Lezione del 17 Dicembre

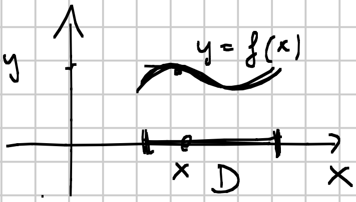
FUNZIONI IN PIÙ VARIABILI

$f: \boxed{D \subseteq \mathbb{R}} \to \mathbb{R}$ funzioni in una variabile

$x \in D \longrightarrow f(x) \in \mathbb{R}$

$y = f(x)$

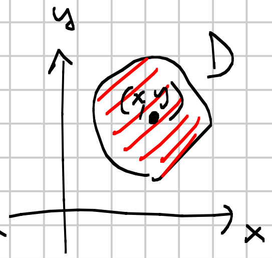
grafico $= \{ (x, f(x)), \ x \in D \}$



funzione in due variabili

$f: D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$

$(x,y) \in D \longrightarrow f(x,y) \in \mathbb{R}$



es. • $f(x,y) = x^2 + 2y^2 + \operatorname{sen} x$ Dominio $= \mathbb{R}^2$

$f(1,3) = 1 + 2 \cdot 9 + \operatorname{sen} 1$

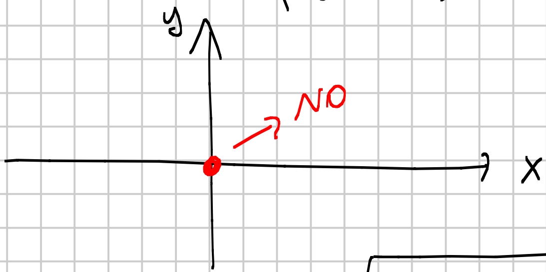
• $f(x,y) = \sqrt{x^2 + y^2}$ Dominio $= \mathbb{R}^2$

$f(-2,-1) = \sqrt{4+1} = \sqrt{5}$

• $f(x,y) = \dfrac{1}{x^2+y^2}$

$x^2 + y^2 = 0 \iff (x,y) = (0,0)$

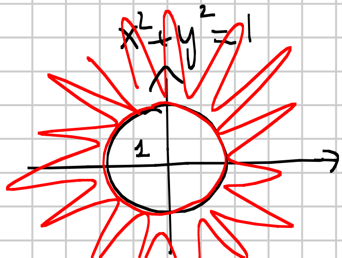
$D = \mathbb{R}^2 \setminus \{(0,0)\}$



• $f(x,y) = \sqrt{x^2 + y^2 - 1}$

$D: \ x^2 + y^2 \geq 1$

tutti i punti fuori della circonf e sulla circonf.

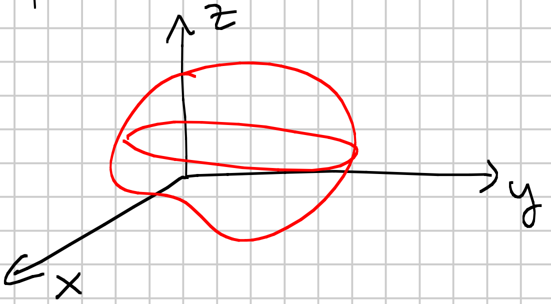


$$f : D \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}$$
$$(x,y,z) \longrightarrow f(x,y,z) \in \mathbb{R}$$

funzioni di 3 variabili

es. • $f(x,y,z) = 3z^2 + 5x + \sin(xyz)$

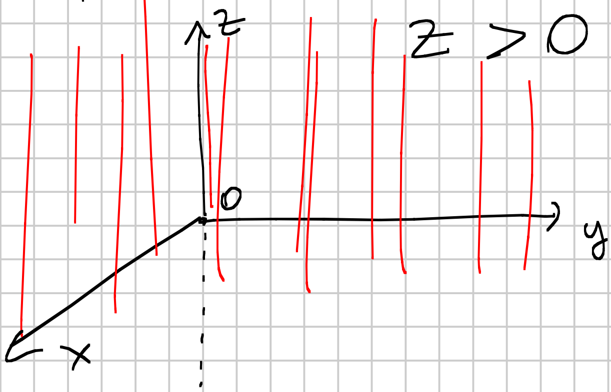
$f(1,2,3) = 3 \cdot 9 + 5 \cdot 1 + \sin(6)$



$D \subseteq \mathbb{R}^3$

$D = \mathbb{R}^3$

• $f(x,y,z) = \log z + 3xy + y^2$



$z > 0$

(x,y,z), $z > 0$

sono i punti che stanno sopra il piano xy

• $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$

$(x_1, x_2, \dots x_n) \longrightarrow f(x_1, x_2, \dots x_n)$

Funzioni di due variabili

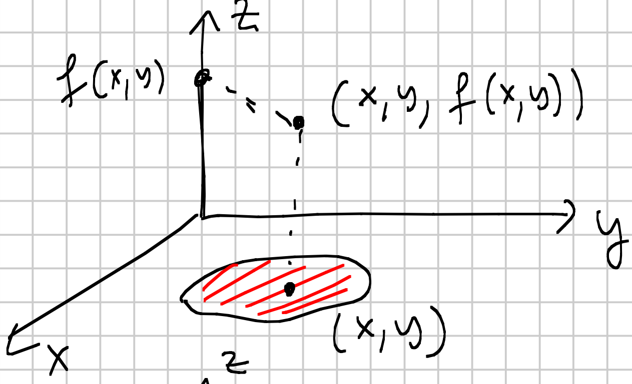
$$f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$$

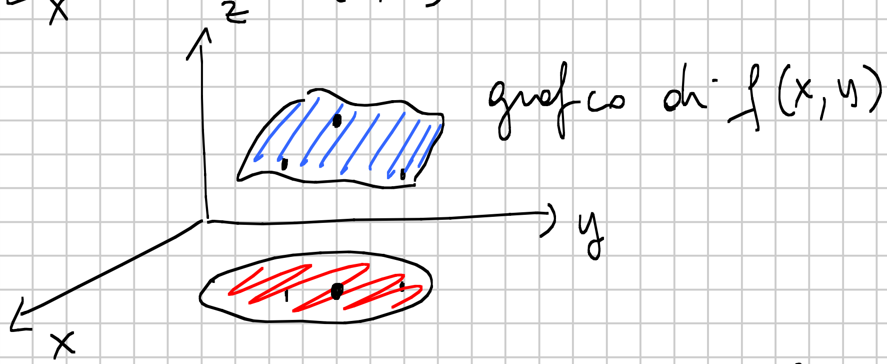
Dominio D si disegna in $\mathbb{R}^2$

$(x,y) \longrightarrow f(x,y)$

$(x,y) \longrightarrow z = f(x,y)$

$(x,y,z) = (x,y,f(x,y)) \in \mathbb{R}^3$





$$\text{graf } f \doteq \{ (x, y, f(x,y)),\ (x,y) \in D \}$$

$$f : D \subseteq \mathbb{R}^3 \to \mathbb{R}$$

$$\text{graf } f = \{ (x,y,z, f(x,y,z)) \in \mathbb{R}^4 \}.$$

NON SI VEDE!

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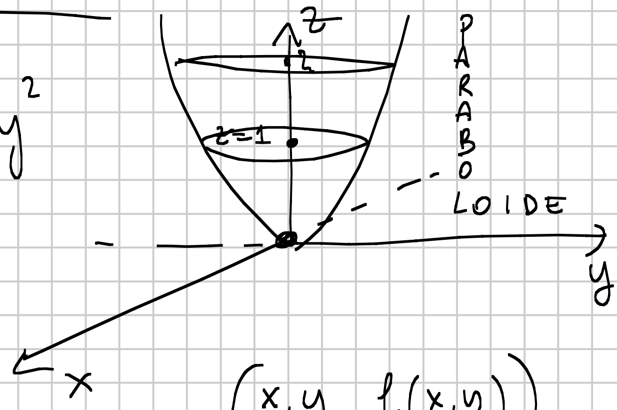
es.

$$f(x,y) = x^2 + y^2$$

$$D = \mathbb{R}^2$$



$f(0,0) = 0$

$(0,0,0) \in \text{graf } f$

$f(-1,5) = 1 + 25 = 26$

$(x, y, f(x,y))$

$(x, y, \underbrace{x^2 + y^2}_{\geq 0})$

il grafico sta sopra il piano orizzontale xy

$f(x,y) = x^2 + y^2$

se pongo $z = 1 = f(x,y)$

$x^2 + y^2 = 1$

$x^2 + y^2 = 2$

Limiti e funzioni continue in + variabili

$f : D \subseteq \mathbb{R} \to \mathbb{R}$ $\quad$ f è continua in $x_0 \in D$

$$\lim_{x \to x_0} f(x) = f(x_0)$$

$$\lim_{x \to x_0} f(x) = \ell \iff \forall V \text{ di } \ell \ \exists \ U \ x_0 : f(x) \in V \ \forall x \in U.$$

per una variabile

$f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$

?

$$\lim_{\boxed{(x,y) \to (x_0,y_0)}} f(x,y) = \ell$$

Devo definire gli intorni U di (x_0, y_0) punto in $\mathbb{R}^2$.

Intorni

prodotto scalare in $\mathbb{R}^2$

$P_1 = (x_1, y_1)$

$P_2 = (x_2, y_2)$

$P_1 \cdot P_2 = (x_1, y_1) \cdot (x_2, y_2) = x_1 x_2 + y_1 y_2$

es. $(-1, 5) \cdot (3, 2) = -1 \cdot 3 + 5 \cdot 2 = 7$

Analog. in $\mathbb{R}^3$

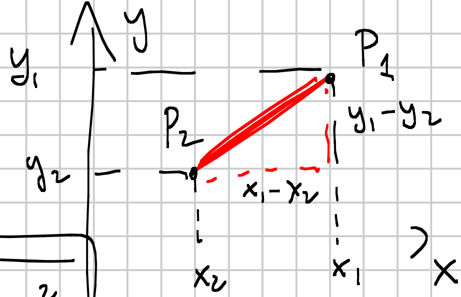
$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = x_1 x_2 + y_1 y_2 + z_1 z_2$

es. $(1, 2, 3) \cdot (-1, 5, 2) = -1 + 10 + 6$
$= 15.$

In $\mathbb{R}^2$ si definisce la distanza tra due punti

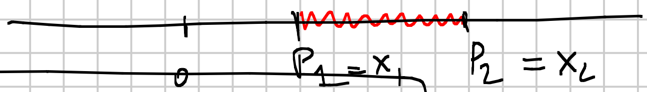
$P_1 = (x_1, y_1)$

$P_2 = (x_2, y_2)$



$$\boxed{d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}$$

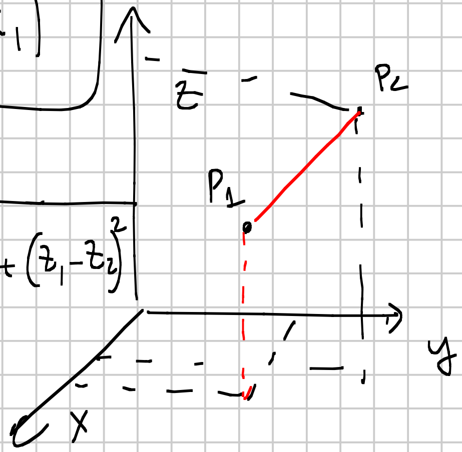
in $\mathbb{R}$



$$\boxed{d(P_1, P_2) = |x_2 - x_1|}$$

in $\mathbb{R}^3$

$$d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$



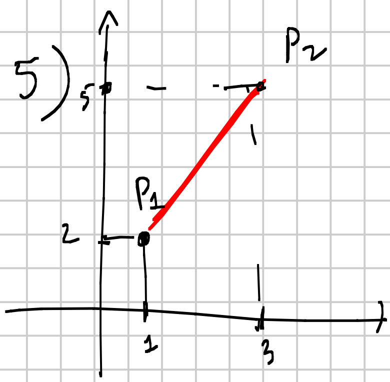
$P_1 = (x_1, y_1, z_1)$

$P_2 = (x_2, y_2, z_1)$

es. in $\mathbb{R}^2$ $P_1 = (1, 2)$, $P_2 = (3, 5)$

$$d(P_1, P_2) = \sqrt{(3-1)^2 + (5-2)^2} =$$

$$= \sqrt{4 + 9} = \sqrt{13}$$



in $\mathbb{R}^3$ $P_1 = (0, 1, 3)$

$P_2 = (-2, -1, 0)$

$$d(P_1, P_2) = \sqrt{4 + 4 + 9} = \dots$$

Intorno (sferico)

$$B_\varepsilon(P) = \{ Q \in \mathbb{R}^2 \quad d(P, Q) < \varepsilon \}$$

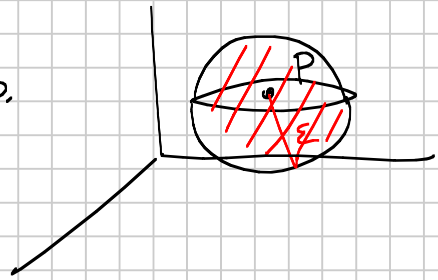
cerchio centrato in P di raggio ε.

ricordiamo gli intorni in $\mathbb{R}$

$$U = (x_0-\varepsilon,\ x_0+\varepsilon)$$

in $\mathbb{R}^3$ $\quad B_\varepsilon(P) = \{ Q \in \mathbb{R}^3 : d(P,Q) < \varepsilon \}$

intorni in $\mathbb{R}^3$.



<u>Def. di limite per funzioni in due variabili</u>

$f : D \subseteq \mathbb{R}^2 \to \mathbb{R} \qquad P_0 = (x_0, y_0)$

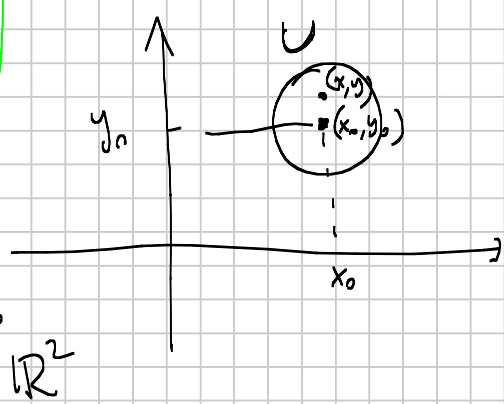
$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \ell \iff \forall\ V$ intorno di ℓ $\exists\ U$ intorno di (x_0,y_0) t.c. $f(x,y) \in V$ se $(x,y) \in U$

<u>esplicitamente</u> $\quad \ell \in \mathbb{R} \qquad V = (\ell-\varepsilon, \ell+\varepsilon)$

$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \ell \iff \forall \varepsilon > 0\ \exists\ \delta$ t.c. $|f(x,y) - \ell| < \varepsilon$ se $d((x,y),(x_0,y_0)) < \delta$

$V = (\ell-\varepsilon, \ell+\varepsilon),\ \ell \in \mathbb{R}$

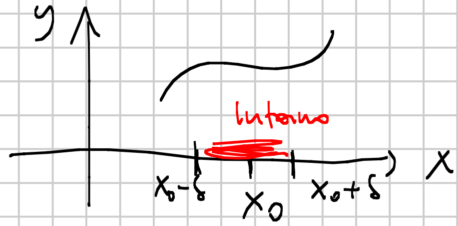
$U = B_\delta((x_0,y_0))$



Differenze tra il calcolo dei limiti in $\mathbb{R}$ e in $\mathbb{R}^2$

$f: D \subseteq \mathbb{R} \to \mathbb{R}$ f. di una variabile

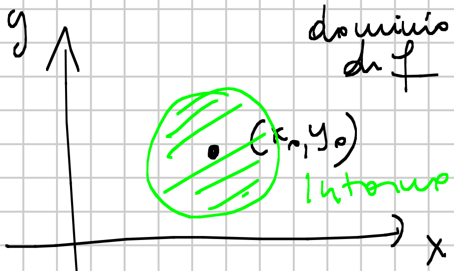
$$\lim_{x \to x_0} f(x) = \ell$$



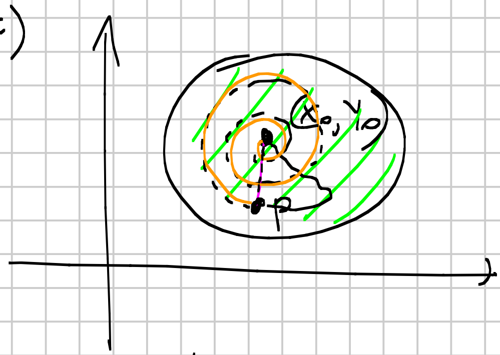
$f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$

$$\lim_{(x,y) \to (x_0,y_0)} f(x,y) = \ell$$

f di due variabili



In questo caso è tutto (x,y) si avvicina a (x_0,y_0) con 2 gradi di libertà (in tutte le direzioni possibili e su tutte le curve possibili)



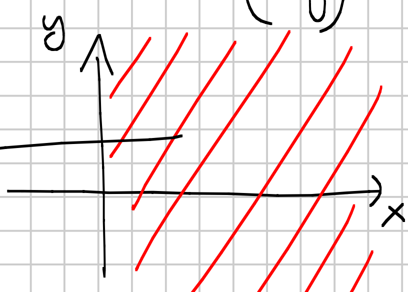
Def. $f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$ è continua in $P_0 = (x_0,y_0)$ se

$$\lim_{(x,y) \to (x_0,y_0)} f(x,y) = f(x_0,y_0)$$

Valgono tutti i teoremi sui limiti e sulle funzioni continue che abbiamo fatto per le f. in una variabile

es. $f(x,y) = x^2 + 3xy + \log x + \operatorname{sen}(xy)$

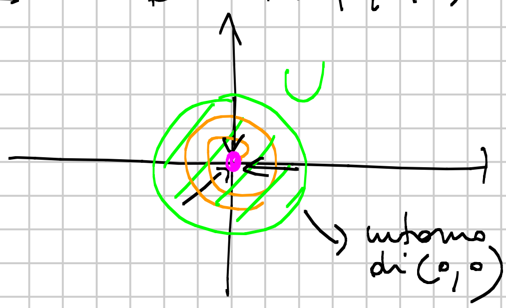
$D = \{(x,y) \in \mathbb{R}^2 : x > 0\}$



Dominio.

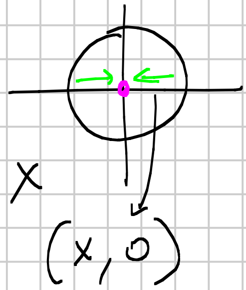
$f(x,y)$ dove è definita è continua perché somma e prodotto di f. continue.

$$f(x,y) = \frac{xy}{x^2+y^2} \qquad D = \mathbb{R}^2 \setminus \{0,0\}$$



$$f(x) = \frac{\operatorname{sen} x}{x} \qquad \lim_{x\to 0} f(x) = ?$$

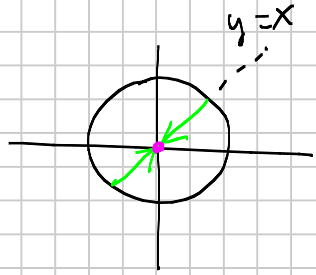
$$? \quad \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} =$$



1) avviciniamoci a $(0,0)$ sull'asse x

$$f(x,y) = \frac{xy}{x^2+y^2}$$

$$f(x,0) = 0 \xrightarrow[x\to 0]{} 0$$



2) avviciniamoci a $(0,0)$ sulla bisettrice del 1° quadrante

$$y = x$$

$$\underbrace{f(x,y)}_{\text{calcolata sulla bisettrice } y=x} = \frac{x \cdot x}{x^2+x^2} = \frac{x^2}{2x^2} = \frac{1}{2}$$

$$f(x,x) = \frac{1}{2} \xrightarrow[x\to 0]{} \frac{1}{2}$$

$$\Rightarrow \nexists \text{ limite } \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

Ho trovato 2 direzioni nelle quali il limite è diverso.