Lezione del 3 Novembre

Limiti di successione

$a_n : \mathbb{N} \to \mathbb{R}$ $\qquad a_n \in \mathbb{R}$

$$\lim_n a_n = \begin{cases} \ell & \text{convergente} \\ \pm\infty & \text{divergente} \\ \nexists & \text{irregolare} \end{cases}$$

- $\{a_n\}$ monotona $\Rightarrow \exists$ limite

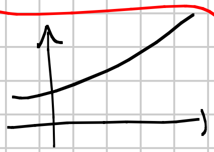
  $\{a_n\}$ monotona, limitata $\Rightarrow \exists$ limite finito

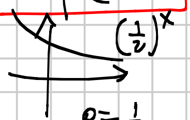
Es. Successione geometrica

$a_n = q^n$, $\quad q \in \mathbb{R}$ ragione della successione

$1, q, q^2, q^3, \dots$

$$\boxed{\lim_{n\to+\infty} q^n} = \begin{cases} +\infty & q > 1 \\ 1 & q = 1 \\ 0 & |q| < 1 \\ \nexists & q \leq -1 \end{cases}$$

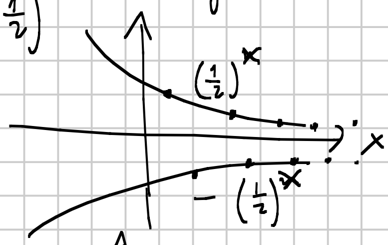




$q = \frac{1}{2}$

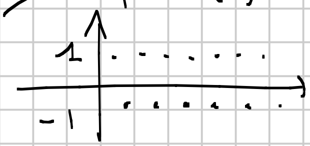
$q = -\frac{1}{2}$

$$a_n = \left(-\frac{1}{2}\right)^n = \begin{cases} \left(\frac{1}{2}\right)^n & n \text{ pari} \\ -\left(\frac{1}{2}\right)^n & n \text{ dispari} \end{cases}$$



$a_n = q^n = (-1)^n$

$\nexists$ limite



$q < -1$

$$a_n = (-3)^n \begin{cases} 3^n & n \text{ pari} \\ -(3)^n & n \text{ dispari} \end{cases}$$

$\lim q^n = \nexists$

$$\boxed{\lim_n \left(1 + \frac{1}{n}\right)^n =: e}$$

di Nepero

$e \notin \mathbb{Q}$

$2 < e < 3$

$\operatorname{senh} x$, $\cosh x$, $\log$ $\ln$

$$\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^x = e$$

$$\lim_{x\to\pm\infty}\left(1+\frac{\alpha}{x}\right)^x = e^\alpha \qquad \alpha\in\mathbb{R}$$

$$\lim_{x\to 0}\frac{\log(1+x)}{x} = 1 \qquad \log(1+x)\underset{x\to 0}{\sim} x$$

$$\lim_{x\to 0}\frac{\log_a(1+x)}{x} = \log_a e$$

$$\lim_{x\to 0}\frac{e^x-1}{x} = 1 \qquad e^x-1\underset{x\to 0}{\sim} x$$

$$\lim_{x\to 0}\frac{a^x-1}{x} = \log a$$

Altro limite notevole

$$\lim_{x\to 0}\frac{(1+x)^\alpha-1}{x} \qquad \alpha\in\mathbb{R},\ \alpha>0$$

$$\frac{(1+x)^\alpha-1}{x} = \frac{e^{\log[(1+x)^\alpha]}-1}{x} =$$

$$= \frac{e^{\alpha\log(1+x)}-1}{x} =$$

$$= \frac{e^{\overbrace{\alpha\log(1+x)}^{y}}-1}{\underbrace{\alpha\log(1+x)}_{y}}\cdot\frac{\alpha\log(1+x)}{x}$$

$$\frac{e^y-1}{y}\xrightarrow[y\to 0]{} 1 \qquad y\to 0 \quad x\to 0 \qquad \frac{\alpha\log(1+x)}{x} \xrightarrow[x\to 0]{} 1$$

$$\boxed{\lim_{x\to 0}\frac{(1+x)^\alpha-1}{x} = \alpha}$$

es. $\displaystyle\lim_{x\to 0}\frac{\sqrt[5]{1+x}-1}{x} = \frac{1}{5}$

es. $\displaystyle\lim_{x\to 0}\frac{\sqrt[3]{1+\operatorname{sen}x}-1}{x} =$

$$= \lim_{x\to 0}\frac{\sqrt[3]{1+\overbrace{\operatorname{sen}x}^{y}}-1}{\underbrace{\operatorname{sen}x}_{y}}\cdot\frac{\operatorname{sen}x}{x} = \frac{1}{3}$$

$\to 1/3$ $\quad \to 1$ $x\to 0$

$x\to 0 \Rightarrow \operatorname{sen}x\to 0$

Scale infinita per le successioni

$\log_b n$ $\qquad b>1$

n^β $\qquad \beta>0$

a^n $\qquad a>1$

$n!$

n^n

$$\lim_n \frac{(\log_b n)^\alpha}{n^\beta} = 0 \qquad \left(\text{es. } \lim_n \frac{(\log n)^{1.000.000}}{\sqrt[1000]{n}} = 0 \right)$$

$$\lim_n \frac{n^\beta}{a^n} = 0 \qquad \forall \beta>0 \quad a>1$$

$$\lim_n \frac{a^n}{n!} = 0$$

$$\lim_n \frac{n!}{n^n} = 0$$

Criteri del rapporto e della radice

(sul libro non c'è).

Teorema $\{a_n\}$ successione t.c. $a_n>0$ e t.c.

$$\lim_n \frac{a_{n+1}}{a_n} = \ell \in \mathbb{R}^*$$

oppure

$$\left(\lim_n \sqrt[n]{a_n} = \ell \in \mathbb{R}^* \right)$$

allora se $\ell > 1 \Rightarrow \lim_n a_n = +\infty$

se $\ell < 1 \Rightarrow \lim_n a_n = 0$

se $\ell = 1 \Rightarrow$ il criterio non dice nulla.

es $\lim_{n\to+\infty} \frac{n^2}{5^n} \Rightarrow a_n$

$$a_{n+1} = \frac{(n+1)^2}{5^{n+1}}$$

criterio del rapporto:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{5^{n+1}} \cdot \frac{5^n}{n^2} =$$

$$= \frac{1}{5}\left(\underbrace{\frac{n+1}{n}}_{\to 1,\ n\to+\infty}\right)^2 \xrightarrow[n\to+\infty]{} \frac{1}{5}$$

$$\lim_n \frac{a_{n+1}}{a_n} = \frac{1}{5} < 1$$ dal criterio del rapporto.

$$\Rightarrow \lim_n \frac{n^2}{5^n} = 0$$

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es. $$\lim_n \frac{n!}{n^n} \to a_n$$

usiamo il criterio del rapporto.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$$

$$= \frac{\cancel{n!}\,\cancel{(n+1)}}{(n+1)^n\cancel{(n+1)}} \cdot \frac{n^n}{\cancel{n!}} = \left(\frac{n}{n+1}\right)^n =$$

$$= \left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{1^n \equiv 1}{\underbrace{\left(1+\frac{1}{n}\right)^n}_{\to e,\ n\to+\infty}} \xrightarrow[n\to+\infty]{} \frac{1}{e}$$

$$\lim_n \frac{a_{n+1}}{a_n} = \frac{1}{e} < 1$$ per il criterio del rapporto.

$$\Rightarrow \lim_n \frac{n!}{n^n} = 0$$

$n^{\frac{1}{n}}$ ∞^0

es. $$\lim_{n\to+\infty} \sqrt[n]{n} =$$

$$= \lim_{n\to+\infty} e^{\log\left(n^{\frac{1}{n}}\right)} = \lim_{n\to+\infty} e^{\frac{1}{n}\log n} = 1$$

$\frac{1}{n}\log n \to 0$ per scala infiniti

$$\boxed{\lim_{n\to+\infty} \sqrt[n]{n} = 1}$$

$\alpha > 0$

es. $$\lim_{n\to+\infty} \sqrt[n]{n^\alpha} =$$

$$= \lim_{n\to+\infty} e^{\frac{1}{n}\log(n^\alpha)} = \lim_{n\to+\infty} e^{\alpha\frac{\log n}{n}} = 1$$

$$\boxed{\lim_{n\to+\infty} \sqrt[n]{n^\alpha} = 1} \qquad \forall \alpha>0.$$

es. $\lim_{n\to+\infty} \sqrt[n]{n^{1000}} = 1$.

es. $\lim_{n} \frac{n^n}{e^{n^2}} \longrightarrow a_n$

$$\sqrt[n]{a_n} = \sqrt[n]{\frac{n^n}{e^{n^2}}} = \frac{n}{\sqrt[n]{e^{n\cdot n}}} =$$

$$= \frac{n}{e^{n\cdot \cancel{n}\cdot\frac{1}{\cancel{n}}}} = \frac{n}{e^n} \xrightarrow[n\to+\infty]{} 0$$

$\lim_{n} \sqrt[n]{a_n} = 0 < 1$ per il criterio della radice

$$\lim_{n} \frac{n^n}{e^{n^2}} = 0.$$

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oss. non sempre sono utili per calcolare i limiti:

es. $\lim_{n\to+\infty} \frac{n}{n+1} = 1$ $\to a_n$

se usassi il criterio del rapporto

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{(n+1)+1}\cdot\frac{n+1}{n} = \frac{(n+1)^2}{(n+2)n}$$

$$= \frac{n^2+2n+1}{n^2+2n} = \frac{\cancel{n^2}\left(1+\frac{2}{n}+\frac{1}{n^2}\right)}{\cancel{n^2}\left(1+\frac{2}{n}\right)} \xrightarrow[n\to+\infty]{} 1$$

$\lim_{n} \frac{a_{n+1}}{a_n} = 1 \Rightarrow$ il criterio del rapporto non ci dice nulla.

Es. $\lim_{x\to+\infty} \left(\frac{x+1}{x-1}\right)^x$ $\qquad 1^\infty$

$$\left(\frac{x+1}{x-1}\right)^x = e^{x \log\left(\frac{x+1}{x-1}\right)}$$

$\infty \cdot 0$

$$\lim_{x\to+\infty} x \log\left(\frac{x+1}{x-1}\right) =$$

$$\lim_{y\to 0} \frac{\log(1+y)}{y} = 1$$

$$= \lim_{x\to+\infty} x \log\left(\frac{x-1+1+1}{x-1}\right) =$$

$$= \lim_{x\to+\infty} x \log\left(1 + \underbrace{\frac{2}{x-1}}_{y}\right)$$

se $x \to +\infty$ $\Rightarrow y \to 0$

$$= \lim_{x\to+\infty} x \frac{\log\left(1+\frac{2}{x-1}\right)}{\frac{2}{x-1}} \frac{2}{x-1}$$

siccome voglio avere $\frac{\log(1+y)}{y} \to 1$ $y \to 0$

$\to 1$

$$= 2.$$

$\Rightarrow \lim (\;) = e^2$

es. (Percorso)

1^∞

$$\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}}$$

$$= \lim_{x\to 0} e^{\frac{1}{x}\log(\cos x)}$$

$$\frac{1}{x^2}\log(1-1+\cos x)$$

$$= \frac{1}{x^2}\log(1+(\cos x - 1))$$

$= \cdot - - --$