right]}_{\rightarrow 1} \underbrace{\left[\frac{e^{(x-2)}}{x-2} \right]}_{\rightarrow +\infty} = +\infty$$

$$\lim_{x \rightarrow -\infty} \underbrace{\left[\frac{|x|}{(x-2)^2} \right]}_{\rightarrow 0} \underbrace{e^{(x-2)}}_{\rightarrow 0} = 0$$

$$\lim_{x \rightarrow 2} \underbrace{\left[\frac{|x|}{(x-2)^2} \right]}_{\rightarrow +\infty} \underbrace{e^{(x-2)}}_{\rightarrow 1} = +\infty$$

3) Asintoti: $y = 0$ è un asintoto orizzontale a $-\infty$
 $x = 2$ è un asintoto verticale

Potrebbe esserci un asintoto obliquo a $+\infty$:

$$\lim_{x \rightarrow +\infty} \frac{|x|}{(x-2)^2} e^{(x-2)} \cdot \frac{1}{x} = +\infty$$

non ci sono asintoti obliqui.

$$4) f(x) = \frac{|x|}{(x-2)^2} e^{(x-2)}$$

$$B = \{x \in \mathbb{R} : f \text{ è derivabile in } x\}$$

f è derivabile $\forall x \neq 2$ e $x \neq 0$ per i teoremi sull'algebra delle derivate e sulle derivazione delle funzioni composte.

$\forall x \neq 0$ e $\forall x \neq 2$ si ha

$$f(x) = \begin{cases} \frac{x}{(x-2)^2} e^{(x-2)} & \text{se } x > 0 \\ \frac{-x}{(x-2)^2} e^{(x-2)} & \text{se } x < 0 \end{cases}$$

se $x > 0$:

$$\begin{aligned} \frac{x}{(x-2)^2} e^{(x-2)} \quad f'(x) &= \left[\frac{(x-2)^2 - 2x(x-2)}{(x-2)^4} + \frac{x}{(x-2)^2} \right] e^{(x-2)} \\ &= \frac{(1+x)(x-2) - 2x}{(x-2)^3} e^{(x-2)} \\ &= \frac{x^2 - x - 2 - 2x}{(x-2)^3} e^{(x-2)} = \\ &= \frac{x^2 - 3x - 2}{(x-2)^3} e^{(x-2)} \end{aligned}$$

se $x < 0$ $f(x) = \frac{-x}{(x-2)^2} e^{(x-2)}$

$$f'(x) = - \frac{x^2 - 3x - 2}{(x-2)^3} e^{(x-2)}$$

Studiamo la derivabilità in $x_0 = 0$

$$\lim_{x \rightarrow 0^+} f'(x) = \frac{1}{4} e^{-2}$$

$$\lim_{x \rightarrow 0^-} f'(x) = -\frac{1}{4} e^{-2}$$

Quindi f non è derivabile in $x = 0$.

$$B = \{x \in \mathbb{R} : x \neq 0 \text{ e } x \neq 2\}$$

5) Studiamo la monotonia di f

se $x > 0$: $f'(x) = \frac{x^2 - 3x - 2}{(x-2)^3} e^{x-2} > 0$

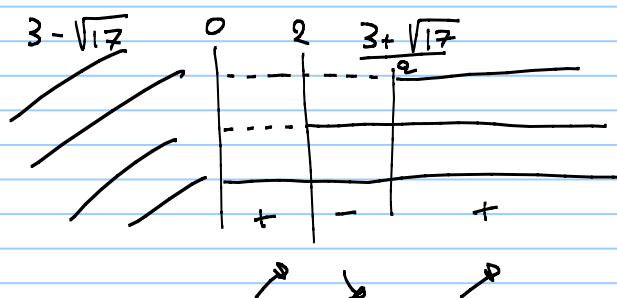
$$x^2 - 3x - 2 > 0$$

$$x_{1,2} = \frac{3 \pm \sqrt{9+8}}{2} = \frac{3 \pm \sqrt{17}}{2}$$

$$(x^2 - 3x - 2) > 0$$

$$(x-2)^3 > 0$$

$$e^{x-2} > 0$$



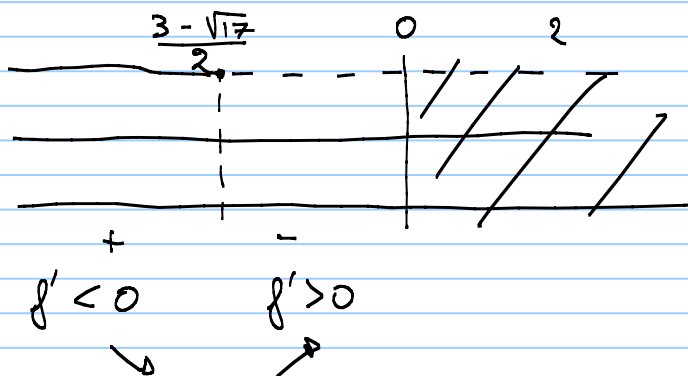
se $x < 0$ $f'(x) = - \frac{(x^2 - 3x - 2)}{(x-2)^3} e^{(x-2)} > 0$

$$\Leftrightarrow \frac{(x^2 - 3x - 2)}{(x-2)^3} e^{(x-2)} < 0$$

$$x^2 - 3x - 2 > 0$$

$$x-2 > 0$$

$$e^{x-2} > 0$$



f è decrescente in $(-\infty, \frac{3 - \sqrt{17}}{2}]$
e in $[2, \frac{3 + \sqrt{17}}{2}]$

f è crescente in $[\frac{3 - \sqrt{17}}{2}, 2]$
e in $[\frac{3 + \sqrt{17}}{2}, +\infty)$

