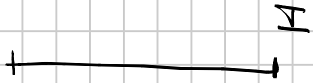
23/10/13

Limiti di funzioni

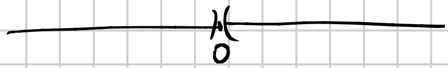
Definizione topologica di limite (p. 119 del libro)

$\lim\limits_{x \to c} f(x)$ $\qquad f : I \to \mathbb{R}$ $\qquad$ I intervallo

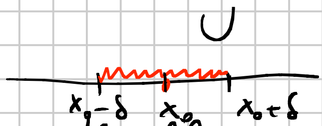


f può essere definita o no nel punto c, che può essere interno ad I oppure agli estremi di I

$f(x) = \frac{1}{x}$ $\qquad \mathbb{R} \setminus \{0\}$



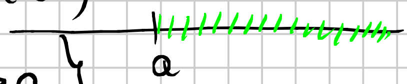
Def. di intorno di un punto



- $x_0 \in \mathbb{R}$ Intorno U di x_0 è un intervallo del tipo $(x_0-\delta, x_0+\delta)$

  $U = \{ x \in \mathbb{R} : |x - x_0| < \delta \} = (x_0-\delta, x_0+\delta)$

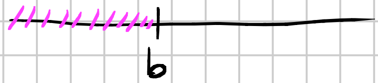
- $+\infty$ Intorno U di $+\infty$ è un intervallo della forma $(a, +\infty)$

  $U = \{ x \in \mathbb{R} : x > a \}$

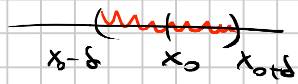
- $-\infty$ Intorno U di $-\infty$ è un intervallo della forma $(-\infty, b)$

  $U = \{ x \in \mathbb{R} : x < b \}$

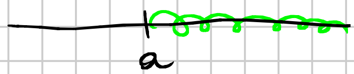
  

Definitivamente: $f(x)$ ha una certa proprietà definitivamente per $x \to c$ se esiste un intorno U di c t.c. vale la proprietà per ogni $x \in U$, $x \neq c$

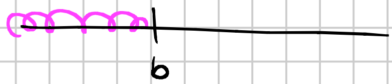
$x \in U_{x_0} \iff |x - x_0| < \delta$



$x \in U_{+\infty} \iff x > a$

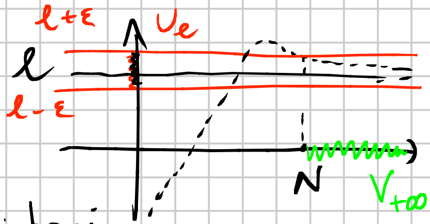


$x \in U_{-\infty} \iff x < b$



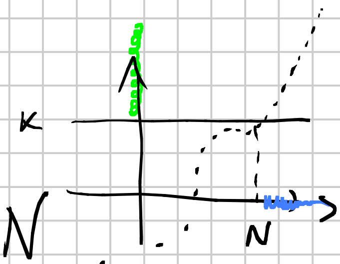
$$\lim_{n\to+\infty} a_n = \ell \iff \forall \varepsilon > 0 \; \exists N : \; |a_n - \ell| < \varepsilon, \; \forall n > N$$

$(\ell-\varepsilon, \ell+\varepsilon) = U_\ell$



con gli intorni

$$\lim_{n\to+\infty} a_n = \ell \iff \forall U_\ell \text{ (intorno di } \ell) \; \exists V_{+\infty} \text{ t.c. } a_n \in U_\ell \; \forall n \in V_{+\infty}$$



$$\lim_{n\to+\infty} a_n = +\infty \iff \forall K > 0 \; \exists N \text{ t.c. } a_n > K, \; \forall n > N$$

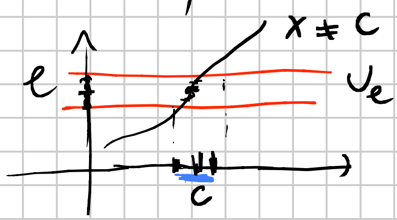
$$\lim_{n\to+\infty} a_n = +\infty \iff \forall U_{+\infty} \; \exists V_{+\infty} \text{ t.c. } a_n \in U_{+\infty}, \; \forall n \in V_{+\infty}.$$

$\mathbb{R}^* = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$

<u>Definizione di limite</u>

Sia $c \in \mathbb{R}^*$, f definita (almeno) definitivamente per $x \to c$. $\ell \in \mathbb{R}^*$

$\lim_{x\to c} f(x) = \ell$ significa che $\forall U_\ell$ intorno di ℓ esiste V_c intorno di c t.c. $f(x) \in U_\ell$, $\forall x \in V_c$, $x \neq c$

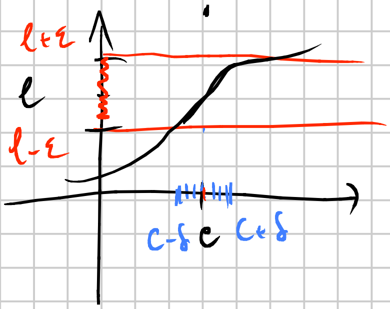


<u>Caso 1</u> $c, \ell \in \mathbb{R}$

$\lim_{x\to c} f(x) = \ell$ $\qquad$ $U_\ell = (\ell-\varepsilon, \ell+\varepsilon)$

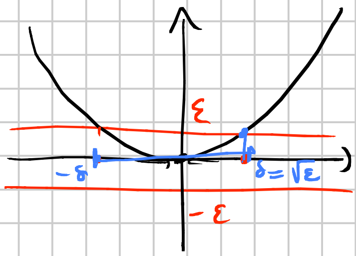
$c, \ell \in \mathbb{R}$ $\qquad$ $V_c = (c-\delta, c+\delta)$

$$\boxed{\lim_{x\to c} f(x) = \ell \text{ significa che } \forall \varepsilon\ \exists \delta \text{ t.c. } |f(x)-\ell|<\varepsilon,\ \forall x\ |x-c|<\delta,\ x\neq c.}$$



oss. Se cambio ε mi cambia δ

oss. non si richiede di conoscere f in $x=c$
non serve $|f(c)-\ell|<\varepsilon$



es. $\lim_{x\to 0} x^2 = 0$

$c=0$

$\ell=0$

$\Updownarrow$?

$\forall \varepsilon>0\ \exists \delta : \quad |x^2|<\varepsilon$

$\forall\ |x|<\delta,\quad x\neq 0$

$x^2<\varepsilon \quad \Rightarrow \quad |x|<\underbrace{\sqrt{\varepsilon}}_{=\delta}$ $\qquad$ $\boxed{\delta=\sqrt{\varepsilon}}$

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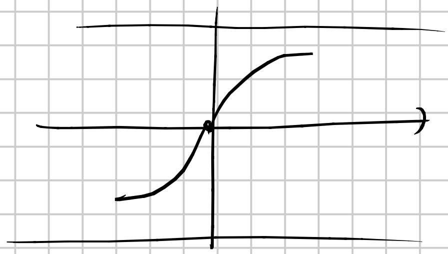
es. $\lim_{x\to 0} \operatorname{sen} x = 0$



$\ell=0$

$c=0$

$\Updownarrow$?

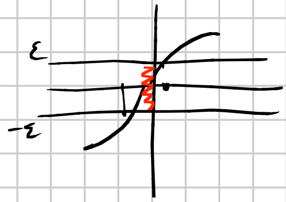
$\forall \varepsilon>0\ \exists \delta>0$ t.c. $|\operatorname{sen} x|<\varepsilon$

$\forall x\ |x|<\delta,\quad x\neq 0$

$|\operatorname{sen} x|<\varepsilon \qquad -\varepsilon<\operatorname{sen} x<\varepsilon$

$\varepsilon > 1$ sempre vero

$\varepsilon < 1$ $\qquad -\varepsilon < \operatorname{sen} x < \varepsilon$



$$\operatorname{arcsen}(-\varepsilon) < x < \operatorname{arcsen}\varepsilon$$

$$-\operatorname{arcsen}\varepsilon < x < \operatorname{arcsen}\varepsilon \qquad \delta$$

$$|x| < \boxed{\operatorname{arcsen}\varepsilon}$$

$l = 1$
$c = 0$

Limite sbagliato $\quad \lim_{x\to 0} x^2 = 1$

$\forall \varepsilon > 0 \quad \exists\, \delta > 0$ t.c. $\quad |x^2 - 1| < \varepsilon \quad$?

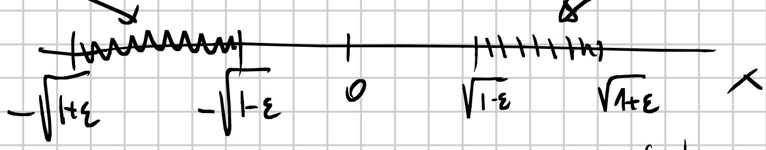
$\forall x: \quad |x| < \delta$

$|x^2 - 1| < \varepsilon \qquad 1 - \varepsilon < x^2 < 1 + \varepsilon$

$$\Downarrow$$

$$\sqrt{1-\varepsilon} < x < \sqrt{1+\varepsilon}$$
$$-\sqrt{1+\varepsilon} < x < -\sqrt{1-\varepsilon}$$

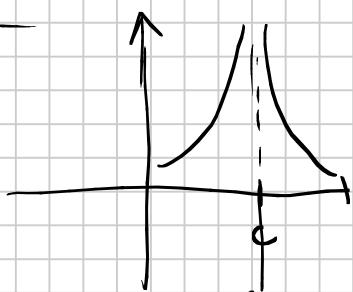
non è intorno di 0!



$\nexists$ intorno di 0 t.c. $f(x) \in U_l$.

Caso 2

$c \in \mathbb{R}$



$$\lim_{x\to c} f(x) = +\infty$$

$$\Updownarrow$$

si dice che f ha un <u>asintoto verticale</u> in $x = c$

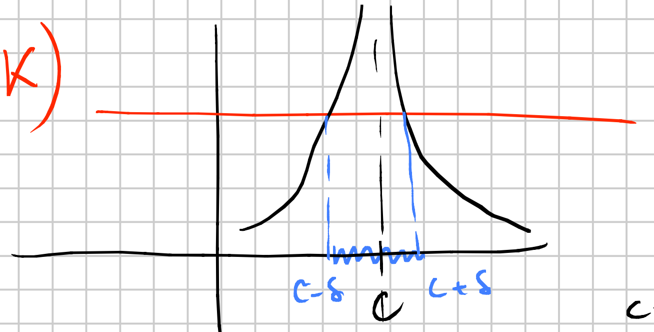
$\forall\, U_{+\infty} \quad \exists\, V_c$ t.c.

$f(x) \in U_{+\infty}, \quad \forall x \in V_c, \quad x \neq c$

$U_{+\infty} = (K, +\infty) \qquad V_c = (c - \delta, c + \delta)$

$$\boxed{\lim_{x\to c} f(x) = +\infty \iff \forall K > 0 \quad \exists\, \delta \text{ t.c. } f(x) > K \quad \forall x: |x - c| < \delta,\ x \neq c}$$

(δ dipende da K)

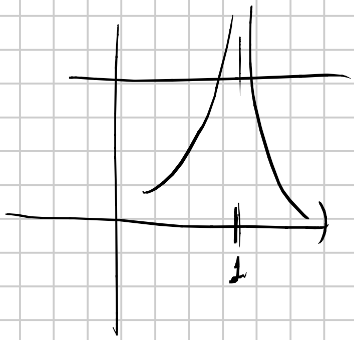


$c = 1$

es. $\lim_{x \to 1} \frac{1}{(x-1)^2} = +\infty$

? $\Downarrow$ $\forall K > 0 \ \exists \delta : \frac{1}{(x-1)^2} > K$

$\forall x : |x - 1| < \delta$



$(x-1)^2 < \frac{1}{K}$

$|x-1| < \frac{1}{\sqrt{K}} = \delta$

Esercizi su successioni

criterio del rapporto

$\frac{b^n}{n!} \to 0 \qquad \frac{n!}{n^n} \to 0$

$\lim_n \frac{n^\alpha}{b^n} \to 0$

$\frac{\log n}{n}$

con il rapporto $\frac{a_{n+1}}{a_n} \to 1$

criterio del rapporto

$\boxed{b > 1}$

$a_n = \frac{n^\alpha}{b^n}$

$\frac{a_{n+1}}{a_n} = \frac{(n+1)^\alpha}{b^{n+1}} \cdot \frac{\cancel{b^n}}{n^\alpha} =$

$\Rightarrow \lim a_n = 0$

$= \underbrace{\left(\frac{n+1}{n}\right)^\alpha}_{\to 1} \frac{1}{b} \to \frac{1}{b} < 1$

Es. $\lim_n \dfrac{a^{n+1} + n^2 + (-1)^{n+1}}{\pi^n - 2n^3 - 2\,\mathrm{sen}\, n}$ $\quad a > 0$

1) $a > 1$

$$= \lim_n \frac{a^{n+1}\left(1 + \overbrace{\frac{n^2}{a^{n+1}}}^{\to 0} + \overbrace{\frac{(-1)^{n+1}}{a^{n+1}}}^{\to 0}\right)}{\pi^n\left(1 - \underbrace{\frac{2n^3}{\pi^n}}_{\to 0} - \underbrace{\frac{2\,\mathrm{sen}\, n}{\pi^n}}_{\to 0}\right)} =$$

infinitesima per limitata

$$= \lim_n a\left(\frac{a}{\pi}\right)^n \frac{(1 + \dots)}{(1 + \dots)} = \begin{cases} +\infty & a > \pi \\ \pi & a = \pi \\ 0 & 1 < a < \pi \end{cases}$$

$$q^n \to \begin{cases} +\infty & q > 1 \\ 1 & q = 1 \\ 0 & 0 < q < 1 \end{cases} \qquad q = \frac{a}{\pi} > 1 \qquad a > \pi$$

2) $a \leq 1$

$$\lim_n \frac{a^{n+1} + n^2 + (-1)^{n+1}}{\pi^n - 2n^3 - 2\,\mathrm{sen}\, n} = \lim_n \underbrace{\frac{n^2}{\pi^n}}_{\to 0} \frac{\left(\overbrace{\frac{a^{n+1}}{n^2}}^{0} + 1 + \overbrace{\frac{(-1)^{n+1}}{n^2}}^{0}\right)}{\left(1 - \underbrace{\frac{2n^3}{\pi^n}}_{0} - \underbrace{\frac{2\,\mathrm{sen}\, n}{\pi^n}}_{0}\right)}$$

$$= 0$$

$$\lim_n (\quad) = \begin{cases} +\infty & a > \pi \\ \pi & a = \pi \\ 0 & a < \pi \\ 0 & a \leq 1 \end{cases}$$

---

es. $\lim_n \dfrac{n^{\sqrt{n}}}{(\sqrt{n})^n}$

$$n^{\sqrt{n}} \quad \infty^{\infty}$$
$$\to +\infty$$

$$n^{\sqrt{n}} = e^{\log\left(n^{\sqrt{n}}\right)} = e^{\overset{(+\infty)(+\infty)}{\sqrt{n}\,\log n}} \to +\infty$$

$$\lim_n \frac{n^{\sqrt{n}}}{(\sqrt{n})^n} = \lim_n \frac{n^{\sqrt{n}}}{(\sqrt{n})^{\sqrt{n}\sqrt{n}}} = \lim_n \left(\frac{n}{(\sqrt{n})^{\sqrt{n}}}\right)^{\sqrt{n}}$$

$$= \lim_n e^{\sqrt{n}\log\left(\frac{n}{(\sqrt{n})^{\sqrt{n}}}\right)} =$$

$$= \lim_n e^{\sqrt{n}\log\left(\frac{(\sqrt{n})^2}{(\sqrt{n})^{\sqrt{n}}}\right)} =$$

$$= \lim_n e^{\overset{\nearrow +\infty}{\sqrt{n}}\underbrace{\log\left(\frac{1}{(\sqrt{n})^{\sqrt{n}-2}}\right)}_{\to -\infty}} \to 0$$

$$e^{-\infty}$$

<u>Es.</u> $$\lim_n \frac{2^n}{e^{n^2}} = \lim_n \frac{2^n}{e^{nn}} =$$

$$= \lim_n \frac{2^n}{(e^n)^n} = \lim_n \left(\frac{2}{e^n}\right)^n$$

$$\left(\frac{2}{e^n}\right)^n = e^{\overset{\nearrow +\infty}{n}\underbrace{\log\left(\frac{2}{e^n}\right)}_{\to -\infty}} \to 0$$

2^n è infinito di ordine inferiore rispetto a e^{n^2}

es.
$$\lim_{n} \frac{4e^{n} - \cosh^2 n}{\cos n + e^{n^2}}$$

Numeratore
$\downarrow$

$$\cosh n = \frac{e^{n} + e^{-n}}{2}$$

$$4e^{n} - \left(\frac{e^{n} + e^{-n}}{2}\right)^2 =$$

$$= 4e^{n} - \left(\frac{e^{2n} + 2 + e^{-2n}}{4}\right) =$$

$$= \frac{16e^{n} - e^{2n} - 2 - e^{-2n}}{4} =$$

$$= \underbrace{-e^{2n}}_{\to -\infty}\left(-\underbrace{\frac{16}{e^{n}}}_{\to 0} + 1 + \underbrace{\frac{2}{e^{2n}}}_{\to 0} + \underbrace{e^{-4n}}_{\to 0}\right) \to -\infty$$

$$\lim_{n} \frac{4e^{n} - \cosh^2 n}{\cos n + e^{n^2}} = \lim_{n} \frac{-e^{2n}(1 + \cdots)}{4 e^{n^2}\Big(\underbrace{\frac{\cos n}{e^{n^2}}}_{\to 0} + 1\Big)} = 0$$

$$\frac{e^{2n}}{e^{n^2}} = \left(\frac{e^{2}}{e^{n}}\right)^{n} \longrightarrow 0$$

Fare voi come es. di prima