Lezione 10 Dicembre

1) Integrazione per parti

$$\int f \cdot g' \, dx = f \cdot g - \int f' g \, dx$$

2) Integrazione per sostituzione

$$\int f(x) \, dx = \int f(\varphi(t)) \, \varphi'(t) \, dt \qquad x = \varphi(t)$$

Esercizi

$$\int \underbrace{1}_{g'} \cdot \underbrace{\arcsin x}_{f} \, dx = x \arcsin x - \int x \frac{1}{\sqrt{1-x^2}} \, dx =$$

$g' = 1 \Rightarrow g = x$

$1 - x^2 = t$

$-2x \, dx = dt$

$x \, dx = -\frac{1}{2} dt$

$$= x \arcsin x - \int \left(-\frac{1}{2}\right) \frac{dt}{\sqrt{t}} = x \arcsin x$$

$$+ \frac{1}{2} \int t^{-1/2} \, dt = x \arcsin x + \cancel{\frac{1}{2}} \frac{t^{1/2}}{\cancel{\frac{1}{2}}}$$

$$= x \arcsin x + \sqrt{1-x^2} + K$$

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es. $$\int x^3 \cdot e^{x^4} \, dx =$$

$x^4 = t$

$4x^3 \, dx = dt$

$x^3 \, dx = \frac{1}{4} dt$

$$= \int e^t \, \frac{1}{4} \, dt = \frac{1}{4} e^{x^4}$$

es. $$\int_2^3 \frac{1}{x \log x} \, dx =$$

$\log x = t$

$\frac{1}{x} \, dx = dt$

$x = 2 \quad t = \log 2$

$x = 3 \quad t = \log 3$

$$= \int_{\log 2}^{\log 3} \frac{1}{t} \, dt =$$

$$= \log|t| \Big/_{\lg 2}^{\lg 3} = \log(\lg 3) - \lg(\lg 2)$$
$$= \log\left(\frac{\lg 3}{\lg 2}\right)$$

PC.

es. $\displaystyle\int \frac{(\log x)^n}{x}\,dx$ $\qquad \lg x = t \quad \dots\dots$

es. $\displaystyle\int \operatorname{sen}^3 x\,dx = \int \operatorname{sen} x \cdot \operatorname{sen}^2 x\,dx =$

$\displaystyle= \int \operatorname{sen} x\,(1-\cos^2 x)\,dx =$ $\qquad \cos x = t$

$\qquad - \operatorname{sen} x\,dx = dt$

$\qquad \operatorname{sen} x\,dx = -dt$

$\displaystyle= -\int (1-t^2)\,dt =$

$\displaystyle= -\left(t - \frac{t^3}{3}\right) = -t + \frac{t^3}{3} + K$

$\displaystyle= -\cos x + \frac{\cos^3 x}{3} + K.$

es. PC. $\displaystyle\int \frac{1}{(3+5x)^6}\,dx, \int \sqrt{x+2}\,dx$

$\displaystyle\int \frac{x}{\sqrt{2-3x^2}}\,dx, \int \frac{x^3}{\sqrt{1+x^4}}$

$\displaystyle\int 3x\,e^{x^2}, \int \frac{\operatorname{sen}\sqrt{x}}{\sqrt{x}}\,dx, \int \frac{1}{1+e^x}\,dx$

es. $\displaystyle\int \frac{\operatorname{sen}\sqrt{x}}{\sqrt{x}}\,dx$ $\qquad \sqrt{x} = t$

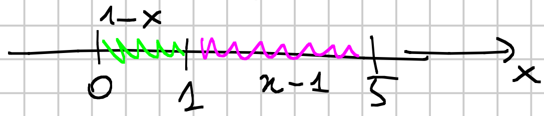
$\displaystyle\qquad \frac{1}{2}\frac{1}{\sqrt{x}}\,dx = dt$

$\displaystyle= \int \operatorname{sen} t\; 2\,dt$ $\qquad \displaystyle\frac{1}{\sqrt{x}}\,dx = 2dt$

$= \dots\dots$

es. $\displaystyle\int_0^5 |x-1|\,dx =$

$$|x-1| = \begin{cases} x-1 & x \geq 1 \\ 1-x & x < 1 \end{cases}$$



$$= \int_0^1 |x-1|\,dx + \int_1^5 |x-1|\,dx =$$

$$= \int_0^1 (1-x)\,dx + \int_1^5 (x-1)\,dx = \dots$$

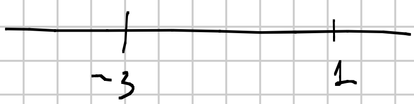
es. $\displaystyle\int_0^2 |x^2+2x-3|\,dx$

$x^2+2x-3=0$

$x = -1 \pm \sqrt{1+3} = -1 \pm 2$

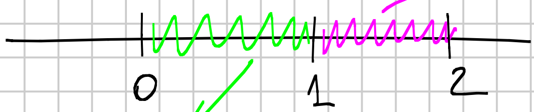
$x=-3$

$x=1$



$x^2+2x-3>0$

$x>1$ o $x<-3$



$|x^2+2x-3| =$
$= x^2+2x-3$

$|x^2+2x-3| = -x^2-2x+3$

$$\int_0^2 |x^2+2x-3|\,dx = \int_0^1 (-x^2-2x+3)\,dx + \int_1^2 (x^2+2x-3)\,dx = \dots$$

Integrazione di alcune funzioni razionali

$$f(x) = \frac{P(x)}{Q(x)}$$ quoziente di due polinomi di qualsiasi grado

$$f(x) = \frac{x^4+1}{x+3x^2} \qquad f(x) = \frac{1}{x}$$

$$f(x) = \frac{x}{3x^5+5}$$

Casi che faremo

1) $f(x) = \dfrac{1}{Ax+B}$

2) $f(x) = \dfrac{1}{Ax^2+Bx+C}$

3) $f(x) = \dfrac{Dx+E}{Ax^2+Bx+C}$

1) $\displaystyle\int \frac{1}{Ax+B}\,dx =$

$Ax+B = t$
$A\,dx = dt$
$dx = \frac{1}{A}dt$

$$= \int \frac{1}{t}\,\frac{1}{A}\,dt = \frac{1}{A}\log|Ax+B| + K$$

se $\displaystyle\int \frac{1}{A\left(x+\frac{B}{A}\right)}\,dx = \frac{1}{A}\int \frac{1}{\boxed{\left(x+\frac{B}{A}\right)}}\,dx =$

$x+\frac{B}{A} = t$

$dx = dt$

$$= \frac{1}{A}\int \frac{1}{t}\,dt = \frac{1}{A}\log\left|x+\frac{B}{A}\right|$$

$$= \frac{1}{A}\log\left|\frac{Ax+B}{A}\right| =$$

$$= \frac{1}{A}\log|Ax+B| - \frac{1}{A}\log|A| + K$$

costante

K

2) $\displaystyle\int \frac{1}{Ax^2+Bx+C}\,dx$

es. 1 $\displaystyle\int \frac{1}{x^2+1}\,dx = \operatorname{arctg} x + K$

$\underbrace{x^2+1}_{>0} \neq 0$ $\boxed{\Delta < 0}$

es. 2 $\displaystyle\int \frac{1}{x^2-1}\,dx = \int \frac{1}{(x-1)(x+1)}\,dx$

$x^2-1=0$

$x^2-1=0 \quad =(x-1)(x+1)$ $\boxed{\Delta > 0}$

$$\int \frac{1}{(x-1)(x+1)} = \int \frac{A}{(x-1)} + \int \frac{B}{(x+1)} =$$

$$= \int \frac{1}{2(x-1)}\,dx - \frac{1}{2}\int \frac{1}{x+1}\,dx$$

$$= \frac{1}{2}\log|x-1| - \frac{1}{2}\log|x+1| =$$

$$= \frac{1}{2}\log\frac{|x-1|}{|x+1|} + K$$

es. 3 $\displaystyle\int \frac{1}{x^2+2x+1}\,dx = \int \frac{1}{(x+1)^2}\,dx$

$x^2+2x+1=0 \Rightarrow \Delta = 0$

$x=-1$

$$\int \frac{1}{(x+1)^2}\,dx = \int \frac{1}{t^2}\,dt$$

$x+1=t$

$dx=dt$

$$= -t^{-1} =$$

$$= -\frac{1}{(x+1)} + K$$

es. $\int \frac{1}{x^2+x+1}\,dx$

cerchiamo di capire se $x^2+x+1=0$ ha 2, 1, 0 soluzioni

$x^2+x+1=0 \qquad \Delta = 1-4 = -3<0$

$x^2+x+1>0 \Rightarrow$ lo voglio ricondurre all'es. 1. x^2+1

$$x^2+x+1 = (\underbrace{\cdots}_{\text{dipende da } x})^2 + a \quad , a>0$$

$$x^2+x+1 = \underbrace{x^2+x+\frac{1}{4}}_{\left(x+\frac{1}{2}\right)^2} - \frac{1}{4}+1$$

$$= \left(x+\frac{1}{2}\right)^2 + \frac{3}{4}$$

$$\int \frac{1}{x^2+x+1}\,dx = \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx =$$

$$= \int \frac{1}{\frac{3}{4}\left(\frac{\left(x+\frac{1}{2}\right)^2}{\frac{3}{4}}+1\right)}\,dx =$$

$$= \frac{4}{3}\int \frac{1}{\frac{4}{3}\left(x+\frac{1}{2}\right)^2+1}\,dx =$$

$$= \frac{4}{3}\int \frac{1}{\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right]^2+1}\,dx =$$

$$\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right) = t$$

$$\frac{2}{\sqrt{3}}\,dx = dt$$

$$dx = \frac{\sqrt{3}}{2}\,dt$$

$$= \frac{4}{3}\int \frac{1}{t^2+1}\,\frac{\sqrt{3}}{2}\,dt =$$

$$= \frac{\cancel{4}^{\,2}}{3}\,\frac{\sqrt{3}}{\cancel{2}}\,\mathrm{arctg}\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right) + K$$

<u>Morale</u> $\displaystyle\int \frac{1}{Ax^2+Bx+C}\,dx$

se $Ax^2+Bx+C=0$ ha $\Delta<0$

si cerca di "riempire il quadrato"

$$\underbrace{Ax^2+Bx+?}_{(\alpha x+\beta)^2}\;\underbrace{-\,?+C}_{\gamma}$$

$$(\alpha x+\beta)^2+\gamma$$

e ci si riconduce a $\displaystyle\int \frac{1}{x^2+1}\,dx$

<u>es.</u> $\displaystyle\int \frac{1}{x^2-5x+6}\,dx$

$x^2-5x+6=0 \qquad \Delta = 25-24 = 1>0$

$$x = \frac{5\pm 1}{2} = \begin{cases} 3 \\ 2 \end{cases}$$

$$x^2-5x+6 = (x-3)(x-2)$$

$$\int \frac{1}{x^2-5x+6}\,dx = \int \frac{1}{(x-3)(x-2)}\,dx$$

uso il metodo dei fratti semplici per decomporre. Voglio trovare A, B

$$\frac{1}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2} =$$

$$\forall x$$

$$= \frac{A(x-2)+B(x-3)}{(x-3)(x-2)} =$$

$$= \frac{Ax-2A+Bx-3B}{(x-3)(x-2)} =$$

$$= \frac{x(A+B)-2A-3B}{(x-3)(x-2)}$$

Voglio che

$$\frac{1}{\cancel{(x-3)}\cancel{(x-2)}} = \frac{x(A+B)-2A-3B}{\cancel{(x-3)}\cancel{(x-2)}} \quad \forall x$$

due polinomi sono uguali per ogni x sse i coefficienti dei termini dello stesso grado sono uguali

$$\begin{cases} A+B=0 & \text{coeff. dei termini di grado 1} \\ -2A-3B=1 & \text{'' '' '' di grado 0.} \end{cases}$$

$$B=-A$$

$$-2A-3(-A)=1 \qquad A=1$$

$$B=-1$$

$$\int \frac{1}{x^2-5x+6}\,dx = \int \frac{1}{(x-2)(x-3)}\,dx = \left(\text{fratti semplici}\right)$$

$$= \int \left(\frac{1}{x-3} - \frac{1}{x-2}\right) dx$$

$$= \int \frac{1}{x-3}\,dx - \int \frac{1}{x-2}\,dx$$

$$= \log|x-3| - \log|x-2| =$$

$$= \log\left|\frac{x-3}{x-2}\right| + K$$

Morale se

$$\int \frac{1}{Ax^2+Bx+C}\,dx$$

con $Ax^2+Bx+C=0$ che ha $\Delta>0$

x_1, x_2 le soluzioni dell'eq.

$$Ax^2+Bx+C = A(x-x_1)(x-x_2)$$

$$\int \frac{1}{Ax^2+Bx+C}\,dx = \int \frac{1}{A(x-x_1)(x-x_2)}\,dx$$

questo integrale si risolve con la decomposizione in fratti semplici

$$\frac{1}{(x-x_1)(x-x_2)} = \frac{\bar{A}}{(x-x_1)} + \frac{\bar{B}}{(x-x_2)}$$

dove $\bar{A}$, $\bar{B}$ sono da determinare

e quindi

$$\int \frac{1}{Ax^2+Bx+C}\,dx = \frac{1}{A}\left(\bar{A}\int \frac{1}{x-x_1}\,dx + \bar{B}\int \frac{1}{x-x_2}\,dx\right)$$

$$= \frac{1}{A}\left(\bar{A}\log|x-x_1| + \bar{B}\log|x-x_2|\right)$$