# Lezione del 16 Dicembre

- $\displaystyle\int_a^b f(x)\,dx$ dove $f(x) \xrightarrow[x \to a]{} \pm\infty$ $(x \to b)$

$\displaystyle\int_0^1 \frac{1}{x^\alpha}\,dx$ converge sse $\alpha < 1$

$\displaystyle\int_0^{1/2} \frac{1}{x^\alpha |\log x|^\beta}\,dx$ converge sse $\begin{cases} \alpha < 1 & \forall \beta \\ \alpha = 1 & \forall \beta > 1 \end{cases}$

- $\displaystyle\int_a^{+\infty} f(x)\,dx \longleftrightarrow \sum_{k=1}^{+\infty} a_k$

$\displaystyle\int_1^{+\infty} \frac{1}{x^\alpha}\,dx$ converge sse $\alpha > 1$ $\left(\sum \frac{1}{k^\alpha} \text{ conv. sse } \alpha > 1\right)$

$\displaystyle\int_1^{+\infty} \frac{1}{x^\alpha (\log x)^\beta}\,dx$ conv. sse $\begin{array}{l} \alpha > 1 \ \forall \beta \\ \alpha = 1 \ \forall \beta > 1 \end{array}$

2 modi per vedere se un integrale generalizzato converge o no.

1°) Con la definizione

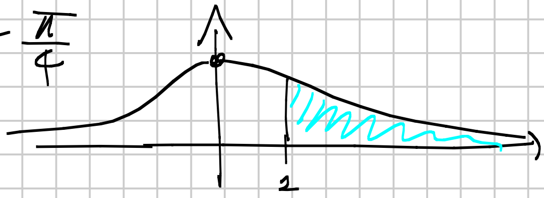
2°) Con i criteri di convergenza.

Es. 1° modo $\displaystyle\int_1^{+\infty} \frac{1}{1+x^2}\,dx = \lim_{\omega \to +\infty} \int_1^{\omega} \frac{1}{1+x^2}\,dx$

$\displaystyle = \lim_{\omega \to +\infty} \left(\operatorname{arctg} x\right)\Big|_1^{\omega} = \lim_{\omega \to +\infty} \operatorname{arctg}\omega - \operatorname{arctg} 1$

$= \frac{\pi}{2} - \frac{\pi}{4}$

$\displaystyle\int_1^{+\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} - \frac{\pi}{4}$



Con questo metodo si trova anche a cosa converge, non solo se converge.

2°) Criteri di integrabilità

$$\int_1^{+\infty} \frac{1}{1+x^2}\,dx \qquad f(x)=\frac{1}{1+x^2}$$

$x\to+\infty \quad f\to 0$

$$f(x)\sim\frac{1}{x^2}$$

$$\int_1^{+\infty}\frac{1}{x^2}\,dx \qquad \alpha=2>1 \text{ converge} \quad \Rightarrow \int_1^{+\infty} f(x)\,dx \text{ converge.}$$

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es. dire se converge

$$\int_0^{+\infty}\frac{1}{\sqrt{x^2+1}}\,dx$$

$$f(x)=\frac{1}{\sqrt{x^2+1}}\sim\frac{1}{x} \qquad \boxed{x\to+\infty}$$

$$\underbrace{\int_1^{+\infty} f(x)\,dx \text{ conv.}}_{\text{non converge.}} \text{ sse } \underbrace{\int_1^{+\infty}\frac{1}{x}\,dx}_{\text{non converge}}$$

$$\int_0^{+\infty}\frac{1}{\sqrt{1+x^2}}\,dx = \underbrace{\int_0^1\frac{1}{\sqrt{1+x^2}}\,dx}_{\substack{\text{è integrale definito}\\ \text{cioè un numero!}}} + \int_1^{+\infty}\frac{1}{\sqrt{1+x^2}}\,dx$$

oss.

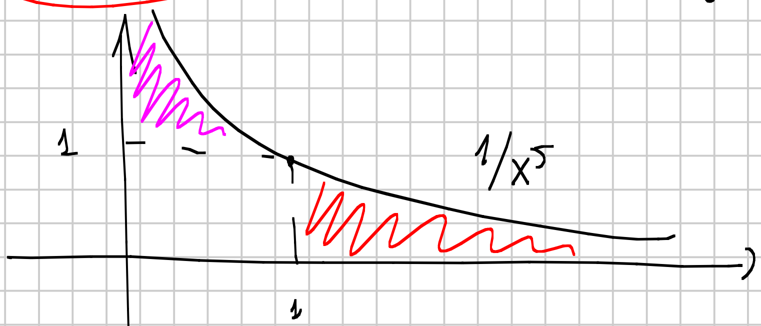
$$\int_0^{+\infty}\frac{1}{x^5}\,dx = \underbrace{\int_0^1\frac{1}{x^5}\,dx}_{\text{diverge}} +$$

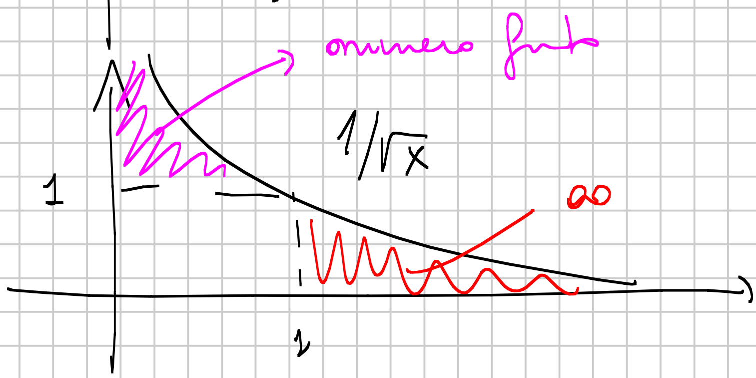
$$+\int_1^{+\infty} \frac{1}{x^5}\,dx$$ converge

$\alpha = 5 > 1$

in totale $\int_0^{+\infty} \frac{1}{x^5}\,dx$ diverge.

$f(x) = \frac{1}{x^5}$

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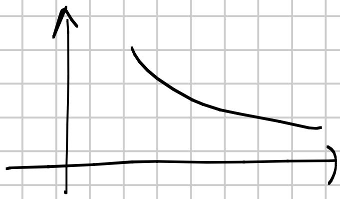
es. $$\int_1^{+\infty} \left(1 - \cos\frac{1}{x}\right) dx$$

f continua in intervallo illimitato.



$f(x) = 1 - \cos\frac{1}{x}$

$x \to +\infty$ $\qquad f \to 0$

$x \to +\infty \qquad \frac{1}{x} \to 0 \qquad \frac{1}{x} = y$

$$\cos y = 1 - \frac{y^2}{2} + o(y^2) \qquad y \to 0$$

$$\Updownarrow$$

$$\cos y \sim 1 - \frac{y^2}{2} \qquad y \to 0$$

$$\cos\frac{1}{x} \sim 1 - \frac{1}{2x^2} \qquad x \to +\infty$$

$$f(x) = 1 - \cos\frac{1}{x} \sim \cancel{1} - \left(\cancel{1} - \frac{1}{2x^2}\right)$$

$x \to +\infty$

$$f(x) \sim \frac{1}{2x^2}$$

$$\int_1^{+\infty} f(x)\,dx \text{ conv.} \iff \int_1^{+\infty} \frac{1}{x^2}\,dx \text{ conv.}$$

Sì!

es. $$\int_1^{+\infty} \underbrace{x^2\left(\operatorname{sen}\frac{1}{x} - \frac{1}{x}\right)}_{f(x)} dx$$

$x \to +\infty \qquad \frac{1}{x} \to 0$

$y = \frac{1}{x} \qquad y \to 0 \qquad x \to +\infty$

$$\operatorname{sen} y = y - \frac{y^3}{3!} + o(y^3)$$

$$\operatorname{sen} y \sim y - \frac{y^3}{6}$$

$$\operatorname{sen}\frac{1}{x} \sim \frac{1}{x} - \frac{1}{6x^3}$$

$$f(x) = x^2\left(\operatorname{sen}\frac{1}{x} - \frac{1}{x}\right) \sim \cancel{x^2}\left(-\frac{1}{6x^{\cancel{3}}}\right) = -\frac{1}{6x}$$

$$f(x) \sim -\frac{1}{6x} \qquad \int_1^{+\infty} \frac{1}{x}\,dx \text{ diverge}$$

$$\Rightarrow \int_1^{+\infty} f(x)\,dx \text{ diverge.}$$

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dire se converge.

$$\int_1^{+\infty} \frac{3x^2+1}{x^5 + e^{1/x} + \operatorname{sen} x}\,dx$$

$$f(x) = \frac{3x^2+1}{x^5 + e^{1/x} + \operatorname{sen} x}$$

bisogna studiare il comportamento di $f$ per $x \to +\infty$

$$f(x) = \frac{3\cancel{x^2}\overbrace{\left(1 + \frac{1}{3x^2}\right)}^{\to 1}}{x^{\cancel{5}\,3}\left(1 + \frac{e^{1/x}}{\ldots} + \frac{\operatorname{sen} x}{\ldots}\right)}$$

$x^5 \qquad x^5$

$x \to +\infty$

$$f(x) \sim 3\,\frac{1}{x^3} \qquad x \to +\infty$$

$$\int_1^{+\infty} \frac{1}{x^3}\,dx \text{ converge} \overset{\alpha = 3 > 1}{\Longrightarrow} \int_1^{+\infty} f(x)\,dx \text{ converge.}$$

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es.

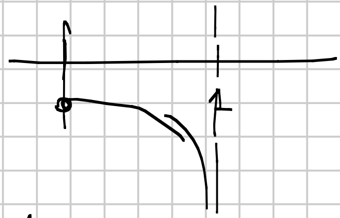
$$\int_2^{+\infty} \frac{1}{x^2-1}\,dx \qquad f(x) \sim \frac{1}{x^2} \quad x \to +\infty$$

$$\int_2^{+\infty} f(x)\,dx \text{ converge.}$$

$$\int_0^1 \frac{1}{x^2-1}\,dx \qquad f(x) = \frac{1}{(x-1)(x+1)}$$

è illimitata in $x = 1$

$$\lim_{x \to 1^-} f(x) = -\infty$$



$$\int_0^1 \frac{1}{(x-1)(x+1)}\,dx \qquad x - 1 = t$$

$x = t + 1$

$dx = dt$

$$= \int_{-1}^{0} \frac{1}{t(t+2)}\,dt$$

$$= -\int_0^{-1} \frac{1}{t(t+2)}\,dt$$

problemi in $t = 0$

$(x = 1)$

$$f(t) = \frac{1}{t(t+2)} \sim \frac{1}{t} \qquad t \to 0$$

$$\int_0^{-1} \frac{1}{t(t+2)}\,dt \text{ converge}$$

sse

$$\Rightarrow \int_0^1 \frac{1}{x^2-1}\,dx \quad \text{diverge,}$$

$\int_0^{-1} \frac{1}{t}\,dt$ converge. no!

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es. Dire per quali $\alpha \in \mathbb{R}$

1) $f(x) = \dfrac{\operatorname{sen} \frac{1}{x}}{x^\alpha \log^2 x}$ tende a zero per $x \to +\infty$

2) è integrabile in $[2, +\infty)$.

1) $\operatorname{sen} \frac{1}{x} \sim \frac{1}{x} \qquad x \to +\infty$

$$f(x) \sim \frac{1}{x \cdot x^\alpha \log^2 x} = \frac{1}{x^{\alpha+1} \cdot \log^2 x}$$

per quali $\alpha$ $\downarrow 0$ ?

a) se $\alpha+1 > 0 \qquad f \to 0$

b) se $\alpha+1 = 0 \qquad f \to 0$

c) se $\alpha+1 < 0 \qquad f \to +\infty$

Quindi $f \to 0 \iff \alpha+1 \geq 0 \iff \boxed{\alpha \geq -1}$

2) Per quali $\alpha$ $f$ è integrabile

Da 1) $f(x) \sim \dfrac{1}{x^{\alpha+1} \log^2 x} \qquad x \to +\infty$

$f$ è integrabile in $(2, +\infty)$ se $\dfrac{1}{x^{\alpha+1} \log^2 x}$ è integrabile in $[2, +\infty)$

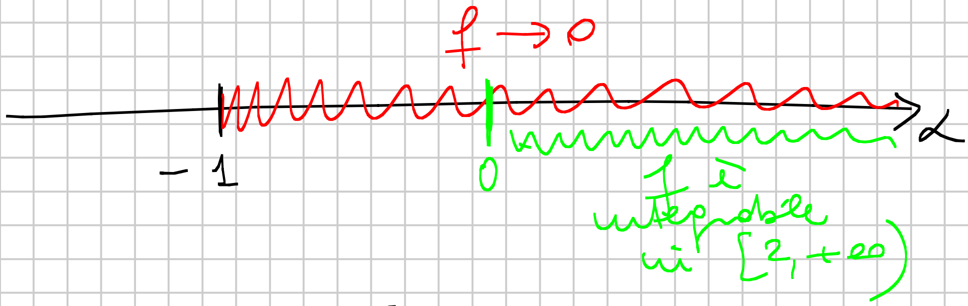
$\Updownarrow$ cioè se

$\beta = 2$

$$\alpha+1 \geq 1 \iff \begin{cases} \alpha+1 > 1 \\ \alpha+1 = 1 \end{cases}$$

$\boxed{\alpha \geq 0}$

$f$ è integrabile in $[2, +\infty) \iff \alpha \geq 0$

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**Es.** $$\int_0^{1/2} \frac{e^x - 1}{x^\alpha \sqrt{1-x}}\, dx$$

Dire per quali $\alpha \in \mathbb{R}$ l'integrale converge.

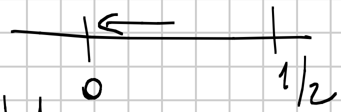
$$f(x) = \frac{e^x - 1}{x^\alpha \sqrt{1-x}}$$

se $\alpha < 0$ $\quad f(x)$ è limitata in $[0,1]$

$\Rightarrow \int_0^1 f(x)\,dx$ è un integrale definito cioè un numero (quindi converge)

se $\alpha = 0$ $\quad$ // $\quad$ // $\quad$ // $\quad$ //

se $\alpha > 0$ $\quad$ integrale generalizzato perché $\lim_{x \to 0^+} f(x) =$ può essere illimitato.




$$f(x) = \frac{e^x - 1}{x^\alpha \underbrace{\sqrt{1-x}}_{\sim 1,\ x \to 0}}$$

$e^x \sim 1 + x \quad x \to 0$

$e^x - 1 \sim x \quad x \to 0$

$x \to 0$

$$f(x) \sim \frac{x}{x^\alpha \cdot 1} = \frac{1}{x^{\alpha - 1}} \quad x \to 0$$

$$\int_0^{1/2} f(x)\,dx \text{ conv.} \iff \int_0^{1/2} \frac{1}{x^{\alpha-1}}\,dx \text{ converge}$$

$$\Downarrow$$

$$\alpha - 1 < 1$$

cioè $\boxed{\alpha < 2}$

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$$\int_1^{+\infty} \frac{(1+x^3)^{1-\alpha}}{x^{2-\alpha}(\log^2 x + 2\log x + 1)} \, dx$$

$$(\log x + 1)^2$$

1) dire per quali $\alpha \in \mathbb{R}$ l'$\int$ converge

2) Calcolarlo per $\alpha = 1$

1) $f(x)$ è continua in $[1, +\infty)$

$$f(x) \sim \frac{x^{3\cdot(1-\alpha)}}{x^{2-\alpha}(\log x)^2} =$$

$$= \frac{1}{x^{2-\alpha-3+3\alpha}(\log x)^2}$$

$$f(x) \sim \frac{1}{x^{2\alpha-1}(\log x)^2} \qquad x \to +\infty$$

$$\int_1^{+\infty} \frac{1}{x^{2\alpha-1}(\log x)^2}$$ converge sse

$$2\alpha - 1 \geq 1$$

$$2\alpha \geq 2$$

$$\alpha \geq 1$$

2) $\alpha = 1$

$$\int_1^{+\infty} \frac{1}{x(\log x + 1)^2} \, dx$$

$\log x = t$

$\frac{1}{x} dx = dt$

$$= \int_0^{+\infty} \frac{1}{(t+1)^2}\,dt =$$

$$= \lim_{\omega \to +\infty} \int_0^{\omega} \frac{1}{(t+1)^2}\,dt$$

$$= \lim_{\omega \to +\infty} \left(-\frac{1}{(t+1)}\right)\Big/_0^{\omega} =$$

$$= \lim_{\omega \to +\infty} \left(-\frac{1}{\omega+1} + 1\right) = 1$$

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PC.

$$\int_1^{+\infty} \frac{e^{\frac{1}{x}} - 1}{\log x}\,dx$$

$$\int_1^{+\infty} \frac{\operatorname{sen} \frac{1}{x}}{x^{\alpha} + x^2}\,dx$$

dire per quali $\alpha \in \mathbb{R}$ conv.

$$\int_1^{+\infty} \frac{x^{2\alpha}}{x^2 + 1}$$

1) per quali $\alpha \in \mathbb{R}$ $f(x) \to 0$

2) per quali $\alpha \in \mathbb{R}$ l'$\int$ converge.

3) calcolarlo per $\alpha = 0$.

$$\int_0^1 \frac{\operatorname{sen}(x^{\alpha})}{x^2 \cos x}\,dx$$

per quali $\alpha \geq 0$ converge.

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Integrazione di alcune funzioni irrazionali

$f(x) = \sqrt{1+x^2}$ , $\sqrt{1-x^2}$ , $\sqrt{x^2-1}$

$$\int \sqrt{1-x^2}\,dx =$$

$\sin^2 t + \cos^2 t = 1$

$1 - \sin^2 t = \cos^2 t$

$x = \sin t$

$dx = \cos t\,dt$

$$= \int \sqrt{1-\sin^2 t}\,\cos t\,dt$$

$$= \int \sqrt{\cos^2 t}\,\cos t\,dt = \int |\cos t| \cos t\,dt$$

$$= \begin{cases} \int \cos^2 t\,dt & \cos t > 0 \\ -\int \cos^2 t\,dt & \cos t < 0 \end{cases}$$

$$\int \cos^2 t\,dt = \int \cos t \cdot \cos t\,dt = \text{per parti} \ldots$$

$$\int \cos^2 t\,dt = F(t)$$

$x = \sin t$

$t = \arcsin x$

$$= F(\arcsin x)$$

$$\int_0^1 \sqrt{1-x^2}\,dx \qquad |x| \le 1$$

Integrali che coinvolgono $\sqrt{1-x^2}$ si fanno con la sostituzione $x = \sin t$

es. $$\int \sqrt{4-x^2}\,dx = \int \sqrt{4\left(1-\left(\frac{x}{2}\right)^2\right)}\,dx$$

$$= 2\int \sqrt{1-\left(\frac{x}{2}\right)^2}\,dx$$

$$\frac{x}{2} = \sin t$$

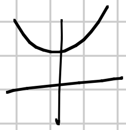
$$\int \sqrt{1+x^2}\, dx =$$

$\cosh^2 t - \operatorname{senh}^2 t = 1$

$\cosh^2 t = 1 + \operatorname{senh}^2 t$

$x = \operatorname{senh} t$

$dx = \cosh t \, dt$



$$= \int \sqrt{1+\operatorname{senh}^2 t}\, \cosh t \, dt =$$

$$= \int \sqrt{\cosh^2 t}\, \cosh t \, dt$$

$\cosh t \geq 0$

$$= \int \cosh t \cdot \cosh t \, dt = \dots\dots$$

per farli per casa

per funzioni del tipo $\sqrt{1+x^2}$ si usa la sostituzione $x = \operatorname{senh} t$

$(1 + \operatorname{senh}^2 t = \cosh^2 t)$

es. $$\int \sqrt{9+x^2}\, dx = 3\int \sqrt{1+\left(\frac{x}{3}\right)^2}\, dx$$

$$\frac{x}{3} = \operatorname{senh} t$$

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$$\int \sqrt{x^2-1}\, dx =$$

def. $|x| \geq 1$

$\cosh^2 t - \operatorname{senh}^2 t = 1$

$\cosh^2 t - 1 = \operatorname{senh}^2 t$

$x = \cosh t$

$dx = \operatorname{senh} t \, dt$

$$= \int \sqrt{\cosh^2 t - 1}\, \operatorname{senh} t \, dt =$$

$$= \int \sqrt{\operatorname{senh}^2 t} \cdot \operatorname{senh} t \, dt$$

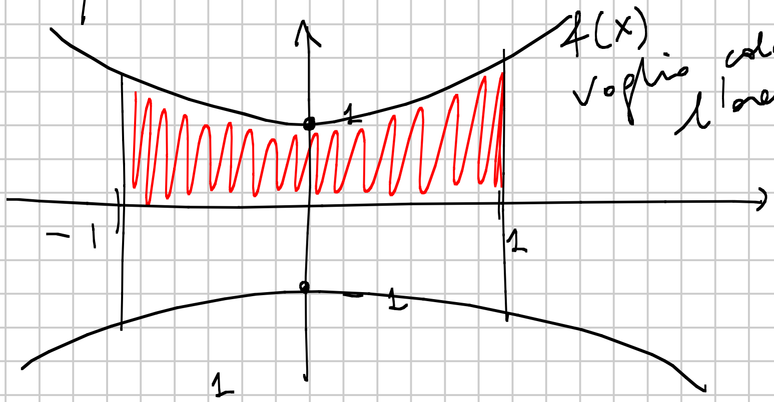
$$= \int |\operatorname{senh} t| \cdot \operatorname{senh} t \, dt = \pm \underbrace{\int \operatorname{senh} t \operatorname{senh} t \, dt}_{\text{per parti due volte (per casa)}}$$

$\boxed{\text{per funzioni del tipo } \sqrt{x^2 - 1} \text{ si usa la sostituzione } x = \cosh t}$

## Area sotto grafici "noti"

Es. Calcolare l'area sotto l'iperbole

$y^2 - x^2 = 1$ con $x \in [-1, 1]$




voglio calcolare l'area della regione ROSSA

$y = f(x)$

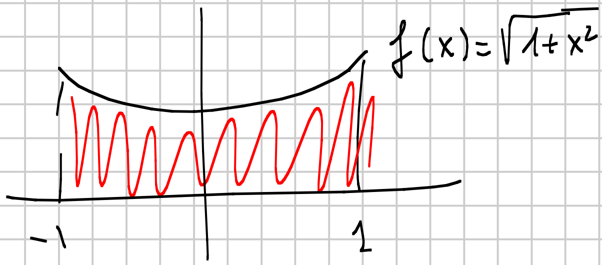
$y^2 - x^2 = 1$

$y^2 = 1 + x^2$

$y = \pm\sqrt{1 + x^2}$

$f(x) = \sqrt{1 + x^2}$

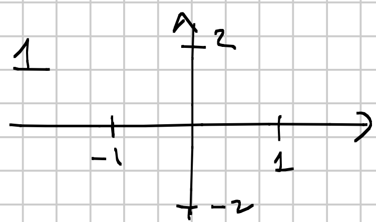
$$\text{Area} = \int_{-1}^{1} \sqrt{1 + x^2} \, dx$$

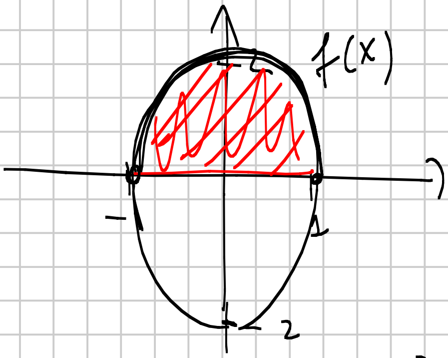



$$\int_{-1}^{1} \sqrt{1 + x^2} \, dx =$$ fate voi il conto

## Area ellisse

$$x^2 + \frac{y^2}{4} = 1$$






Area regione rossa

$$= \int_{-1}^{1} \sqrt{4 - 4x^2}\, dx =$$

$$x^2 + \frac{y^2}{4} = 1 \qquad \frac{y^2}{4} = 1 - x^2$$

$$y^2 = 4 - 4x^2$$

$$y = \pm\sqrt{4 - 4x^2}$$

$$f(x) = \sqrt{4 - 4x^2}$$

$$\int_{-1}^{1} 2\sqrt{1 - x^2}\, dx$$