Lezione del 16 Ottobre

$$\lim_n a_n = \begin{cases} l \in \mathbb{R} & \text{convergente} \\ \pm\infty & \text{divergente} \\ \nexists & \text{indeterminata} \end{cases}$$

OSS. $\{a_n\}$ limitata $\quad$ $\{a_n\}$ ammette limite finito

$m \leq a_n \leq M \;, \forall n$

Prop. $\{a_n\}$ è convergente $\Rightarrow$ $\{a_n\}$ è limitata

$a_n \to l \;, l \in \mathbb{R}$

Dim. $\forall \varepsilon > 0 \;\exists N : |a_n - l| < \varepsilon \;, n > N$

$$\Downarrow$$

$$l - \varepsilon < a_n < l + \varepsilon \;, \forall n > N$$

per gli altri, quelli con $n \leq N$

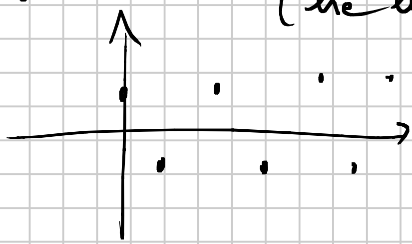
$a_0, a_1, \dots a_N$ sono numeri $\Rightarrow \exists\, m, M$ t.c.

$\Rightarrow \quad m \leq a_n \leq M \;, \forall n$ $\qquad$ #

Ma non vale il viceversa

$\{a_n\}$ limitata $\not\Rightarrow$ $\{a_n\}$ è convergente (ha limite)

es. $\quad a_n = (-1)^n$



Calcolo dei limiti

Teorema $\quad a_n \to a \;, b_n \to b \;, a, b \in \mathbb{R}$

- $a_n \pm b_n \to a \pm b$
- $a_n \cdot b_n \to a \cdot b$
- $\dfrac{a_n}{b_n} \to \dfrac{a}{b} \;, \quad (b_n, b \neq 0)$
- $(a_n)^{b_n} \to a^b \quad (a_n, a > 0)$

Dim. • Ts $a_n + b_n \to a+b$

Hp. $a_n \to a$ $\quad \forall \varepsilon \; \exists N_1 : |a_n - a| < \varepsilon, \; n > N_1$ $\quad$ 1)

 $b_n \to b$ $\quad \forall \varepsilon \; \exists N_2 : |b_n - b| < \varepsilon, \; n > N_2$ $\quad$ 2)

$N = \max\{N_1, N_2\}$, prendo $n > N \Rightarrow$ vale 1) e 2)

$$|(a_n + b_n) - (a+b)| = |\underbrace{a_n - a}_{} + \underbrace{b_n - b}_{}| \leq |a_n - a| +$$

$$+ |b_n - b| < \varepsilon + \varepsilon = 2\varepsilon = \tilde{\varepsilon}$$

$$\forall \tilde{\varepsilon} \; \exists N : |(a_n + b) - (a+b)| < \tilde{\varepsilon}, \; n > N.$$

• Hp. $a_n \to a$ $\qquad$ Ts. $a_n b_n \to a \cdot b$

 $b_n \to b$

$$|a_n b_n - ab| = |a_n b_n - a_n b + a_n b - ab| =$$

$$= |a_n (b_n - b) + b(a_n - a)| \leq |a_n (b_n - b)| +$$

$$+ |b(a_n - a)| = \underbrace{|a_n|}_{\leq L} \underbrace{|b_n - b|}_{< \varepsilon} + |b| \underbrace{|a_n - a|}_{< \varepsilon} \leq$$

$$\leq L\varepsilon + |b|\varepsilon = \tilde{\varepsilon}$$

$|a_n| \leq L$ perché se è convergente è limitata

$$\forall \tilde{\varepsilon} \; \exists N : |a_n b_n - ab| < \tilde{\varepsilon}, \; \forall n > N$$

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Es. provare a dim $\frac{a_n}{b_n} \to \frac{a}{b}$, $b_n, b \neq 0$.

esemp. $\quad a_n \to a \Rightarrow \underset{b_n}{C} a_n \to Ca$

$$\lim_n \frac{1}{n} = 0 \Rightarrow \lim_n \frac{-82}{n} = 0$$

$$\lim_n \frac{c}{n} = 0$$

$$\lim_n \frac{1}{n^2} = \lim_n \frac{1}{n} \cdot \frac{1}{n} = 0$$

$$a_n^{b_n} \to a^b$$

$$\lim_n 3^{1/n} = 3^0 = 1$$

$$\lim_n 3^{1/n} + \frac{1}{n^2} - \frac{87}{n} = 1 + 0 - 0 = 1$$

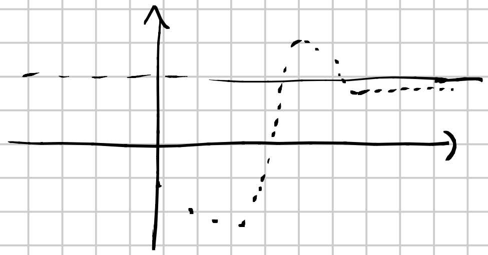
Teorema di permanenza del segno

$a_n \to a$ e $a > 0$, allora

$a_n > 0$, definitivamente $\left(\exists N : a_n > 0 \ \forall n > N\right)$

Dim.

$\forall \varepsilon > 0 \ \exists N : |a_n - a| < \varepsilon \quad \forall n > N$



$\boxed{a - \varepsilon < a_n} < a + \varepsilon$

$a_n > a - \varepsilon$

Poiché $a > 0$ e la disuguaglianza vale $\forall \varepsilon$ scelgo ε t.c. $a - \varepsilon > 0$

$\Rightarrow a_n > 0, \ \forall n > N$

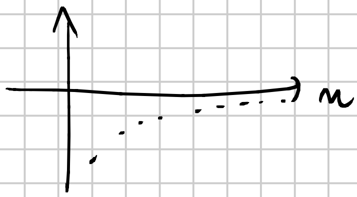
$a_n \to a, \ a > 0 \Rightarrow a_n > 0$ definitivamente

oss. non vale se si scrive così:

$a_n \to a, \ a \geq 0 \Rightarrow a_n \geq 0$ definitivamente

$a_n \to a, \ a = 0 \overset{?}{\Rightarrow} a_n \geq 0$ No!

$a_n = -\frac{1}{n}$



T. permanenza segno

$a_n \to a, \ a > 0 \Rightarrow a_n > 0$ definitivamente

Non vale il viceversa

$a_n \to a, \ a_n > 0$ definitivamente $\Rightarrow a > 0$?

es. $a_n = \frac{1}{n} > 0$ ma $a = 0$ non è > 0

OSS. $a_n \to a, \quad a < 0 \Rightarrow a_n < 0$ definitivamente

2° modo per scrivere il teo. di permanenza del segno

$$a_n \to a \quad e \quad a_n \geq 0 \Rightarrow a \geq 0$$

dim. x assurdo $\quad a < 0 \Rightarrow a_n < 0$ definitivamente assurdo perché $a_n \geq 0$ definitivamente.

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1) $a_n \to a \quad a > 0 \quad \Rightarrow a_n > 0$ definitiv.

2) $a_n \to a \quad a_n \geq 0 \Rightarrow a \geq 0$

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Corollario

$a_n \to a$, $b_n \to b$, $\quad a_n \geq b_n$ definitivamente $\quad \Rightarrow \quad a \geq b$

dim $\quad (a_n - b_n) \geq 0 \underset{\text{per } 2)}{\Rightarrow} (a-b) \geq 0 \Rightarrow a \geq b$

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Alcuni limiti $\qquad \alpha \in \mathbb{R}$

$$\lim_n n^\alpha = \begin{cases} +\infty & \alpha > 0 \\ 1 & \alpha = 0 \\ 0 & \alpha < 0 \end{cases}$$

si verifica con la def. di limite

$$\lim_n \frac{n^{1/2} + n^2 + 8}{n^{3/2} + n^5} = \lim_n \frac{\cancel{n^2}\left(\frac{1}{n^{3/2}} + 1 + \frac{8}{n^2}\right)}{n^{\cancel{5}\,3}\left(\frac{1}{n^{5-\frac{3}{2}}} + 1\right)}$$

$$= \lim_n \underbrace{\frac{1}{n^3}}_{\to 0} \cdot \frac{\left(\overbrace{\frac{1}{n^{3/2}}}^{\to 0} + 1 + \overbrace{\frac{8}{n^2}}^{\to 0}\right)}{\left(\underbrace{\frac{1}{n^{7/2}}}_{\to 0} + 1\right)} = 0 \cdot \frac{1}{1} = 0$$

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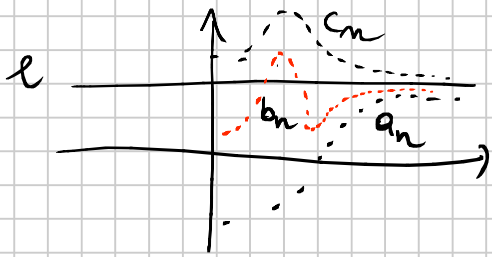
Teorema del confronto (dei "carabinieri")

Se $a_n \le b_n \le c_n$ definitivamente

se $a_n \to \ell$, $c_n \to \ell$, $\ell \in \mathbb{R}$

$\Rightarrow b_n \to \ell$



Dim. $a_n \to \ell$

$c_n \to \ell$

$\forall \varepsilon > 0 \ \exists N_1 : \ |a_n - \ell| < \varepsilon \quad n > N_1$

$\forall \varepsilon > 0 \ \exists N_2 : \ |c_n - \ell| < \varepsilon \quad n > N_2$

$\ell - \varepsilon < a_n < \ell + \varepsilon \qquad \ell - \varepsilon < c_n < \ell + \varepsilon$

$$\ell - \varepsilon < a_n \le b_n \le c_n < \ell + \varepsilon$$

$|b_n - \ell| < \varepsilon \quad \forall n > N = \max\{N_1, N_2\}$

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Teo. del confronto

$$\begin{array}{ccccc} a_n & \le & b_n & \le & c_n \\ \downarrow & & \downarrow & & \downarrow \\ \ell & & \ell & & \ell \end{array}$$

Si usa così:

A) • $|b_n| \le c_n$ (per n suff. grande)

$c_n \to 0 \quad \Rightarrow \quad b_n \to 0$

$$\begin{array}{ccccc} -c_n & \le & b_n & \le & c_n \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array}$$

B) • $b_n \to 0$ $\qquad |c_n \cdot b_n| \le K |b_n| \to 0$

$|c_n| \le K$

$\Rightarrow \quad c_n \cdot b_n \to 0$

Ho dim. che

<u>Prop.</u> Il prodotto di una successione che tende a zero per una successione limitata, tende a zero.

se $b_n \to 0$ dico che b_n è <u>infinitesima</u>

Il prodotto di una successione <u>infinitesima</u> per una limitata è una successione <u>infinitesima</u>

<u>es.</u> $$\lim_n \frac{\operatorname{sen} n}{n} = \lim_n \underbrace{(\operatorname{sen} n)}_{\substack{\text{limitata} \\ \text{ma irregolare}}} \cdot \underbrace{\frac{1}{n}}_{\text{infinitesima}} = 0$$

oppure dico

$$-\frac{1}{n} \leq \frac{\operatorname{sen} n}{n} \leq \frac{1}{n}$$

per il confronto.

$$\begin{array}{ccc} -\frac{1}{n} & \frac{\operatorname{sen} n}{n} & \frac{1}{n} \\ \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 \end{array}$$

<u>Algebra dei limiti nel caso infinito</u>

<u>Teo.</u>
$$\begin{array}{l} a_n \to a \\ b_n \to +\infty \end{array} \Rightarrow \begin{array}{l} a_n + b_n \to +\infty \\ a_n - b_n \to -\infty \end{array} \qquad \begin{array}{l} (a + \infty = +\infty) \\ (a - (+\infty) = -\infty) \end{array}$$

<u>Dim</u> no

<u>es.</u> $\frac{1}{n} + n^3 \to +\infty$

$\frac{1}{n} - n^3 \to -\infty$

<u>Teo.</u>
$$\begin{array}{l} a_n \to +\infty \\ b_n \to +\infty \end{array} \Rightarrow \begin{array}{l} a_n + b_n \to +\infty \\ -a_n - b_n \to -\infty \\ a_n \cdot b_n \to +\infty \end{array} \qquad \begin{array}{l} +\infty + \infty = +\infty \\ -\infty - \infty = -\infty \\ (+\infty)(+\infty) = +\infty \end{array}$$

$$\begin{array}{l} a_n \to a \neq 0 \\ b_n \to +\infty \end{array} \qquad a_n \cdot b_n \to \pm\infty$$

a seconda del segno di a

$$b_n \to 0 \qquad \frac{1}{b_n} \to \pm\infty$$

a seconda del segno di b_n per n grande

$$\begin{cases} b_n > 0 & \frac{1}{b_n} \to +\infty \\ b_n < 0 & \frac{1}{b_n} \to -\infty \end{cases}$$

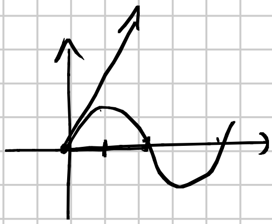
$$b_n \to \pm\infty \quad \Rightarrow \quad \frac{1}{b_n} \to 0$$

<u>Dim.</u> no (si fa con la def. di limite)

<u>es.</u>

$$\underbrace{\left(3 - \frac{1}{n}\right)}_{\downarrow \, 3} \cdot \underbrace{n^3}_{\downarrow \, +\infty} \to +\infty$$

$$\underbrace{\left(\frac{1}{n} - 3\right)}_{\downarrow \, -3} \cdot \underbrace{n^3}_{\downarrow \, +\infty} \longrightarrow -\infty$$



<u>es.</u>

$$\underset{\downarrow \, 0}{0} \le \underset{\downarrow \, 0}{\operatorname{sen}\frac{1}{n}} \le \underset{\downarrow \, 0}{\frac{1}{n}} \qquad n = 1, 2, \dots$$

$\left(\frac{1}{n} \in [0, \pi]\right.$ $\qquad \operatorname{sen} x \le x$

$\operatorname{sen}\frac{1}{n} \to 0$

<u>es.</u>

$$\frac{1}{\operatorname{sen}^2\left(\frac{1}{n}\right)} \to +\infty$$

$\operatorname{sen}^2\frac{1}{n} \to 0$

Inoltre $\operatorname{sen}^2\frac{1}{n} > 0$

$$\frac{1}{\operatorname{sen}\left(\frac{1}{n}\right)} \to +\infty$$

$\operatorname{sen}\frac{1}{n} \to 0$

$\operatorname{sen}\frac{1}{n} \geq 0$

<u>es.</u>

$$\frac{1}{3^n} \to 0 \qquad 3^n \to +\infty$$

<u>OSS.</u> Nei teoremi precedenti mancano le regole per calcolare alcuni limiti

$$\begin{matrix} a_n \to +\infty \\ b_n \to +\infty \end{matrix} \Rightarrow a_n - b_n \; ?$$

- $+\infty - \infty$

- $0 \cdot (\pm\infty)$ $\qquad a_n \to 0,\ b_n \to \pm\infty \qquad a_n \cdot b_n \to ?$

- $\dfrac{0}{0}$ $\qquad a_n \to 0,\ b_n \to 0 \qquad \dfrac{a_n}{b_n} \to ?$

- $\dfrac{\pm\infty}{\pm\infty}$ $\qquad a_n \to \pm\infty,\ b_n \to \pm\infty \Rightarrow \dfrac{a_n}{b_n} \to ?$

si chiamano forme d'indecisione (o indeterminate) perché può succedere qualsiasi cosa.

esercizi

- $$\lim_n n^3 - n^2 + 5 - n =$$
$$+\infty - \infty + 5 - \infty$$
forma di indecisione $\infty - \infty$

$$= \lim_n n^3\Big(\underset{\downarrow\atop 1}{1} - \underset{\downarrow\atop 0}{\frac{1}{n}} + \underset{\downarrow\atop 0}{\frac{5}{n^3}} - \underset{\downarrow\atop 0}{\frac{1}{n^2}}\Big) = +\infty$$

- $$\lim_n \sqrt{n} - \sqrt{n+1} = \lim_n \frac{(\sqrt{n}-\sqrt{n+1})(\sqrt{n}+\sqrt{n+1})}{\sqrt{n}+\sqrt{n+1}}$$
$$+\infty - \infty$$
$$= \lim_n \frac{\cancel{n} - \cancel{n} - 1}{\sqrt{n}+\sqrt{n+1}} = \lim_n \frac{-1}{\sqrt{n}+\sqrt{n+1}} = 0$$

- $$\lim_n \frac{n^5 + \operatorname{sen} n}{2n^5 + \frac{1}{n}} = \lim_n \frac{\cancel{n^5}\left(1 + \overset{\to 0}{\frac{\operatorname{sen} n}{n^5}}\right)}{\cancel{n^5}\left(2 + \underset{\to 0}{\frac{1}{n^6}}\right)} = \frac{1}{2}$$

- $$\lim_n \frac{1}{\underbrace{\sqrt{n} - \sqrt{n+1}}_{<0}} \to -\infty \qquad \frac{1}{0}$$
$$\lim_n \sqrt{n} - \sqrt{n+1} = 0 \qquad \text{(fatto prima)}$$

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Esercizi su domini di funzioni

$$f(x) = \cosh\left(\log|x^2-1|\right)$$

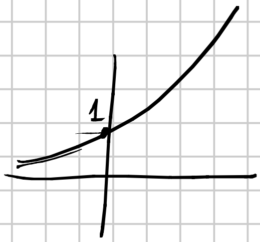
$\cosh y \qquad \forall y \in \mathbb{R}$

$|x^2-1| > 0 \qquad |x^2-1| \geq 0$

$|x^2-1| \neq 0 \qquad x^2-1 \neq 0 \qquad x \neq \pm 1$

$$D = \mathbb{R} \setminus \{\pm 1\}$$

$$f(x) = \arcsin\left(e^{\frac{|x+2|}{x}}\right)$$



$\arcsin y \qquad |y| \leq 1$

$x \neq 0 \qquad \left|e^{\frac{|x+2|}{x}}\right| \leq 1 \qquad e^{y} \leq 1 \Leftrightarrow y \leq 0$

$$e^{\frac{|x+2|}{x}} \leq 1$$

$\dfrac{|x+2|}{x} \leq 0 \qquad x<0$

$$D = \{x \in \mathbb{R} : x<0\}$$

es. $\sqrt{2^{|x-3|}-8} + \sqrt{3^{x^2+x+2}-9} = f(x)$

$$\begin{cases} 2^{|x-3|} \geqslant 2^3 \\ 3^{x^2+x+2} \geqslant 3^2 \end{cases} \qquad \begin{cases} |x-3| \geqslant 3 \\ x^2+x+2 \geqslant 2 \end{cases}$$

Finire $\qquad \mathbb{R}. \qquad x \leq -1$ o $x \geqslant 0$