3 Dicembre 2013

$$\int_a^b f(x)\,dx = G(b) - G(a)$$

$\Downarrow$ G è una primitiva di f

$G' = f$

$$\int f(x)\,dx = \text{tutte le primitive di } f$$

1) Integrazione per sostituzione $\quad x = \varphi(t)$

$dx = \varphi'(t)dt$

$$\int f(x)\,dx = \int f(\varphi(t))\,\varphi'(t)\,dt$$

2) Integrazione per parti

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx$$

es $\int x \operatorname{sen} x\,dx$, $\int e^x \operatorname{sen} x\,dx = I$

$\Downarrow$ $I = h(x) - I$

$2I = h(x) \Rightarrow I = \dfrac{h(x)}{2}$

$\int 1 \cdot \log x$, $\int \operatorname{arctg} x$, $\int \operatorname{arcsen} x$ $\int \arccos x$

es $$\int \underbrace{x^2}_{f} \underbrace{\operatorname{sen} x}_{g'}\,dx = x^2(-\cos x) - \int 2x(-\cos x)\,dx$$

$$= -x^2 \cos x + 2\int \underbrace{x}_{f} \underbrace{\cos x}_{g'}\,dx = -x^2\cos x + 2x \operatorname{sen} x$$

$$-2\int \operatorname{sen} x\,dx = -x^2\cos x + 2x\operatorname{sen} x + 2\cos x + K$$

es. $$\int x^n \operatorname{sen} x\,dx$$

es. $$\int \operatorname{sen}^2 x\,dx = \int \underbrace{\operatorname{sen} x}_{f} \cdot \underbrace{\operatorname{sen} x}_{g'}\,dx =$$

$$= \operatorname{sen} x(-\cos x) - \int \cos x \cdot (-\cos x)\,dx =$$

$$= -\operatorname{sen} x\cos x + \int \underbrace{\cos^2 x}_{1-\operatorname{sen}^2 x}\,dx = -\operatorname{sen} x\cos x +$$

$$+\int (1-\operatorname{sen}^2 x)\,dx = -\operatorname{sen} x\cos x + x - \int \operatorname{sen}^2 x\,dx$$

$$2\int \operatorname{sen}^2 x\,dx = x - \operatorname{sen} x\cos x$$

$$\int \operatorname{sen}^2 x\,dx = \frac{x - \operatorname{sen} x\cos x}{2} + K$$

es. $$\int \underbrace{1}_{g'} \cdot \underbrace{(\log x)^2}_{f}\,dx = x\lg^2 x - \int \cancel{x}\, 2\log x \frac{1}{\cancel{x}}\,dx$$

$$= x\lg^2 x - 2\underbrace{\int \log x\,dx}_{\text{la volta scorsa}}$$

es. $$\int x^5 e^{x^2}\,dx = \int x^4 \cdot x e^{x^2}\,dx =$$

$$x^2 = y$$

$$2x\,dx = dy$$

$$= \int y^2 e^y \frac{1}{2}\,dy$$

$$= \frac{1}{2} \int \underbrace{y^2}_{f} \underbrace{e^y}_{g'} \, dy = \text{due volte per parti}$$

<u>Integrazione di funzioni razionali</u>

$$\text{funzione razionale} = \frac{P_n(x)}{Q_m(x)}$$

$P_n(x)$ polinomio di grado n

$Q_m(x)$ " di grado m

es. $\frac{3x^2+1}{x^5+5}$, $\frac{1}{2x^2-8}$, $\frac{x^8}{x^3+2x+3}$

1) $n \geqslant m$ si fa la divisione tra i due polinomi e si trova

$$\frac{P_n}{Q_m} = \text{polinomio} + \frac{\text{polinomio di grado} < m}{Q_m}$$

resto nella divisione

$$\int \frac{P_n}{Q_m} dx = \int \text{polinomio} + \int \frac{R_p}{Q_m}$$

polinomio di grado $p < m$

$p < m$

Noi considereremo solo i casi

1) $f(x) = \frac{1}{\text{polinomio di grado } 1}$

2) $f(x) = \frac{\text{polinomio di grado } 1}{\text{polinomio di grado } 2}$

1) $\displaystyle\int \frac{1}{Ax+B}\,dx =$

$Ax+B$ polinomio di grado 1

$Ax+B = y$

$A\,dx = dy$

$\displaystyle = \int \frac{1}{y}\,\frac{1}{A}\,dy =$

$\displaystyle = \frac{1}{A}\log|Ax+B| + K$

2) $\displaystyle\int \frac{Ax+B}{ax^2+bx+c}\,dx$

polinomio di grado 1

polinomio di grado 2

$A=0$ $\displaystyle\int \frac{1}{ax^2+bx+c}\,dx$

$ax^2+bx+c=0$

$\Delta>0$

$\Delta<0$

$\Delta=0$

es. $\displaystyle\int \frac{1}{x^2-5x+6}\,dx$

$x^2-5x+6=0$ $\Delta = 25-24 = 1$

$\displaystyle x = \frac{5\pm 1}{2} \begin{cases} 3 \\ 2 \end{cases}$

$x^2-5x+6 = (x-3)(x-2)$

$$\int \frac{1}{x^2-5x+6}\,dx = \int \frac{1}{(x-3)(x-2)}\,dx$$

cerco A e B t.c.

$$\underbrace{\frac{1}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}}_{\text{è vero } \forall x} = \frac{Ax-2A+Bx-3B}{(x-3)(x-2)}$$

$$1 = Ax - 2A + Bx - 3B$$

$$1 = x(A+B) - 2A - 3B$$

uguaglio i coefficenti dei termini dello stesso grado

$$\begin{cases} A+B=0 \\ -2A-3B=1 \end{cases} \qquad \begin{array}{l} B=-A \\ -2A+3A=1 \Rightarrow A=1 \\ \qquad\qquad\qquad\quad B=-1 \end{array}$$

$$\frac{1}{(x-3)(x-2)} = \frac{1}{x-3} - \frac{1}{x-2}$$

$$\int \frac{1}{(x-3)(x-2)}\,dx = \int \frac{1}{x-3}\,dx - \int \frac{1}{x-2}\,dx =$$

$$= \log|x-3| - \log|x-2|$$

se $ax^2+bx+c=0$ è t.c. $\Delta>0$, x_1, x_2

$$ax^2+bx+c = a(x-x_1)(x-x_2)$$

$$\int \frac{1}{ax^2+bx+c}\,dx = \frac{1}{a}\int \frac{1}{(x-x_1)(x-x_2)}\,dx$$

$$= \frac{1}{a}\left(\int\left(\frac{A}{x-x_1} + \frac{B}{x-x_2}\right)dx\right) =$$

dove A e B sono da determinare come fatto nell'esempio

$$= \frac{1}{a}\left(A \log|x-x_1| + B \log|x-x_2| \right)$$

• $\Delta < 0$

es. $\displaystyle\int \frac{1}{x^2+4}\,dx = \int \frac{1}{4\left(\left(\frac{x}{2}\right)^2+1\right)}\,dx =$

$x^2+4=0 \qquad \Delta<0$ $\Rightarrow \displaystyle\frac{1}{4}\int \frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx =$

$\frac{x}{2}=y \qquad \frac{1}{2}\,dx = dy$ $\qquad = \displaystyle\frac{1}{4}\int \frac{1}{y^2+1}\,2dy$

$= \frac{2}{4}\operatorname{arctg}\left(\frac{x}{2}\right) = \frac{1}{2}\operatorname{arctg}\left(\frac{x}{2}\right)$

es. $\displaystyle\int \frac{1}{x^2+x+1}\,dx$

$x^2+x+1=0 \qquad \Delta = 1-4<0$

$$x^2+x+1 = \underbrace{x^2+x+\frac{1}{4}}_{\left(x+\frac{1}{2}\right)^2} - \frac{1}{4} + 1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4}$$

$$\int \frac{1}{x^2+x+1}\,dx = \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx =$$

$x+\frac{1}{2}=y \qquad dx = dy$ $\qquad = \displaystyle\int \frac{1}{y^2+\frac{3}{4}}\,dy =$

$$= \int \frac{1}{\frac{3}{4}\left(\frac{4}{3}y^2+1\right)}\,dy = \frac{4}{3}\int \frac{1}{\left(\frac{2}{\sqrt{3}}y\right)^2+1}\,dy$$

$\frac{2}{\sqrt{3}}y = z$

$\frac{2}{\sqrt{3}}dy = dz$

$dy = \frac{\sqrt{3}}{2}dz$

$$= \frac{\cancel{4}^{\,2}}{\sqrt{3}\,\cancel{3}}\int \frac{1}{z^2+1}\,\frac{\cancel{\sqrt{3}}}{\cancel{2}}\,dz$$

$$= \frac{2}{\sqrt{3}}\,\text{arctg}\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)$$

Quindi se

$$\int \frac{1}{ax^2+bx+c}\,dx$$

e $ax^2+bx+c=0$ ha $\Delta<0$ cioè non ha soluzioni reali

Allora

se uso il metodo del "riempimento del quadrato" cioè faccio in modo che

$$ax^2+bx+c = (Ax+B)^2 + C$$

con A, B, C da determinare

$C>0$

$$\int \frac{1}{ax^2+bx+c}\,dx = \int \frac{1}{(Ax+B)^2+C}\,dx$$

$$= \int \frac{1}{C\left(\left(\frac{(Ax+B)}{\sqrt{C}}\right)^2+1\right)}\,dx$$

$\frac{Ax+B}{\sqrt{C}} = z$

$\Downarrow$

integrale del tipo $(\text{costante})\,\text{arctg}\left(\frac{Ax+B}{\sqrt{C}}\right)$

- $\Delta = 0$

es. $\displaystyle\int \frac{1}{x^2+2x+1}\,dx = \int \frac{1}{(x+1)^2}\,dx =$

$x^2+2x+1=0 \qquad \Delta = 1-1=0$

$\Downarrow \quad (x+1)^2=0$

$(x+1)=y$

$dx = dy$

$\displaystyle = \int \frac{1}{y^2}\,dy = -\frac{1}{(x+1)}$

se $ax^2+bx+c=0 \qquad \Delta = 0$

$ax^2+bx+c = (Ax+B)^2$ A, B da determinare

$\displaystyle\int \frac{1}{ax^2+bx+c}\,dx = \int \frac{1}{(Ax+B)^2}\,dx$

$Ax+B=y$

$\displaystyle = -\frac{1}{Ax+B} + K.$

Abbiamo fatto tutti i casi del tipo

$$\int \frac{1}{ax^2+bx+c}\,dx$$

Ora facciamo il caso

$$\int \frac{x}{ax^2+bx+c}\,dx \longrightarrow ax^2+bx+c=0 \begin{cases} \Delta>0 \\ \Delta<0 \\ \Delta=0 \end{cases}$$

es. $\displaystyle\int \frac{x}{x^2-5x+6}\,dx$

$x^2-5x+6=0 \qquad \Delta>0 \quad \begin{cases} x=2 \\ x=3 \end{cases}$

$x^2-5x+6=(x-2)(x-3)$

$$\int \frac{x}{(x-2)(x-3)}\,dx \qquad \frac{x}{(x-2)(x-3)}=\frac{A}{(x-2)}+\frac{B}{(x-3)}$$

$$=\frac{Ax-3A+Bx-2B}{(x-2)(x-3)}$$

$$1x = x(A+B)-3A-2B \qquad \forall x$$

$$\begin{cases} A+B=1 \\ -3A-2B=0 \end{cases} \qquad A=-\frac{2}{3}B \qquad -\frac{2}{3}B+B=1 \Rightarrow \frac{1}{3}B=1 \qquad B=3 \quad A=-2$$

$$\frac{x}{x^2-5x+6}=\frac{-2}{x-2}+\frac{3}{x-3}$$

abbiamo decomposto in fratti semplici

$$\int \frac{x}{x^2-5x+6}\,dx=-2\int\frac{1}{x-2}\,dx+3\int\frac{1}{x-3}\,dx$$

$$=-2\log|x-2|+3\log|x-3|$$

Quindi se $\Delta>0$ $\dfrac{x}{ax^2+bx+c}$ si decompone sempre con i fratti semplici

$$\int\frac{x}{ax^2+bx+c}\,dx=\int\frac{A}{(x-x_1)}\,dx+\int\frac{B}{(x-x_2)}\,dx$$

Se $\Delta < 0$ $\qquad x^2+x+1=0 \quad \Delta<0$

es. $\frac{1}{2}\int \frac{2x+1-1}{x^2+x+1}\,dx = \frac{1}{2}\left(\int \frac{2x+1}{x^2+x+1} - \frac{1}{x^2+x+1}\right) dx$

$x^2+x+1=y$

$(2x+1)\,dx = dy$

$\int \frac{2x+1}{x^2+x+1}\,dx = \int \frac{1}{y}\,dy =$

$= \log(x^2+x+1)$

l'abbiamo fatto prima (metodo del riempimento del quadrato ... arctg(...))

$\int \frac{x}{x^2+x+1}\,dx = \log(x^2+x+1) + \frac{2}{\sqrt{3}}\,\mathrm{arctg}\left(\frac{2}{\sqrt{3}}(x+1)\right) + K$

Quindi se $\Delta < 0$

$\int \frac{x}{ax^2+bx+c}\,dx = \int \frac{2ax+b-b}{2a(ax^2+bx+c)}\,dx$

$(ax^2+bx+c)' = 2ax+b$

$= \frac{1}{2a}\int \frac{2ax+b}{ax^2+bx+c}\,dx - \frac{1}{2a}\int \frac{b}{ax^2+bx+c}\,dx$

$= \frac{1}{2a}\log(ax^2+bx+c) - \frac{1}{2a}\,\mathrm{arctg}(\dots)$

$\Delta = 0$ $\qquad (x^2+2x+1)' = 2x+2$

es. $\frac{1}{2}\int \frac{(2x+2)-2}{x^2+2x+1}\,dx$

$$= \frac{1}{2}\int \frac{2x+2}{x^2+2x+1}\,dx + \frac{1}{\cancel{2}}\int \frac{-\cancel{2}}{x^2+2x+1}\,dx$$

$$= \frac{1}{2}\log\left|x^2+2x+1\right| - \int \frac{1}{(x+1)^2}\,dx$$

fatto prima

$$= \frac{1}{2}\log\left|x^2+2x+1\right| + \frac{1}{(x+1)} + K$$

se $x \neq -1$

$x^2+2x+1>0$

se $\Delta = 0$

$$\boxed{\int \frac{x}{ax^2+bx+c}\,dx = \int \cdots \qquad (ax^2+bx+c)' = 2ax+b}$$

come il caso precedente ($\Delta<0$)

$$\int \frac{Ax+B}{ax^2+bx+c}\,dx = A\int \frac{x}{ax^2+bx+c}\,dx + B\int \frac{1}{ax^2+bx+c}\,dx$$

es
$$\int \frac{x-1}{x^2+x} = \int \frac{x-1}{x(x+1)}\,dx$$

$x^2+x = x(x+1)$

$$\frac{x-1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

e si determine A e B e poi si integra.

es
$$\int \frac{3x-8}{x^2+5}\,dx = \underbrace{3\int \frac{x}{x^2+5}\,dx}_{} - 8\int \frac{1}{x^2+5}\,dx$$

$$= \frac{3}{2}\int \frac{2x}{x^2+5}\,dx - \frac{8}{5}\int \frac{1}{\left(\frac{x}{\sqrt{5}}\right)^2+1}\,dx$$

$$= \frac{3}{2}\log\left(x^2+5\right) - \frac{8}{5}\sqrt{5}\,\operatorname{arctg}\left(\frac{x}{\sqrt{5}}\right)$$

Es. $$\int \frac{x^3+x}{x^2+x+1}\,dx$$

$$\begin{array}{rrrr|l}
x^3 & +0x^2 & +x & +0 & x^2+x+1 \\
\cline{5-5}
x^3 & +x^2 & +x & +0 & x-1 \\
\hline
0 & -x^2 & 0 & 0 & \\
 & -x^2 & -x & -1 & \\
\hline
 & 0 & x & +1 & \text{resto}
\end{array}$$

$$x^3+x = \left(x^2+x+1\right)(x-1) + (x+1)$$

$$\frac{x^3+x}{x^2+x+1} = (x-1) + \frac{x+1}{x^2+x+1}$$

$$\int \frac{x^3+x}{x^2+x+1}\,dx = \int (x-1)\,dx + \underbrace{\int \frac{x+1}{x^2+x+1}\,dx}_{\text{rientra nei casi precedenti.}}$$

Es. $$\int \frac{e^x}{e^{2x}+2e^x+5}\,dx =$$

$$e^x = y$$

$$e^x\,dx = dy$$

$$= \int \frac{1}{y^2 + 2y + 5}\, dy$$

$y^2 + 2y + 5 = 0 \qquad \Delta = +1 - 5 < 0$

$y^2 + 2y + 1 - 1 + 5 = (y+1)^2 + 4$

$$= \int \frac{1}{(y+1)^2 + 4}\, dy = \frac{1}{4} \int \frac{1}{\left(\frac{y+1}{2}\right)^2 + 1}\, dy$$

$$= \frac{1}{4}\, 2 \arctg\left(\frac{e^x + 1}{2}\right) + k \qquad \text{(controllare)}.$$

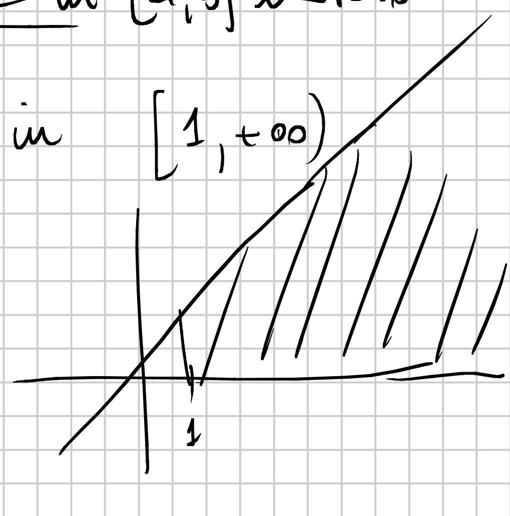
Integrali generalizzati

$\int_a^b f(x)dx$ sono definiti se $[a,b]$ è limitato
e f è limitata

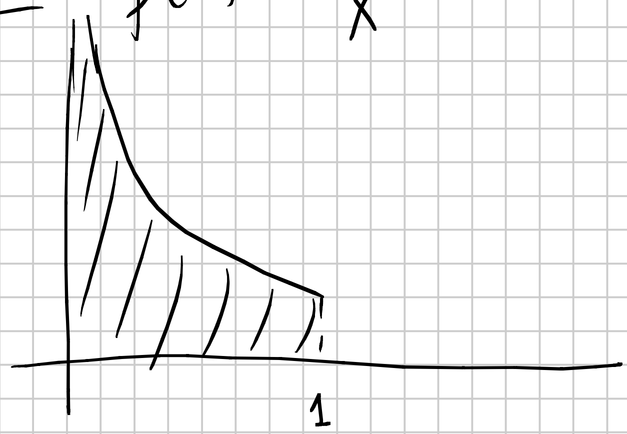
Cosa succede se $[a,b]$ non è limitato oppure
se f non è limitata in $[a,b]$ limitato

es. $f(x) = 3x$ in $[1, +\infty)$

$$\int_1^{+\infty} 3x\, dx = ?$$



es. $f(x) = \frac{1}{x}$ in $(0,1)$



l'intervallo è limitato ma f non è limitata in $(0,1)$

$$\int_0^1 \frac{1}{x}\,dx = ?$$

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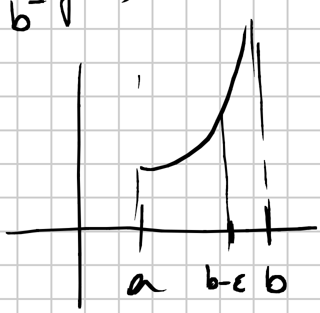
1) Integrazione di funzioni non limitate

$f : [a,b) \to \mathbb{R}$ continua ma $\lim_{x\to b^-} f(x) = +\infty$

$$\int_a^b f(x)\,dx \doteq \lim_{\varepsilon\to 0^+} \underbrace{\int_a^{b-\varepsilon} f(x)\,dx}$$

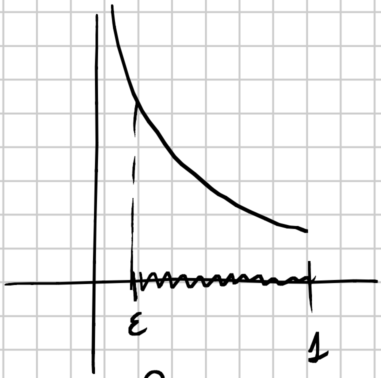


lo so calcolare ed è ben definito perché in $[a, b-\varepsilon]$ la funzione è limitata.

Def. • Se il limite $\exists$ finito, f si dice integrabile in $[a,b]$ oppure si dice $\int_a^b f(x)\,dx$ converge

• Se il limite è $= \pm\infty$ si dice che l'integrale diverge

• Se il limite $\nexists \Rightarrow$ si dice che l'integrale non esiste es. $f(x) = \frac{1}{x}$ in $(0,1]$



$\lim_{x\to 0^+} \frac{1}{x} = +\infty$ cioè è illimitata in $(0,1]$

$f(x) = \frac{1}{x}$ è integrabile in $(0,1]$?

cioè $\int_0^1 \frac{1}{x}\,dx$ converge o no ?

$$\int_0^1 \frac{1}{x}\,dx = \lim_{\varepsilon\to 0^+} \int_\varepsilon^1 \frac{1}{x}\,dx = \lim_{\varepsilon\to 0^+} \left(\log x \Big|_{x=\varepsilon}^{x=1} \right) =$$

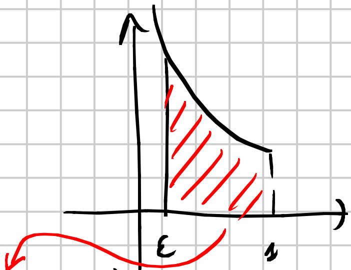
$$= \lim_{\varepsilon\to 0^+} \left(-\log \varepsilon \right) = +\infty$$

Quindi $f(x) = \frac{1}{x}$ non è integrabile tra $(0,1)$

cioè $\int_0^1 \frac{1}{x}\,dx$ diverge

es. $f(x) = \frac{1}{\sqrt{x}}$ in $(0,1]$ $\quad \lim_{x\to 0^+} \frac{1}{\sqrt{x}} = +\infty$

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx \doteq \lim_{\varepsilon\to 0^+} \int_\varepsilon^1 \frac{1}{\sqrt{x}}\,dx =$$

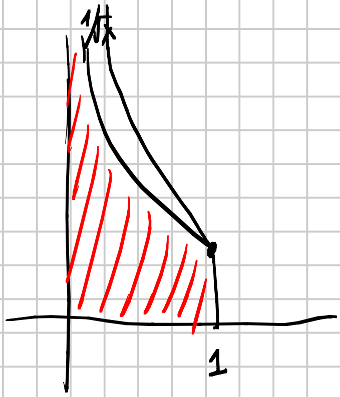


$$= \lim_{\varepsilon\to 0^+} \left(2\sqrt{x} \right)\Big|_\varepsilon^1 = \lim_{\varepsilon\to 0^+} \left(2 - 2\sqrt{\varepsilon} \right) = 2$$

$\lim_{\varepsilon\to 0^+} (\ \ )$ $\exists$ finito $=2$

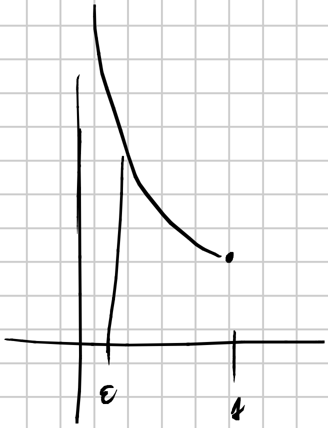
$\frac{1}{\sqrt{x}}$ è integrabile in $(0,1]$ cioè

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx \quad \text{converge}$$



$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2$$

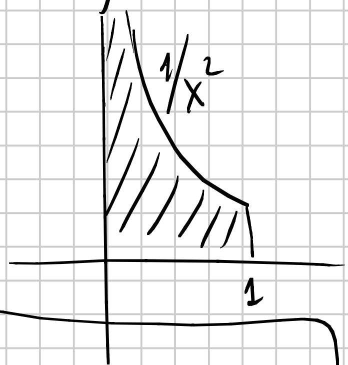
$$\int_0^1 \frac{1}{x}\,dx = +\infty$$



es. $$\int_0^1 \frac{1}{x^2}\,dx = \lim_{\varepsilon\to 0^+} \int_\varepsilon^1 \frac{1}{x^2}\,dx$$

$$= \lim_{\varepsilon\to 0^+} \left(-\frac{1}{x}\right)\Big|_\varepsilon^1 = \lim_{\varepsilon\to 0^+} \left(-1 + \frac{1}{\varepsilon}\right) = +\infty$$

$\frac{1}{x^2}$ non è integrabile in $(0,1]$



In generale

$f(x) = \frac{1}{x^\alpha}$ è integrabile se $\alpha < 1$

non è integrabile $\alpha \geq 1$