Lezione del 21 Ottobre

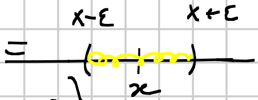
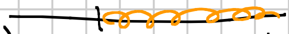
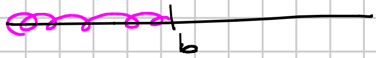
Sul sito WEB ⇒ regole d'esame

1° appello : prima giorni di febbraio
2° appello ~ 3ª settimana di febbraio

Scritto < parte A: 2 o 3 domande di teoria (20 minuti)
parte B: 4 esercizi (portare il foglio delle regole)
2 ore di tempo

Limiti di funzione

Intorni di punti

- $x \in \mathbb{R}$ $\quad \mathcal{U} = (x-\varepsilon, x+\varepsilon) = $ 
$= \{ y : |y-x| < \varepsilon \}$
- $+\infty$ $\quad \mathcal{U} = (a, +\infty) = \{ y : y > a \}$ 
- $-\infty$ $\quad \mathcal{U} = (-\infty, b) = \{ y : y < b \}$ 

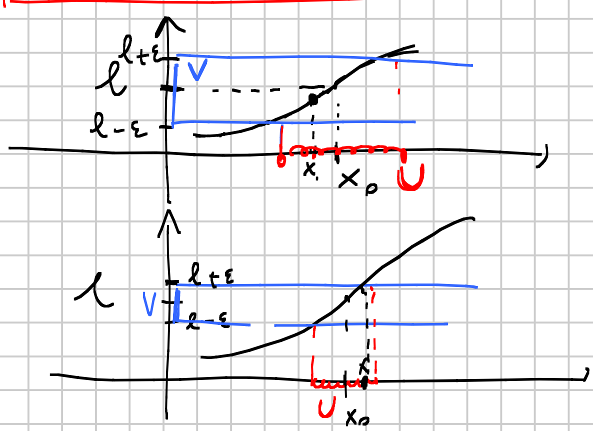
- x_0 p.to di accumulazione per E se
$\forall\, U$ di x_0 $\exists\, y \neq x$, $y \in U \cap E$

Definizione generale di limite di funzione

$f : X \subseteq \mathbb{R} \to \mathbb{R}$, x_0 punto di accumul. per X, $\ell \in \mathbb{R}^*$

$$\boxed{\lim_{x \to x_0} f(x) = \ell \iff \forall\, V \text{ intorno di } \ell \;\; \exists\, \mathcal{U} \text{ intorno di } x_0 \text{ tale che } \forall x \in \mathcal{U} \cap X,\ x \neq x_0 \text{ si ha } f(x) \in V}$$

$\ell, x_0 \in \mathbb{R}$



$\forall$ intorno V di ℓ
$\exists$ intorno U di x_0
t.c. $\forall x \in U \cap X$
$f(x) \in V$

OSS. l'intorno U di x_0 dipende da come scegliamo V, intorno di ℓ.

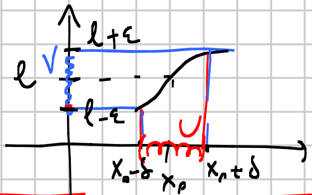
La def di limite esprime rigorosamente il concetto:

mi posso avvicinare "quanto voglio" a ℓ se mi avvicino "opportunamente" a x_0

$\forall V$ $\quad \exists U$

Riscriviamo la def. scrivendo esplicitamente gli intorni.

$\lim_{x \to x_0} f(x) = \ell \iff \forall V$ intorno di ℓ $\exists U$ intorno di x_0 t.c. se $x \in U \cap X, x \neq x_0$ si ha $f(x) \in V$



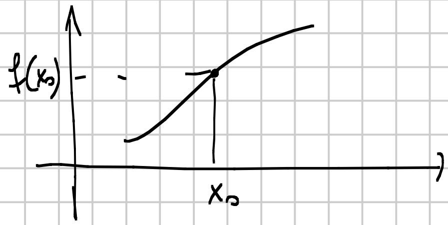
$V = (\ell - \varepsilon, \ \ell + \varepsilon)$

$U = (x_0 - \delta, \ x_0 + \delta)$

Nel caso $x_0, \ell \in \mathbb{R}$

$$\lim_{x \to x_0} f(x) = \ell \iff \forall \varepsilon > 0 \ \exists \delta > 0 \text{ t.c. se } |x - x_0| < \delta, \ x \in X, \ x \neq x_0 \text{ si ha } |f(x) - \ell| < \varepsilon$$

OSS. 1 Se si cambia $\varepsilon \Rightarrow$ cambia δ $\quad (\delta = \delta(\varepsilon))$



OSS. si prende, nella definizione, $x \neq x_0$

Abbiamo scritto $\forall x \in U \cap X$, $x \neq x_0$.

non si richiede che $|f(x_0) - \ell| < \varepsilon$ ma solo che $|f(x) - \ell| < \varepsilon$ per gli x "vicini" a x_0

Es.

$f(x_0) \notin V$

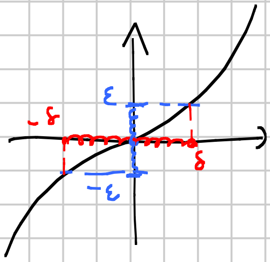
$\lim_{x \to x_0} f(x)$

oss. x_0 può non appartenere al dominio X di f ma deve essere di accumulazione per X. Così $\forall\, U$ intorno di x_0 troviamo $x \in X$ e

$$(\overset{x}{\cdot}\underset{x_0}{|}\ )$$

quindi possiamo calcolare $f(x)$

Es. $\lim\limits_{\underset{x_0}{x \to 0}} x^3 = \underset{\ell}{0}$



$$\Updownarrow$$

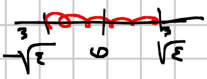
$\forall\, \varepsilon > 0 \ \exists\, \delta$ t.c. se $|x - x_0| < \delta$ si ha $|f(x) - \ell| < \varepsilon$

$\Updownarrow$ nel mio caso diventa

$$\left(\forall \varepsilon > 0 \ \exists\, \delta > 0 \text{ t.c. se } |x| < \delta \text{ si ha } \underbrace{|x^3| < \varepsilon}_{\text{risolvo in } x} \right)$$

siccome ε è fissato

$$|x^3| < \varepsilon$$
$$\Downarrow$$
$$|x| < \sqrt[3]{\varepsilon}$$
$$\Updownarrow$$
$$-\sqrt[3]{\varepsilon} < x < \underbrace{\sqrt[3]{\varepsilon}}_{=\delta}$$

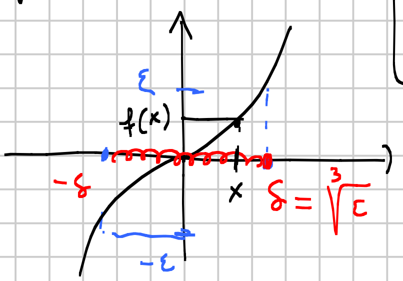


ho dim che

se $x \in (-\sqrt[3]{\varepsilon}, \sqrt[3]{\varepsilon}) = (-\delta, \delta)$

cioè se $|x| < \delta \Rightarrow |x^3| < \varepsilon$

$\forall\, \varepsilon > 0$ ho trovato il δ corrispondente $\boxed{\delta = \sqrt[3]{\varepsilon}}$



$x_0, \ell \in \mathbb{R}^*$

$\lim\limits_{x \to x_0} f(x) = \ell \iff \forall\, V$ intorno di ℓ $\exists\, U$ intorno di x_0 : $x \in U \cap X$, $x \neq x_0$ si ha $f(x) \in V$

Applicando al caso

$x_0 \in \mathbb{R}$

$\ell = +\infty$

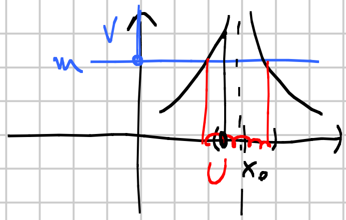
$\lim\limits_{x \to x_0} f(x) = +\infty$

V intorno di $+\infty$

$V = (m, +\infty)$

U intorno di x_0

$U = (x_0 - \delta, x_0 + \delta)$



$$\boxed{\lim_{x \to x_0} f(x) = +\infty \iff \forall m \ \exists \delta > 0 \text{ t.c. se } |x - x_0| < \delta,\ x \neq x_0 \text{ si ha } f(x) > m}$$

$x_0 \in \mathbb{R}$

$l = +\infty$

oss. dire $f(x) > m$ significa $f(x) \in (m, +\infty)$ cioè $f(x) \in V$

oss. δ dipende da m

si dice che f è divergente o infinita quando $x \to x_0$ (anche quando $l = -\infty$)

es. Verificare che

$$\lim_{x \to 1} \frac{1}{(x-1)^2} = +\infty$$

$x_0 = 1$

$l = +\infty$

$\Updownarrow$

$\forall m \ \exists \delta$ t.c. se $|x - 1| < \delta$ ho $\dfrac{1}{(x-1)^2} > m$?

Per verificare risolvo $\dfrac{1}{(x-1)^2} > m$ in x

$m > 0$

$(x-1)^2 < \dfrac{1}{m} \Rightarrow |x - 1| < \dfrac{1}{\sqrt{m}} = \delta$

$-\dfrac{1}{\sqrt{m}} < (x-1) < \dfrac{1}{\sqrt{m}}$

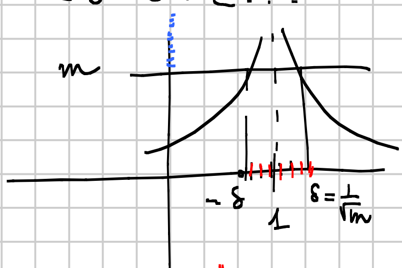
Se scelgo $\boxed{\delta = \dfrac{1}{\sqrt{m}}}$

trovo che se $|x - 1| < \delta \Rightarrow \dfrac{1}{(x-1)^2} > m$

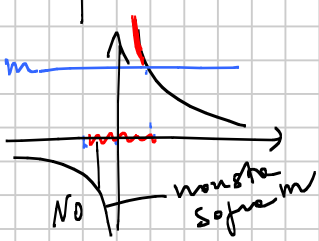
cioè $f(x) > m$

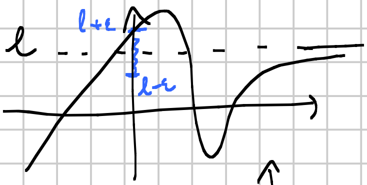
Ho dim. che $\forall m \ \exists \delta$ t.c. ...

$\delta = \dfrac{1}{\sqrt{m}}$



oss. $\displaystyle\lim_{x \to 0} \frac{1}{x} \ \nexists$



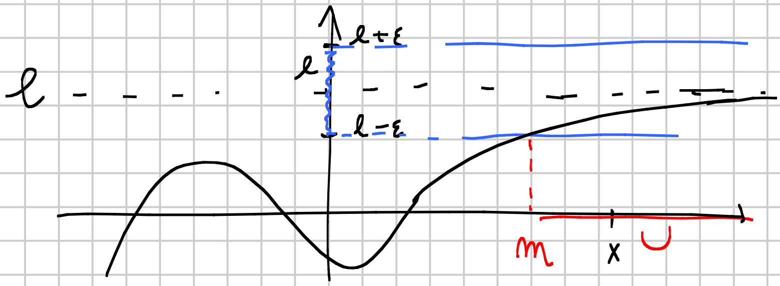


3° caso

$$\lim_{\underbrace{x\to+\infty}} f(x)=\ell \iff \forall\varepsilon>0\ \exists m \text{ t.c. se } x>m \text{ si ha } |f(x)-\ell|<\varepsilon$$

$x_0=+\infty$ $U=(m,+\infty)$ $x\in U$

$\ell\in\mathbb{R}$ $V=(\ell-\varepsilon,\ell+\varepsilon)$



es. $\lim_{x\to+\infty}\underbrace{\frac{x}{x+1}}_{f(x)}=1$

? $\forall\varepsilon>0\ \exists m$ t.c. $x>m \Rightarrow \underbrace{|f(x)-1|<\varepsilon}$?

$$\left|\frac{x}{x+1}-1\right|<\varepsilon$$ risolvo in x

$\Downarrow$

$$\left|\frac{\cancel{x}-\cancel{x}-1}{x+1}\right|<\varepsilon \Rightarrow \left|\frac{1}{x+1}\right|<\varepsilon$$

prendo $x>-1$ (tanto devo trovare $m: x>m$)

così $\frac{1}{x+1}<\varepsilon \Rightarrow x+1>\frac{1}{\varepsilon} \Rightarrow$

$x>\underbrace{\frac{1}{\varepsilon}-1}_{m}$ quindi se $x>m$ $\Rightarrow |f(x)-1|<\varepsilon$

$$m=\frac{1}{\varepsilon}-1$$

limite sbagliato

$$\lim_{x\to+\infty}\frac{x}{x+1}=5$$?

? $\forall\varepsilon>0\ \exists m$ t.c. $x>m$ $\left|\frac{x}{x+1}-5\right|<\varepsilon$

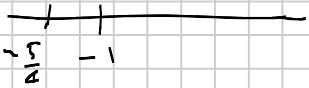
$$\left|\frac{x-5x-5}{x+1}\right|<\varepsilon$$

$$\left|\frac{-4x-5}{x+1}\right|<\varepsilon$$

$$\left|\frac{4x+5}{x+1}\right| < \varepsilon$$

$x+1>0$

$x>-1$



$x > -\frac{5}{4}$

$$\frac{4x+5}{x+1} < \varepsilon$$

$$4x+5 < \varepsilon(x+1)$$

$$4x - \varepsilon x < \varepsilon - 5$$

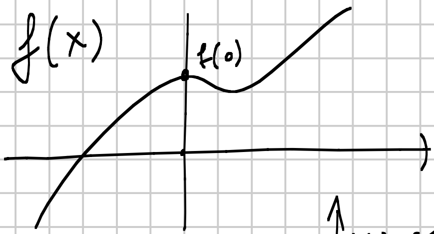
$$x(4-\varepsilon) < \varepsilon - 5$$

se $\varepsilon < 4$

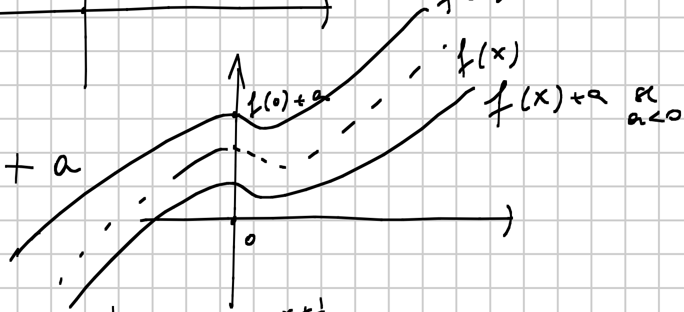
$$x < \frac{\varepsilon - 5}{4 - \varepsilon} \quad ?$$

non riesco a trovare che per $x > m$ vale $|f(x) - \overline{5}| < \varepsilon$

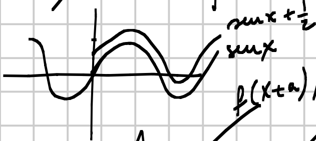
Operazioni sui grafici di funzioni



• $f(x) + a$

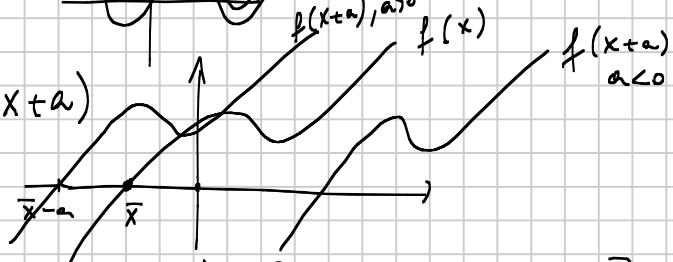


sen x



sen x + $\frac{1}{2}$

$f(x+a)$

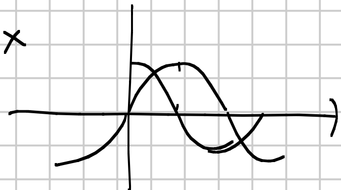


$f(\bar{x}) = 0$

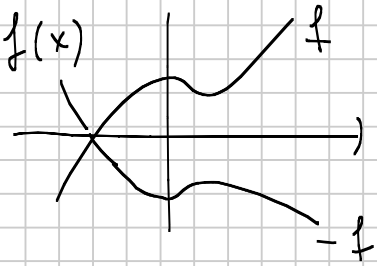
$x + a = \bar{x}$

$x = \bar{x} - a$

$\operatorname{sen}\left(x + \frac{\pi}{2}\right) = \cos x$

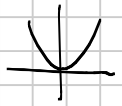


• $f(x)$

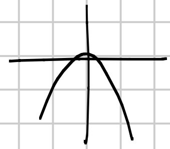


$-f(x)$

$f(x)=x^2$



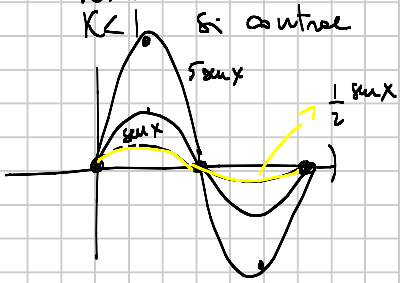
$f(x)=-x^2$



- $f(x) \Rightarrow K f(x)$ $\quad K>1$ si stira, $K<1$ si contrae

es. $f(x)= 5\,\mathrm{sen}\,x$

$f(x)= \frac{1}{2}\,\mathrm{sen}\,x$



- $f(Kx)$

$\mathrm{sen}(3x)$

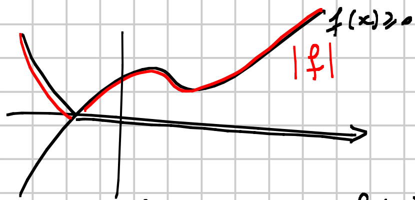
$\mathrm{sen}(3x)=0$ $\qquad 3x=0$

$3x=\pi \qquad x=\frac{\pi}{3}$

$3x=2\pi \qquad x=\frac{2\pi}{3}$

si stira o si contrae orizzontalmente.

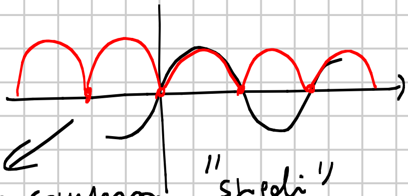
- $f(x)$



$|f| \geq 0$

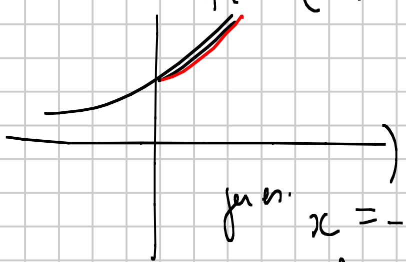
$$|f(x)| = \begin{cases} f(x) & f(x)\geq 0 \\ -f(x) & f(x)<0 \end{cases}$$

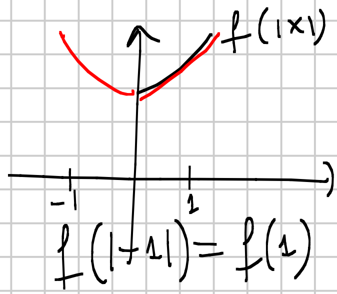
es. $|\mathrm{sen}\,x|$



sono comparsi "spigoli" dove la $f(x)$ cambia segno.

- $$f(|x|) = \begin{cases} f(x) & x\geq 0 \\ f(-x) & x<0 \end{cases}$$





per es. $x=-1$ $\quad f(|-1|)=f(1)$

funzione pari

es. disequazione

$$f(x) = \log\left(\cosh^2 x - 4\cosh x + 5\right)$$

1) dominio $\qquad \cosh x \quad \forall x \in \mathbb{R}$

$$\cosh^2 x - 4\cosh x + 5 > 0 \qquad \cosh x = y$$

$$y^2 - 4y + 5 > 0$$

$$\Delta = 4 - 5 < 0$$

$y^2 - 4y + 5 > 0$ è verificata $\forall y = \cosh x$

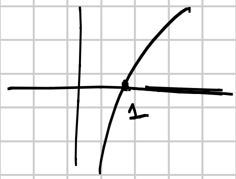
Dominio $f = \mathbb{R}$

$(\cosh x)^2 \qquad \cosh(x^2)$

2) segno

$$\log\left(\cosh^2 x - 4\cosh x + 5\right) > 0$$

$$\cosh^2 x - 4\cosh x + 5 > 1$$



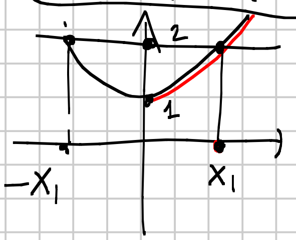
$$y = \cosh x$$

$$y^2 - 4y + 5 > 1$$

$$y^2 - 4y + 4 > 0$$

$$(y-2)^2 > 0 \iff y \neq 2$$

$$\boxed{\cosh x \neq 2} \implies$$



$x > 0$

$x_1 = \text{settcosh}\,2$

e

anche

$= -\text{settcosh}\,2$

$$f(x) > 0 \iff \cosh x \neq 2$$

$$\iff \forall x \neq \pm\,\text{settcosh}\,2$$

e $f(x) = 0$ se $x = \pm\,\text{settcosh}\,2$