Soluzioni esercizi su limiti, Parte 2

a.a. 2013/2014

1) $\lim_{x \to 0^+} \dfrac{x^a - \operatorname{sen} x^2 + 1 - e^{-x^2}}{x - \operatorname{tg}(x + x^3) + 4^{-1/3x}}$

<u>NUM.</u> $= x^a - (\cancel{x^2} - \frac{1}{6}x^6 + o(x^6)) +$

$+ \cancel{1} - (\cancel{1} - \cancel{x^2} + \frac{x^4}{2} - \frac{x^6}{6} + o(x^6))$

$= x^a - \frac{x^4}{2} + \frac{1}{3}x^6 + o(x^6)$

<u>Denom.</u> $= x - (x + x^3 + \frac{1}{3}(x + x^3)^3$

$+ o(x^3)) + o(x^3) =$

$= -\frac{4}{3}x^3 + o(x^3)$

infinitesimo di ordine 3

<u>nota</u>: $4^{-1/3x} = o(x^\beta)$ $x \to 0^+$ $\forall \beta > 0$

Quindi

$\lim_{x \to 0^+} \frac{\text{NUM.}}{\text{DEN.}} = \lim_{x \to 0^+} \dfrac{x^a - \frac{x^4}{2} + \frac{x^6}{3}}{-\frac{4}{3}x^3}$

$$= \begin{cases} -\infty & \text{se } a < 3 \\ -\dfrac{3}{4} & \text{se } a = 3 \\ 0 & \text{se } a > 3 \end{cases}$$

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2) $$\lim_{x \to 2^+} \frac{e^{x-2} - x - \cos(x-2) + (x-2)^4 \operatorname{sen}\left(\frac{1}{x-2}\right)}{\log(x-1) - (x-2)}$$

$x - 2 = y$

$$= \lim_{y \to 0^+} \frac{e^y - (y+2) + \cos y + y^4 \operatorname{sen}\frac{1}{y}}{\log(1+y) - y}$$

<u>Num.</u> $\cancel{1} + \cancel{y} + \cancel{\frac{y^2}{2}} + \frac{y^3}{3!} + o(y^3) - \cancel{y}$

$- \cancel{2} + \cancel{1} - \cancel{\frac{y^2}{2}} + \frac{y^4}{4!} + o(y^4) +$

$+ y^4 \operatorname{sen}\frac{1}{y}$

$y^4 \operatorname{sen}\frac{1}{y} = o(y^3)$ (verificare!)

Quindi

$$= \frac{y^3}{6} + o(y^3)$$

infinitesimo di ordine 3

$$\underline{\text{Denom}} = \cancel{y} - \frac{y^2}{2} + o(y^2) - \cancel{y} =$$

$$= -\frac{y^2}{2} + o(y^2)$$

Quindi

$$\lim (\qquad) = \lim_{y \to 0^+} \frac{\frac{y^3}{6}}{-\frac{y^2}{2}} = 0$$

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3)

$$\lim_{x \to 0^+} \frac{3^{-\frac{1}{2x}} + e^{x - \frac{x^2}{4}} - 1 - x}{x^a - \log(1 - x^2) - 2\operatorname{tg} x}$$

$\underline{\text{oss.}}$ $$3^{-\frac{1}{2x}} = o(x^\beta) \quad \forall \beta > 0$$

$$\underline{\text{Num}} = \cancel{1} + \left(\cancel{x} - \frac{x^2}{4}\right) + \frac{1}{2}\left(x - \frac{x^2}{4}\right)^2 + o(x^2) \cancel{-1} - \cancel{x} =$$

$$= -\frac{x^2}{4} + \frac{x^2}{2} + o(x^2) = \frac{x^2}{4} + o(x^2)$$

infinitesimo di ordine 2

$$\underline{\text{Den.}} \; x^a - \left(-x^2 - \frac{x^4}{2} + o(x^4)\right) - 2\left(x + \frac{x^3}{3} + o(x^3)\right)^2 =$$

$$= x^a + x^2 - 2x^2 + \frac{x^4}{2} - \frac{4}{3}x^4 + o(x^4) = x^a - x^2 - \frac{5}{6}x^4 + o(x^4)$$

$$\begin{cases} \text{infinitesimo di ordine } a & \text{se } a<2 \\ \text{ordine } 2 & \text{se } a>2 \\ \text{ordine } 4 & \text{se } a=2 \end{cases} \quad x \to 0^+$$

Quindi

$$\lim(\ ) = \lim_{x\to 0^+} \frac{\frac{1}{4}x^2}{x^a - x^2 - \frac{5}{6}x^4} =$$

$$= \begin{cases} = 0 & a<2 \\ = -\infty & a=2 \\ -1/4 & a>2 \end{cases}$$

4)

$$\lim_{x\to 0^+} \frac{\alpha^2 x^2 \lg x + x \operatorname{sen} x}{x^4 \log(1+x) + e^{x^2+x} - 1 - x - \alpha x}, \quad \alpha \in \mathbb{R}.$$

N. $\alpha^2 x^2 \lg x + x\left(x - \frac{x^3}{6} + o(x^3)\right) = \alpha^2 x^2 \lg x + x^2 - \frac{x^4}{6} + o(x^4)$

D. $x^4\left(x - \frac{x^2}{2} + o(x^2)\right) + x^2 + \cancel{x} + \frac{1}{2}(x^2+x)^2 + \frac{1}{6}(x^2+x)^3 + o(x^3) - \cancel{x} - \alpha x =$

$$= -\alpha x + x^2 + \frac{1}{2} x^2 + o(x^2) =$$

$$= -\alpha x + \frac{3}{2} x^2 + o(x^2)$$

$$\frac{N.}{D.} = \frac{\alpha^2 x^2 \log x + x^2 + o(x^2)}{-\alpha x + \frac{3}{2} x^2 + o(x^2)} =$$

se $\alpha \neq 0$ dividendo per x

$$= \frac{\alpha^2 x \log x + x + o(x)}{-\alpha + \frac{3}{2} x + o(x)} \to 0$$

se $\alpha = 0$ $\quad \frac{N.}{D.} = \frac{x^2 + o(x^2)}{\frac{3}{2} x^2 + o(x^2)} \to \frac{2}{3}$

Es. 5) $2^y = e^{y \log 2} = 1 + y \log 2 + y^2 \frac{\log^2 2}{2} + y^3 \frac{\log^3 2}{6} + o(y^3)$

$$2^{x^2 - x} - 2^x + 2(\log 2)x + x^3 \operatorname{sen} \frac{1}{x} =$$

$$= \cancel{1} + (x^2 - \cancel{x}) \log 2 + (x^2 - x)^2 \frac{\log^2 2}{2} + (x^2 - x)^3 \frac{\log^3 2}{6} +$$

$$+ o(x^3) - \cancel{1} - x \cancel{\log 2} - x^2 \frac{\log^2 2}{2} - x^3 \frac{(\log 2)^3}{6} +$$

$$+ o(x^3) + \cancel{2(\log 2)x} + x^3 \operatorname{sen} \frac{1}{x} =$$

$$= x^2 \log 2 + x^2 \cancel{\frac{\log^2 2}{2}} - x^2 \cancel{\frac{\log^2 2}{2}} + o(x^2)$$

N.B. $\quad x^3 \operatorname{sen} \frac{1}{x} = o(x^2)$

Quindi $\quad N. = x^2 \log 2 + o(x^2)$ infinitesimo di ordine 2 per $x \to 0^+$

D. $\quad \operatorname{sen}(\alpha x^2) + (\cos x - 1)^2 + e^{-3/x^2} =$

$$= \alpha x^2 - \frac{1}{6} (\alpha x^2)^3 + o(\alpha x^6) + \left(-\frac{x^2}{2} + \frac{x^4}{4!} + \right.$$

$$+ o(x^4)\big)^2 + e^{-3/x^2} =$$

N.B. $e^{-3/x^2} = o(x^4)$

$$= \alpha x^2 - \frac{x^4}{4} + o(x^4)$$

$$\frac{N.}{D.} = \frac{x^2 \log 2 + o(x^2)}{\alpha x^2 + \frac{x^4}{4} + o(x^4)} \to \begin{cases} \dfrac{\log 2}{\alpha} & \text{se } \alpha \neq 0 \\ +\infty & \text{se } \alpha = 0 \end{cases}$$

Es. 6

$$\underline{N.} = x^{3a}\log x + \left(1 - e^{\arcsin^2 x}\right)^2 = x^{3a}\log x$$

$$+ \left(-(\arcsin x)^2 - \frac{1}{2}(\arcsin x)^4 + o\left((\arcsin x)^4\right)\right)^2$$

$$= x^{3a}\log x + \left(-\left(x + \frac{x^3}{6} + o(x^3)\right)^2 - \frac{1}{2}\left(x + o(x)\right)^4 +\right.$$

$$\left. + o(x^4)\right)^2 = x^{3a}\log x + x^4 + o(x^4)$$

$$\underline{D.}\quad 1 - \cos x - x - \lg(1 - \sinh x) = \frac{x^2}{2} - \frac{x^4}{4!}$$

$$+ o(x^4) - x - \left(-\sinh x - \frac{1}{2}(\sinh x)^2 + o(\sinh^2 x)\right)$$

$$= \frac{x^2}{2} - \frac{x^4}{4!} + o(x^4) - \cancel{x} - \left(-\cancel{x} - \frac{x^3}{6} - \frac{1}{2}x^2 + o(x^2)\right)$$

$$= \frac{x^2}{2} + \frac{x^2}{2} + o(x^2) = x^2 + o(x^2)$$

$$\frac{N.}{D.} = \frac{x^{3a}\log x + x^4 + o(x^4)}{x^2 + o(x^2)} \underset{\text{divido per } x^2}{=}$$

$$= \frac{x^{3a-2}\lg x + x^2 + o(x^2)}{1 + o(1)}$$

$$\lim_{x \to 0^+} \frac{N.}{D.} = \begin{cases} 0 & \text{se } a > 2/3 \\ -\infty & a = 2/3 \end{cases}$$

$\searrow -\infty \qquad a < 2/3$

N.B. $\lim_{x\to 0^+} x^\alpha \log x = \begin{cases} 0 & \text{se } \alpha > 0 \\ -\infty & \text{se } \alpha \le 0 \end{cases}$

N.B. $x \lg x$ è infinitesimo di ordine inferiore a x ma superiore a qualsiasi $x^{1-\varepsilon}$, $\forall\varepsilon$, $x\to 0^+$.

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es. 7 $\qquad f(0) = 2a \qquad$ (la parte a) si può fare anche con l'Hôpital)

a)

$$\lim_{x\to 0^+} \frac{\operatorname{sen} x + \cos x - e^{x^2/2}}{2x} = \lim_{x\to 0^+} \frac{x - \frac{x^3}{6} + o(x^3) \cancel{+1} - \frac{x^2}{2} + o(x^2) \cancel{-1} - \frac{x^2}{2}}{2x} = \lim_{x\to 0^+} \frac{x + o(x)}{2x} = \frac{1}{2}$$

f è continua in $x=0 \iff 2a = \frac{1}{2} \iff a = 1/4$

Negli altri punti $x \in \mathbb{R}$, $x \neq 0$ la f è continua per definizione.

f è continua in $\mathbb{R} \iff a = 1/4$

b) trovo $b \in \mathbb{R}$ t.c. $f(x) = \begin{cases} \frac{\operatorname{sen} x + \cos x - e^{x^2/2}}{2x} & x>0 \\ \frac{1}{2}e^x - 3bx & x \le 0 \end{cases}$

è derivabile e la derivata è continua (C^1)

$$f'(x) = \begin{cases} \frac{2x\left(\cos x - \operatorname{sen} x - x e^{\frac{x^2}{2}}\right) - 2\left(\operatorname{sen} x + \cos x - e^{x^2/2}\right)}{4x^2} & x>0 \\ \frac{1}{2}e^x - 3b & x<0 \end{cases}$$

$f'(x)$ è continua $\forall x \neq 0$

Verifichiamo che in $x=0$ $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x)$

$$\lim_{x\to 0^-} f'(x) = \frac{1}{2} - 3b$$

$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} \frac{2x\left(\cancel{1}-\frac{x^2}{2}-x+\frac{x^3}{6}-x\left(1+\frac{x^2}{2}\right)+o(x^3)\right.}{4x^2}$$

$$\frac{-2\left(\cancel{x}-x^2-\frac{x^3}{6}+o(x^3)\right)}{} =$$

$$= \lim_{x\to 0^+} \frac{-x^3-2x^2-\cancel{2x^2}+\cancel{2x^2}+\frac{x^3}{3}+o(x^3)}{4x^2}$$

$$= -\frac{1}{2}$$

$f'(x)$ $\exists$ ed è continua in $x=0$ $\Leftrightarrow -\frac{1}{2} = \frac{1}{2} - 3b$

$\Rightarrow 3b = 1 \qquad b = 1/3 \quad$ (e $a = 1/4$ per il quale f era continua in $x=0$)

Es. 8

$$\left(\frac{5n! - \sqrt{5}\, n^{2+n^3}}{n^3 \lg\left(1+\frac{1}{\sqrt{n}}\right) - 2n^2 \lg(1+n)}\right)\frac{1}{n^{n^3}} =$$

$$= \frac{-\sqrt{5}\, n^2 \cancel{n^{n^3}}\left(1 - \frac{5\, n!}{\sqrt{5}\, n^2 n^{n^3}}\right)}{\cancel{n^{n^3}}\left(n^3\left(\frac{1}{\sqrt{n}} - \frac{1}{2n} + \frac{1}{3}\frac{1}{n^{3/2}} + o\left(\frac{1}{n^{3/2}}\right)\right) - 2n^2 \lg(1+n)\right)}$$

$$\frac{n!}{n^2\, n^{n^3}} \to 0 \qquad \frac{1}{n^2}\left(\frac{n!}{(n^{n^2})^n}\right) \to 0$$

si fa con le gerarchie degli infiniti

Quindi $\lim_n (\quad) = \lim_n \frac{-\sqrt{5}\, n^2 \left(1 + (\quad)\right)}{n^{5/2} - \frac{1}{2}n^2 + \frac{1}{3}n^{3/2} - 2n^2 \log n}$

(N.B. $\log(1+n) \underset{n\to+\infty}{\sim} \log n$)

$$= \lim_n \frac{-\sqrt{5}\, n^2 \left(1 + (\ )\right)}{n^{5/2}\left(1 - \underbrace{\frac{1}{2n^{1/2}}}_{\searrow 0} + \underbrace{\frac{1}{3n}}_{\searrow 0} - \underbrace{\frac{2 \lg n}{n^{1/2}}}_{\searrow 0}\right)} = 0$$

Es. 9

N. $2^x = e^{x \lg 2}$

$$2^x - \operatorname{sen}(\alpha x) - 1 + x^3 \operatorname{sen}\frac{1}{x} = \cancel{1} + x \lg 2 + \frac{x^2}{2} \lg^2 2$$

$$+ o(x^2) - \alpha x + \alpha^3 \frac{x^3}{6} + o(\alpha^3 x^3) - \cancel{1} + x^3 \operatorname{sen}\frac{1}{x} =$$

$$= x(\lg 2 - \alpha) + \frac{x^2}{2} \log^2 2 + o(x^2)$$

D. $2 - \cos(\sqrt{x}) - \sqrt{1+x} = \cancel{2} - \cancel{1} + \cancel{\frac{x}{2}} - \frac{x^2}{4!} -$

$$-\left(\cancel{1} + \cancel{\frac{1}{2} x} - \frac{1}{8} x^2\right) = x^2\left(+\frac{1}{8} - \frac{1}{24}\right) + o(x^2)$$

$$= \frac{1}{4} x^2 + o(x^2)$$

$$\frac{N.}{D.} = \frac{x(\lg 2 - \alpha) + \frac{x^2}{2} \lg^2 2 + o(x^2)}{\frac{1}{4} x^2 + o(x^2)}$$

$$\lim (\ ) = \begin{cases} +\infty & \text{se } \alpha < \lg 2 \\ -\infty & \text{se } \alpha > \lg 2 \\ 2(\log 2)^2 & \text{se } \alpha = 2\lg 2 \end{cases}$$

Es. 10

$$1 + \operatorname{tg}^3\left(\frac{1}{n}\right) - e^{\operatorname{sen}^3\left(\frac{1}{n}\right)} = \cancel{1} + \left(\frac{1}{n} + \frac{1}{3n^3} + o\left(\frac{1}{n^3}\right)\right)^3$$

$$- \cancel{1} - \operatorname{sen}^3 \frac{1}{n} - \frac{1}{2} \operatorname{sen}^6 \frac{1}{n} =$$

$$= \cancel{\frac{1}{n^3}} + \frac{1}{n^5} + o\left(\frac{1}{n^5}\right) - \left(\cancel{\frac{1}{n}} - \frac{1}{6}\frac{1}{n^3}\right)^3 - \frac{1}{2}\left(\frac{1}{n} + o\left(\frac{1}{n}\right)\right)^6$$

$$= \frac{1}{n^5} + \frac{1}{n^5}\frac{1}{2} + o\left(\frac{1}{n^5}\right) = \frac{3}{2}\frac{1}{n^5}$$

Infinitesimo di ordine 5 rispetto a $1/n$
$n \to +\infty$

$$\underline{D.}\quad \frac{1}{n^{3+a}}\left(\cancel{1} + \sin^2\left(\frac{2}{n}\right) + \frac{1}{2}\sin^4\left(\frac{2}{n}\right) - \cancel{1} - \frac{1}{n^2} - \right.$$

$$\left. - \frac{1}{2}\frac{1}{n^4} + o\left(\frac{1}{n^4}\right)\right) = \frac{1}{n^{3+a}}\left(\left(\frac{2}{n} - \frac{1}{6}\frac{8}{n^3} + o\left(\frac{1}{n^3}\right)\right)^2\right.$$

$$\left. + \frac{1}{2}\left(\frac{2}{n} + o\left(\frac{1}{n}\right)\right)^4 - \frac{1}{n^2} - \frac{1}{2}\frac{1}{n^4} + o\left(\frac{1}{n^4}\right)\right)$$

$$= \frac{1}{n^{3+a}}\left(\frac{4}{n^2} - \frac{1}{n^2} + o\left(\frac{1}{n^2}\right)\right) =$$

$$= \frac{3}{n^{5+a}} + o\left(\frac{1}{n^{5+a}}\right)$$

$$\lim_n \frac{N.}{D.} = \lim_n \frac{n^{5+a}}{\cancel{3}}\,\frac{\cancel{3}}{2}\,\frac{1}{n^5} = \lim_n \frac{n^a}{2} =$$

$$= \begin{cases} +\infty & a > 0 \\ 1/2 & a = 0 \\ 0 & a < 0 \end{cases}$$

Segnalate eventuali errori, grazie!