Lezione del 9 Dicembre

- Def. di $\int_a^b f(x)\,dx$ attraverso le somme sup. e inferiori.

- G è primitiva di f se G è derivabile e se $G'(x) = f(x)$ $\forall x \in (a,b)$

Proprietà
Se G è una primitiva di f tutte e sole le primitive di f sono del tipo
$G + K$, $K \in \mathbb{R}$.

Notazione $\int f(x)\,dx = \{\text{primitive di } f\}$
Integrale indefinito

- si collega - attraverso il teo. fond. del calcolo integrale

$$F(x) = \int_c^x f(t)\,dt \qquad \text{funzione integrale di } f$$

è t.c. $F'(x) = f(x)$ $\forall x \in (a,b)$
cioè $F(x)$ è una primitiva di f.

Tutte le primitive di $f = \int_c^x f(t)\,dt + K$

Corollario

$$\int_a^b f(x)\,dx = G(b) - G(a)$$

dove G è una primitiva di f.

$f(x) = \operatorname{sen} x$ — $G(x) = -\cos x$

$f(x) = \cos x$ — $G(x) = \operatorname{sen} x$

$f(x) = x^\alpha$ — $G(x) = \dfrac{x^{\alpha+1}}{\alpha+1}$

$f(x) = \dfrac{1}{x}$ — $G(x) = \log x$

$f(x) = \dfrac{1}{1+x^2}$ — $G(x) = \operatorname{arctg} x$

$f(x) = \dfrac{1}{\sqrt{1-x^2}}$ — $G(x) = \operatorname{arcsen} x$

$f(x) = \log x \longrightarrow G(x) = ?$

$f(x) = e^{-x^2} \rightarrow$ non si trova la funzione

$f(x) = \dfrac{\operatorname{sen} x}{x} \Rightarrow$ //

$f(x) = \operatorname{senh} x \Rightarrow G(x) = \cosh x$

$ = \cosh x G(x) = \operatorname{senh} x$

Metodi di integrazione

Integrazione per parti

<u>Oss.</u> $\displaystyle\int f'(x)\,dx = f(x) + K$

<u>Teo.</u> f, g derivabili in $[a,b]$

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx$$

e analog.

$$\int_a^b f(x)\,g'(x)\,dx = f(x)\,g(x)\Big/_a^b - \int_a^b f'(x)\,g(x)\,dx$$

<u>Dim.</u> $(f\cdot g)' = f'g + f\cdot g'$

$$\int (f\cdot g)'\,dx = \int f'g\,dx + \int f g'\,dx$$

$\downarrow$ dall'oss. sopra

$$f\cdot g = \int f'g\,dx + \int f g'\,dx \qquad \#$$

Come si usa $\displaystyle\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx$

fattore da derivare ↓   ↓ fattore da integrare

es.

Calcolare la funzione di $x \operatorname{sen} x$

$$\int \underbrace{x}_{f}\, \underbrace{\operatorname{sen} x}_{g'}\, dx = x \cdot (-\cos x) - \int 1 \cdot (-\cos x)\, dx =$$

$g' = \operatorname{sen} x \Rightarrow g = -\cos x$

$f' = 1$

$$= -x\cos x + \int \cos x\, dx =$$

$$= -x\cos x + \operatorname{sen} x + K$$

$K \in \mathbb{R}$

Riprova: deve essere che (Fatelo voi!)

$$\left(-x\cos x + \operatorname{sen} x\right)' = x \operatorname{sen} x$$

es. $\int \underset{g'}{x}\, \underset{f}{\operatorname{sen} x}\, dx$

cosa succede se si fa l'altra scelta

$$\boxed{\int f g' = f g - \int f' g}$$

$g'(x) = x \qquad g(x) = \frac{x^2}{2}$

$$\int x \operatorname{sen} x\, dx = \operatorname{sen} x \cdot \frac{x^2}{2} - \int \cos x \cdot \frac{x^2}{2}\, dx$$

$$\boxed{\int f g' = f g - \int f' g}$$

es. $\int \underset{f}{x^2}\, \underset{g'}{\operatorname{sen} x}\, dx =$

$g = -\cos x$

$$= x^2 \cdot (-\cos x) - \int 2x \cdot (-\cos x)\, dx =$$

$$= -x^2 \cos x + 2\int \underset{f}{x}\, \underset{g'}{\cos x}\, dx =$$

$g = \operatorname{sen} x$

$$= -x^2\cos x + 2\left(x \operatorname{sen} x - \int 1\cdot \operatorname{sen} x\,dx\right)$$

$$= -x^2\cos x + 2x\operatorname{sen}x + 2\cos x + K$$

riprova (fatelo voi)

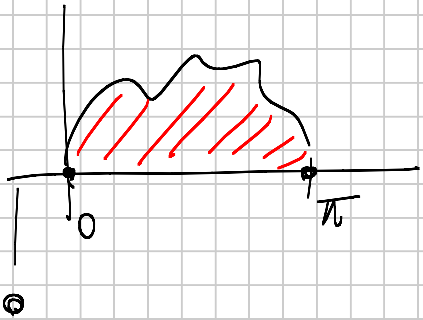
esercizio per casa $\int x^3 \operatorname{sen} x\,dx$

es. $$\int_0^{\pi} x\operatorname{sen}x = \left(-x\cos x+\operatorname{sen}x\right)\Big|_0^{\pi} =$$

$$= \left(-\pi\cos\pi + 0\right) - \left(0+0\right)$$

$$= \pi$$

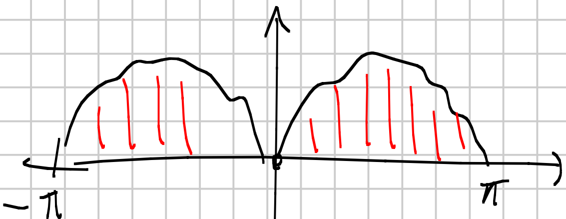
$$h(x) = x\operatorname{sen}x \geq 0$$



$$\Rightarrow \int_0^{\pi} x \operatorname{sen}x\,dx > 0$$

oss. $$\int_{-\pi}^{\pi} x\operatorname{sen}x\,dx = \left(-x\cos x + \operatorname{sen}x\right)\Big|_{-\pi}^{\pi} = \dots$$

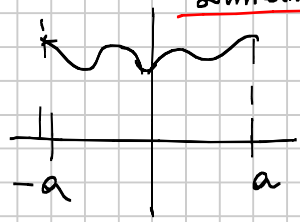
$$h(x) = x\operatorname{sen}x$$
$$h(-x) = (-x)\operatorname{sen}(-x) = x\operatorname{sen}x = h(x)$$
è pari



$$\int_{-\pi}^{\pi} x\operatorname{sen}x\,dx = 2\int_0^{\pi} x\operatorname{sen}x\,dx$$

In generale se $f(x)$ è pari in intervallo simmetrico

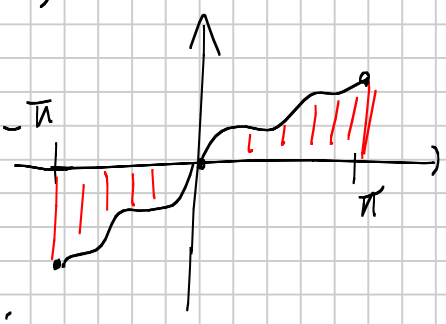
$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$



oss. $h(x) = x\cos x$

$$\int_{-\pi}^{\pi} x\cos x\,dx = \Big(\text{primitiva (con integrazione per parti)}\Big)\Big|_{-\pi}^{\pi}$$

$$h(-x) = -x\cos(-x) = -x\cos x = -h(x)$$



$$\Rightarrow \int_{-\pi}^{\pi} x\cos x\,dx = 0$$

Quindi

$f(x)$ dispari in un intervallo simmetrico

$$\int_{-a}^{a} f(x)\,dx = 0$$

fare voi verifica con i conti $\int_{-\pi}^{\pi} x\cos x\,dx = 0$

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es. $$\int \underset{g'}{e^x}\cdot \underset{f}{\operatorname{sen} x}\,dx =$$

$\int fg' = f\cdot g - \int f'g$

$g' = e^x$
$g = e^x$

$$= \operatorname{sen} x\cdot e^x - \int \underset{f}{\cos x}\; \underset{g'}{e^x}\,dx =$$

$$= \operatorname{sen} x \cdot e^x - \left(\cos x\, e^x - \int (-\operatorname{sen} x)\, e^x\, dx \right) =$$

$$= \operatorname{sen} x \cdot e^x - \cos x\, e^x - \int \operatorname{sen} x\, e^x\, dx$$

$$\underbrace{\int e^x \operatorname{sen} x\, dx}_{I} = \operatorname{sen} x\, e^x - \cos x\, e^x - \int e^x \operatorname{sen} x\, dx$$

$$I = \operatorname{sen} x \cdot e^x - \cos x\, e^x - I$$

$$2 \int e^x \operatorname{sen} x\, dx = \operatorname{sen} x \cdot e^x - \cos x \cdot e^x$$

$$\int e^x \operatorname{sen} x\, dx = \frac{\operatorname{sen} x \cdot e^x - \cos x \cdot e^x}{2} + K \quad K \in \mathbb{R}.$$

(controllare che sia giusto)

<u>Es.</u> $\displaystyle\int_0^{\pi/4} \operatorname{sen}^2 x\, dx$ = numero > 0

$$\int \operatorname{sen}^2 x\, dx = \int \underset{\substack{\downarrow \\ f}}{\operatorname{sen} x} \cdot \underset{\substack{\downarrow \\ g'}}{\operatorname{sen} x}\, dx =$$

$g(x) = -\cos x$

$$= \operatorname{sen} x \cdot (-\cos x) - \int \cos x \cdot (-\cos x)\, dx =$$

$$= -\operatorname{sen} x \cos x + \int \cos^2 x\, dx =$$

$\cos^2 x = 1 - \operatorname{sen}^2 x$

$$= -\operatorname{sen} x \cos x + \int (1 - \operatorname{sen}^2 x)\, dx =$$

$$= -\operatorname{sen} x \cos x + \int 1\, dx - \int \operatorname{sen}^2 x\, dx =$$

$$= -\operatorname{sen} x \cos x + x - \int \operatorname{sen}^2 x\, dx =$$

$$2\int \operatorname{sen}^2 x \, dx = x - \operatorname{sen} x \cos x$$

$$\int \operatorname{sen}^2 x \, dx = \frac{x - \operatorname{sen} x \cos x}{2} + K$$

$$\int_0^{\pi/4} \operatorname{sen}^2 x \, dx = \frac{\frac{\pi}{4} - \operatorname{sen}\frac{\pi}{4}\cos\frac{\pi}{4}}{2} - 0$$

$$= \frac{\frac{\pi}{4} - \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}}{2} = \frac{\frac{\pi}{4} - \frac{1}{2}}{2} =$$

$$= \frac{\pi - 2}{8} > 0$$

$$\int_{-\pi/4}^{\pi/4} \operatorname{sen}^2 x \, dx = 2\int_0^{\pi/4} \operatorname{sen}^2 x \, dx$$

perché $\operatorname{sen}^2 x$ è pari.

PC. $\int \cos^2 x \, dx = \dots$

$$\int \log x \, dx = \int \underbrace{1}_{g'} \cdot \underbrace{\log x}_{f} \, dx =$$

$g = x$

$f' = \frac{1}{x}$

$$= x \log x - \int \cancel{x} \cdot \frac{1}{\cancel{x}} \, dx =$$

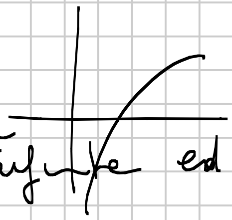
$$= x \log x - x + K$$

$$\int_1^2 \log x \, dx = \left(x \log x - x \right)\Big|_1^2$$

$[a, b]$ in cui $\log x$ è definito

$$\int_0^1 \log x \, dx$$

in $x = 0$ la funzione non è definita ed è



illimitate vicino a zero.

non l'abbiamo (ancora) definito

finché $\int_a^b f(x)\,dx$ con $f(x)$ illimitata in $[a,b]$.

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es. $\int \arcsin x \, dx$ = vi aspetto ......

PC. $\int x e^x \, dx$, $\int x^2 \log x \, dx$,

+ esercizi sul libro.

Integrazione per sostituzione

Teorema $f(x)$ integrabile in $[a,b]$.
Sia $x = \varphi(t)$, $t \in [\alpha, \beta]$
derivabile, con derivata continua in $[\alpha, \beta]$
t.c. $\varphi(\alpha) = a$, $\varphi(\beta) = b$. Allora

$$\int f(x)\,dx = \int f(\varphi(t))\,\varphi'(t)\,dt$$

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt$$

Dim. no.

$$\int f(x)\,dx = \int f(\varphi(t))\,\varphi'(t)\,dt$$

$x = \varphi(t)$ cambio di variabili

$x = \varphi(t)$

$1 \cdot dx = \varphi'(t)\,dt$

$$\int f(x)\,dx = \int f(\varphi(t))\varphi'(t)\,dt$$

es

$$\int \operatorname{sen}(3x)\,dx = \qquad 3x = t$$

$$x = \underbrace{\tfrac{1}{3}\cdot t}_{\varphi(t)}$$

$$\varphi'(t) = \frac{1}{3}$$

$$= \int \operatorname{sen} t \cdot \frac{1}{3}\,dt$$

$$= \frac{1}{3}\int \operatorname{sen} t\,dt =$$

$$= \frac{1}{3}(-\cos t) \overset{t=3x}{=} -\frac{1}{3}\cos(3x)$$

Formalmente

$$\boxed{3x = t}$$

$$3\,dx = dt$$

$$dx = \frac{1}{3}\,dt$$

$$\int \operatorname{sen}(3x)\,dx =$$

$$= \int \operatorname{sen} t\,\frac{1}{3}\,dt$$

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es.

$$\int \operatorname{tg} x\,dx = \int \frac{\operatorname{sen} x}{\cos x}\,dx =$$

$$\cos x = t$$

$$-\operatorname{sen} x\,dx = 1\,dt$$

$$\operatorname{sen} x\,dx = -dt$$

$$= \int \frac{-1}{t}\,dt =$$

$$= -\int \underbrace{\frac{1}{t}}_{\forall t \neq 0}\,dt = -\log|t| =$$

$$= -\log|\cos x| + K$$

es.

$$\int \frac{\operatorname{sen} x}{\cos x}\,dx$$

$$\operatorname{sen} x = t$$

$$\cos x\,dx = dt$$

$$dx = \frac{dt}{\cos x}$$

$$\cos x = \pm\sqrt{1-\operatorname{sen}^2 x}$$

$$\cos x = \pm\sqrt{1-t^2}$$

$$= \int \frac{t}{\pm\sqrt{1-t^2}}\,\frac{dt}{(\pm\sqrt{1-t^2})}$$

$$= \int \frac{t}{1-t^2}\,dt$$

+ complicato ...

Regola

$$\boxed{\int \frac{f'(x)}{f(x)}\, dx = \int \frac{1}{t}\, dt = \log|f(x)| + k}$$

$f(x) = t \qquad f'(x)\, dx = 1\, dt$

es.

$$\int \frac{x}{\sqrt{x^2-1}}\, dx = \int \frac{1}{2\sqrt{t}}\, dt =$$

$x^2 - 1 = t$

$2x\, dx = dt \qquad x\, dx = \frac{1}{2} dt$

$$= \frac{1}{2}\int (t)^{-1/2}\, dt = \frac{1}{2}\, \frac{(t)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = \frac{\cancel{1}}{\cancel{2}}\, \frac{\sqrt{t}}{\frac{1}{\cancel{2}}} =$$

$$\int t^{\alpha}\, dt = \frac{t^{\alpha+1}}{\alpha+1} \qquad = \boxed{\sqrt{x^2-1}}$$

es.

$$\int_2^3 \frac{x}{\sqrt{x^2-1}} = \sqrt{x^2-1}\,\Big|_2^3 \qquad \text{oppure}$$

$$= \frac{1}{2}\int_3^8 (t)^{-1/2}\, dt$$

$t = x^2 - 1$

$x \in [2, 3] \qquad x = 2 \rightarrow t = 3$

$\qquad\qquad\qquad x = 3 \rightarrow t = 8$

$\sim . \sim . \sim . \sim . \sim$

es.

$$\int \frac{1}{2x+1}\, dx = \int \frac{1}{t}\, \frac{1}{2}\, dt =$$

$2x + 1 = t \qquad 2\, dx = dt \qquad dx = \frac{1}{2} dt$

$$= \frac{1}{2} \log |t| = \frac{1}{2} \log |2x+1| + K$$

es. $\displaystyle\int \frac{1}{2x+1}\,dx = \int \frac{1}{2\left(x+\frac{1}{2}\right)}\,dx =$

$\displaystyle= \frac{1}{2}\int \frac{1}{x+\frac{1}{2}}\,dx = \frac{1}{2}\int \frac{1}{t}\,dt =$

$x + \frac{1}{2} = t$

$dx = dt$

$= \frac{1}{2} \log \left|x+\frac{1}{2}\right|$

es. $\displaystyle\int \operatorname{sen}^5 x \cdot \cos x \, dx = \int t^5\,dt =$

$\operatorname{sen} x = t \qquad \cos x\,dx = dt$

$$= \frac{t^6}{6} = \frac{(\operatorname{sen} x)^6}{6} + K$$

es $\displaystyle\int \frac{1}{x^2+4}\,dx$

$\displaystyle\int \frac{1}{x^2+1}\,dx = \operatorname{arctg} x + K$ integrale immediato.

$$\int \frac{1}{x^2+4}\,dx = \frac{1}{4}\int \frac{1}{\frac{x^2}{4}+1}\,dx =$$

$$= \frac{1}{4}\int \frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx =$$

$\frac{x}{2} = t$

$x = 2t$

$$= \frac{1}{\cancel{4}_{2}} \int \frac{1}{t^2+1} \cancel{2}\,dt = \frac{1}{2}\int \frac{1}{t^2+1}\,dt \qquad dx = 2dt$$

$$= \frac{1}{2}\, \operatorname{arctg}\left(\frac{x}{2}\right) + K$$

es. $\displaystyle\int \frac{1}{x^2-1}\,dx = \int \frac{1}{(x-1)(x+1)}\,dx$

$x^2 - 1 = 0 \qquad x^2-1 = (x-1)(x+1)$

$$\boxed{\frac{1}{(x-1)(x+1)} \overset{?}{=} \frac{A}{x-1} + \frac{B}{x+1}}$$

Cerco A e B t.c. valga l'uguaglianza per ogni valore di x

$$\boxed{\frac{1}{\cancel{(x-1)}\cancel{(x+1)}}} = \frac{A(x+1)+B(x-1)}{(x-1)(x+1)} =$$

$$= \frac{Ax + A + Bx - B}{(x-1)(x+1)} =$$

$\forall x$ ~~$\not\equiv$~~

$$= \boxed{\frac{x(A+B)+(A-B)}{\cancel{(x-1)}\cancel{(x+1)}}}$$

$$1 = x(A+B) + (A-B)$$

deve valere $\forall x$

i due polinomi sono uguali $\forall x$ sse

$$\begin{cases} A+B = 0 \\ A - B = 1 \end{cases} \qquad \begin{matrix} B = -A \\ A + A = 1 \end{matrix} \qquad 2A = 1$$

$A = 1/2 \qquad B = -1/2$

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} = \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$$

$$\int \frac{1}{(x-1)(x+1)}\,dx = \frac{1}{2}\int \frac{1}{x-1}\,dx - \frac{1}{2}\int \frac{1}{x+1}\,dx$$

$x-1=t$ $\qquad$ $x+1=t$

controllare $\rightarrow$

$$= \frac{1}{2}\log|x-1| - \frac{1}{2}\log|x+1| + K$$

Decomposizione in fratti semplici.