Lezione del 10 ottobre

funzioni invertibili

$$f : D \to f(D)$$

Def. f è invertibile in D se $\forall y \in f(D)\ \exists$ uno e un solo $x \in D : f(x) = y$

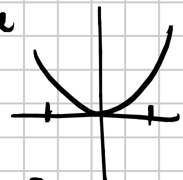
la f deve essere iniettiva:

$$\forall x_1, x_2 \in D \quad x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

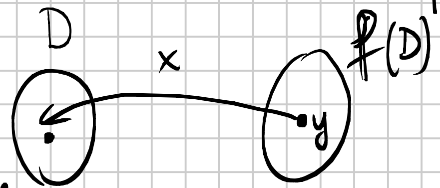
es. $f(x) = x^3$ è iniettiva

$f : \mathbb{R} \to \mathbb{R} \quad \Rightarrow$ è invertibile

$f(x) = x^2 \quad f : \mathbb{R} \to [0, +\infty)$



non è invertibile.



Se f è invertibile

$f^{-1} : y \longrightarrow x \quad : \quad f(x) = y$

$$f^{-1} : f(D) \longrightarrow D$$

$$x = f^{-1}(y) \iff y = f(x)$$

— · —

oss. $f^{-1} \circ f(x) = f^{-1}(f(x)) = x \quad , \ \forall x \in D$

$$f^{-1} \circ f = Id$$

$$f \circ f^{-1}(y) = f(f^{-1}(y)) = y \ , \ \forall y \in f(D)$$

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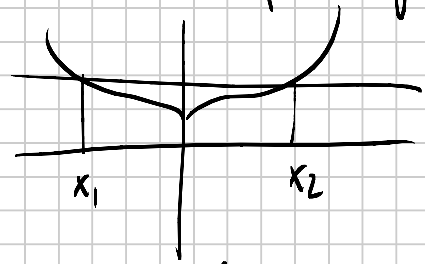
Oss. se f è pari non è invertibile



se f è periodica non è invertibile

Teorema $f: D \to \mathbb{R}$ strettamente monotona in D è invertibile in D

Dim sup. f sia strett. crescente
$(x_1 > x_2 \Rightarrow f(x_1) > f(x_2))$
dim. che f è iniettiva

Ts. $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$?

$x_1 \neq x_2$ — $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$
— $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

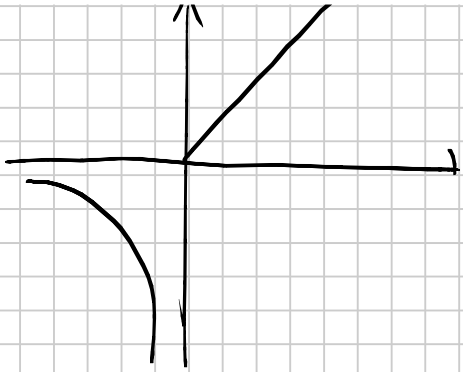
quindi mai può non essere $f(x_1) = f(x_2)$ #

Teo. Se f è strett. monotona $\Rightarrow$ anche f^{-1} è strett. monotona

Dim. no

Oss. f è strett. monotona $\Rightarrow$ f è invertibile
$\not\Leftarrow$

$\exists$ f. invertibili che non sono strettamente monotone:



→ f è iniettiva quindi invertibile $\mathbb{R}\setminus\{0\}$
ma non è monotona in tutto $\mathbb{R}\setminus\{0\}$

es. $y = a^x \;,\; a>1$

$f: \mathbb{R} \to (0,+\infty)$ è strett. monotone
è invertibile

$f^{-1}: (0,+\infty) \to \mathbb{R}$

$y \longrightarrow x \qquad f^{-1}(y) =: \log_a y$

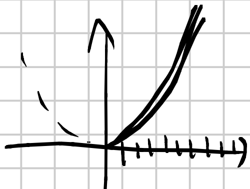
$y = a^x$

es. $f(x) = x^3 \qquad f: \mathbb{R}\to\mathbb{R}$ strett. crescente

$f^{-1}: \mathbb{R}\to\mathbb{R} \qquad f^{-1}(y) =: \sqrt[3]{y}$

$y \to x \;:\; y = x^3$

es. $f(x) = x^2$



$f: \mathbb{R} \to [0,+\infty)$ non è invertibile

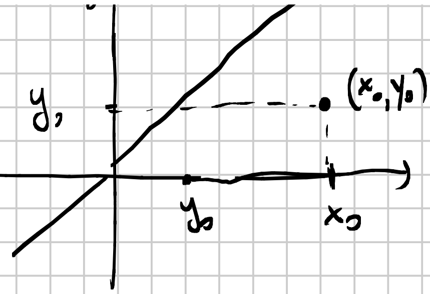
$f: \{x\geq 0\} \longrightarrow [0,+\infty)$ è invertibile

$f^{-1}: [0,+\infty) \longrightarrow \{x\geq 0\}$

$y \longrightarrow x \;:\; y = x^2$

$f^{-1}(y) =: \sqrt{y} = x$

Grafico di f^{-1}



$(x_0,y_0) \in \text{graf}\, f \Leftrightarrow f(x_0)=y_0$

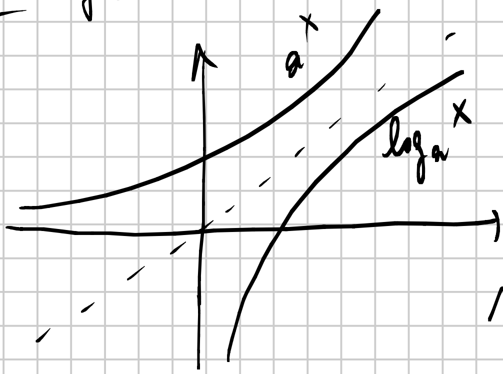
$(y_0,x_0) \in \text{graf}\, f^{-1} \Leftrightarrow f^{-1}(y_0)=x_0$

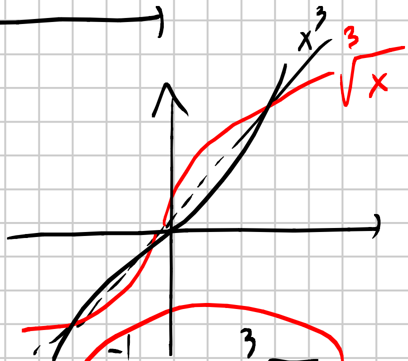
(x_0,y_0) e (y_0,x_0) sono p.ti simmetrici rispetto alla bisettrice del 1° e 3° quadrante

es. $f(x)=a^x$ $\quad a>1$ $\qquad f^{-1}(y)=\log_a y$

$\downarrow$

$f^{-1}(x)=\log_a x$





es. $f(x)=x^3$

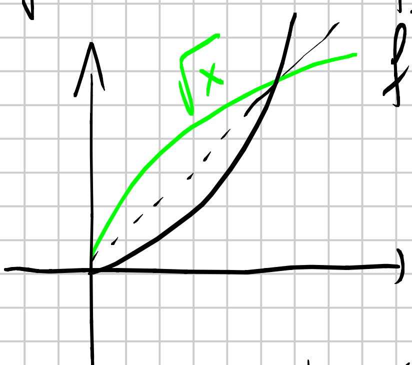
$f^{-1}(y)=\sqrt[3]{y}$

$\hookrightarrow$ $f^{-1}(x)=\sqrt[3]{x}$

es $f(x)=x^2$ $\qquad f:[0,+\infty)\to[0,+\infty)$

$f^{-1}:[0,+\infty)\to[0,+\infty)$

$f^{-1}(y)=\sqrt{y}$



es. $f(x)=x^2$

$f:\mathbb{R}\to[0,+\infty)$

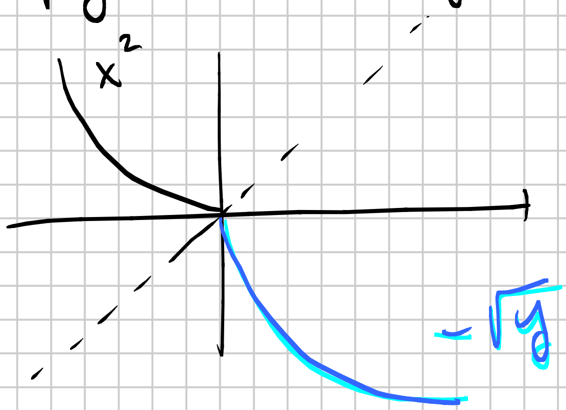
$f(x)=x^2$ $\qquad f:(-\infty,0]\to[0,+\infty)$

$\exists$ $\qquad f^{-1}:[0,+\infty)\to(-\infty,0]$

$$y \longrightarrow x$$

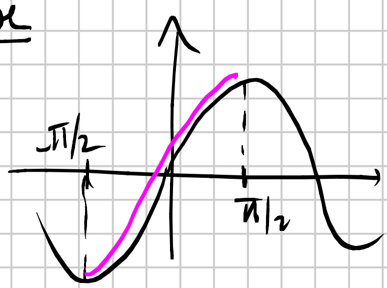
$$y = x^2$$

$$f^{-1}(y) = -\sqrt{y} = x \qquad y = x^2$$



Funzioni trigonometriche inverse

$y = \operatorname{sen} x$ è periodica



$$\operatorname{sen} x : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$$

In tale dominio $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ f è strett. crescente

$\Rightarrow$ f è invertibile

$$f^{-1} : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

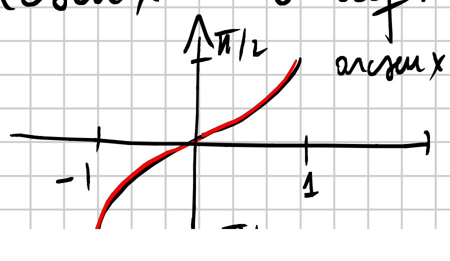
$$y \longrightarrow x \quad : \quad y = \operatorname{sen} x$$

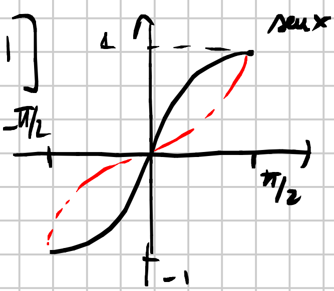
$$\boxed{f^{-1}(y) =: \operatorname{arcsen} y}$$

$$x = \operatorname{arcsen} y \iff y = \operatorname{sen} x$$

$$y \in [-1, 1] \qquad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$\operatorname{arcsen} x$ è def. in $[-1, 1]$



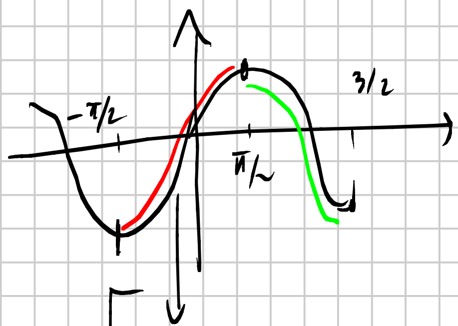


$\arcsin(-x) = -\arcsin x$ dispari

es. $f(x) = \arcsin\left(\log_5(x-1)\right)$

$$\begin{cases} x-1>0 \\ |\log_5(x-1)| \le 1 \end{cases}$$

oss. Cosa succede se invece di prendere l'intervallo $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ prendo un altro intervallo in cui $\operatorname{sen} x$ è monotona



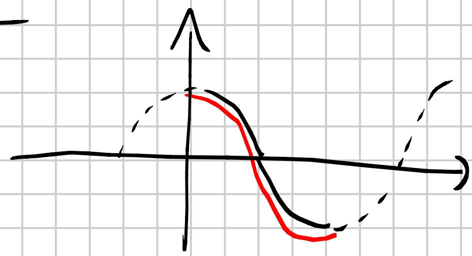
se prendo $x \in \left[\frac{\pi}{2}, \frac{3}{2}\pi\right]$

$\operatorname{sen} x$ è invertibile

$\exists\ f^{-1}(y)$ ma $f^{-1}(y) \ne \arcsin y$ $\left[f^{-1}(y) =: \arcsin y \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \right]$

$f^{-1}(y) = \pi - \arcsin y$ (si può dim.)

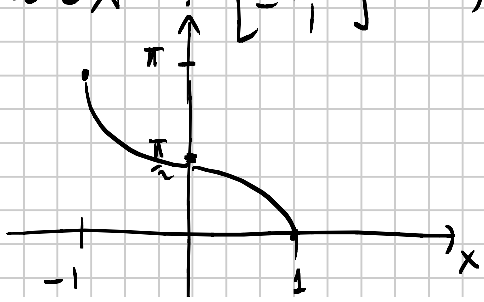
Analogamente



$y = \cos x$
$x \in [0, \pi]$

$f^{-1}(y) =: \arccos y = x \iff y = \cos x$
$y \in [-1, 1]$ $\qquad x \in [0, \pi]$

$\arccos x : [-1, 1] \to [0, \pi]$



$$f(x) = \arccos(x^3 - 8)$$ dominio

$$|x^3 - 8| \leq 1$$

$$-1 \leq x^3 - 8 \leq 1$$

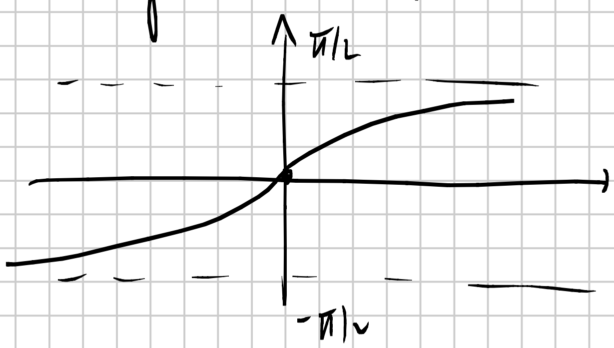
$$\begin{cases} x^3 \leq 9 \\ x^3 \geq 7 \end{cases}$$

Analog. $\operatorname{tg} x : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$

$\operatorname{arctg} y = x \iff y = \operatorname{tg} x$

$y \in \mathbb{R}$ $\qquad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\operatorname{arctg} x : \mathbb{R} \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$



$\operatorname{arctg}(-x) = -\operatorname{arctg} x$
dispari

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oss. $f \circ f^{-1}(y) = y$

$f^{-1} \circ f(x) = x$

$\downarrow f(x) = \operatorname{sen} x$

$$\arcsin(\operatorname{sen} x) = x \iff x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

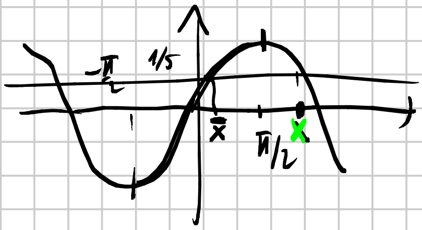
sennò $\arcsin(\operatorname{sen} x) \neq x$

es. $x = \pi$

$0 = \arcsin(0) \neq \pi$

es. Trovare le soluzioni di

$\operatorname{sen} x = \frac{1}{5}$



$\bar{x} = \arcsin \frac{1}{5}$

$x = \pi - \arcsin \frac{1}{5}$

$x = \arcsin \frac{1}{5} + 2k\pi$

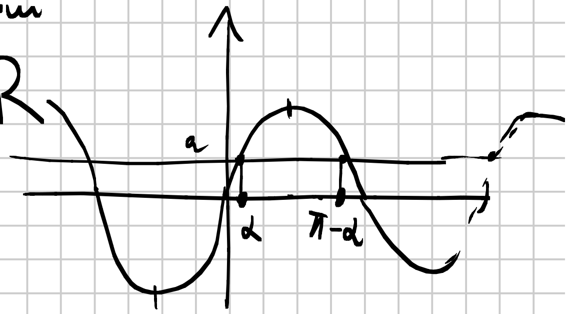
$x = \pi - \arcsin \frac{1}{5} + 2k\pi$

es. $\operatorname{sen} x > a, \quad a \in \mathbb{R}$

1) $a \geq 1$ $\quad \nexists$ soluzioni

2) $a < -1$ $\quad \forall x \in \mathbb{R}$

3) $a \in [-1, 1)$



$\operatorname{sen} \alpha = a$

$\alpha = \arcsin a$ $\quad$ perché $\alpha \in [-\pi/2, \pi/2]$

$x \in (\alpha, \pi - \alpha) \cup (\alpha + 2\pi, \pi - \alpha + 2\pi) \cup \dots$

$x \in (\arcsin a + 2k\pi, \pi - \arcsin a + 2k\pi)$

$k \in \mathbb{Z}$

es. $\operatorname{sen} x < a$ $\quad$ fa cose