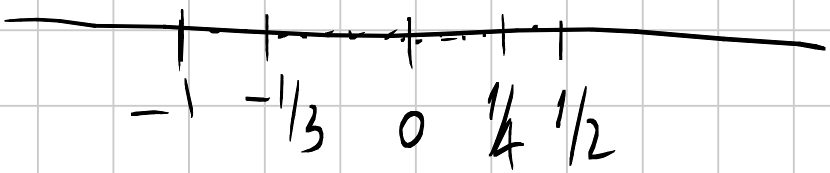
31013

Esercizio su max, min, sup, inf

$$A = \left\{ x \in \mathbb{R},\ x = \frac{(-1)^n}{n},\ n \in \mathbb{N},\ n \neq 0 \right\}$$

$\max A = 1/2$ fate voi



$\min A = -1$

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n.b. non confondere con i limiti

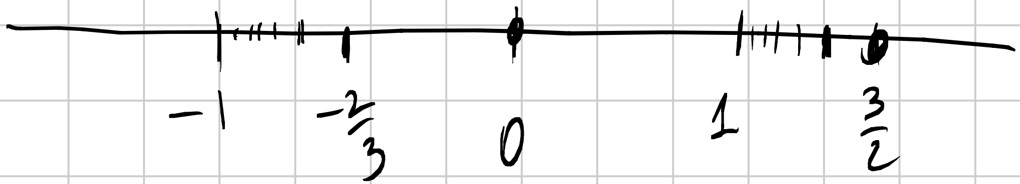
1) $-1 \in A$ si prende $n=1 \Rightarrow x=-1$

2) ? $x \geq -1$ $\forall x \in A$

? $\dfrac{(-1)^n}{n} \geq -1$

- n pari: $\dfrac{1}{n} \geq -1$ sì!
- n dispari: $-\dfrac{1}{n} \geq -1$ $\quad \dfrac{1}{n} \leq 1$ sì!

Es. $E = \left\{ x \in \mathbb{R} : \ x = \dfrac{1}{n} + (-1)^n \right\}$



n pari

$x = 1 + \dfrac{1}{n}$

$n=2 \qquad x = 3/2$

n dispari $\quad x = +\frac{1}{n} - 1$

$n = 1 \quad x = 0$

$n = 3 \quad x = -\frac{2}{3}$

$n = 5$

$\max E = \frac{3}{2}$ (fate voi!)

$E = \left\{ x = \frac{1}{n} + (-1)^n \right\}$

$\inf E = -1$

1) ?. $x \geq -1 \quad \forall x \in E$

?. $\frac{1}{n} + (-1)^n \geq -1$

- n pari: $\frac{1}{n} + 1 \geq -1$ sì!
- n dispari: $\frac{1}{n} \cancel{-1} \geq \cancel{-1}$ sì!

2) $\forall \varepsilon > 0 \quad \exists\, x \in E$ t.c. $\quad x < -1 + \varepsilon$

$\forall \varepsilon > 0 \quad \exists\, n \in \mathbb{N}$ t.c. $\boxed{\frac{1}{n} + (-1)^n < -1 + \varepsilon}$

prendo n dispari $\quad \frac{1}{n} \cancel{-1} < \cancel{-1} + \varepsilon$

$n > \frac{1}{\varepsilon}$ $\qquad$ n dispari, $n > \frac{1}{\varepsilon}$

$\varepsilon = \frac{1}{100} \qquad n > 100$

$\varepsilon = \frac{1}{10\,000} \qquad n > 10.000$

Valore assoluto (o modulo)

$a \in \mathbb{R}$

$$|a| := \begin{cases} a & \text{se } a \geq 0 \\ -a & \text{se } a < 0 \end{cases}$$

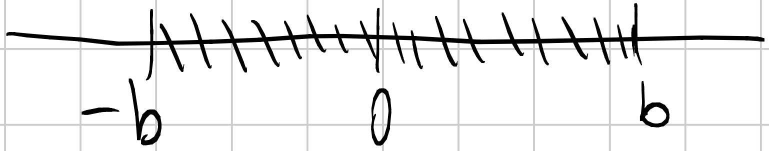
$|a| \geq 0$

$|a| = 0 \iff a = 0$

$-|a| \leq a \leq |a|$

1) $|x| \leq b$

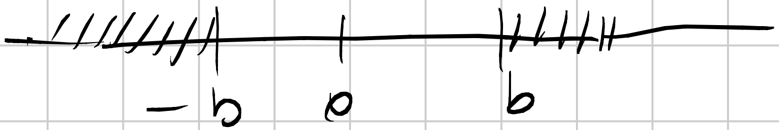
- $b < 0$ mai verificata
- $b \geq 0$
  - $x \geq 0$: $\boxed{x \leq b}$
  - $x < 0$: $-x \leq b$, $\boxed{x \geq -b}$



$$|x| \leq \underset{b \geq 0}{b} \iff -b \leq x \leq b \iff \{x \in \mathbb{R} : x \in [-b, b]\}$$

2) $|x| \geq b$

- $b < 0 \Rightarrow \forall x \in \mathbb{R}$
- $b \geq 0$
  - $x \geq 0$: $x \geq b$
  - $x < 0$: $-x \geq b$, $x \leq -b$



$$|x| \geq \underset{b \geq 0}{b} \iff x \geq b \text{ o } x \leq -b \iff$$

$$\Leftrightarrow \left\{ x \in \mathbb{R} \quad x \in (-\infty, -b] \cup [b, +\infty) \right\}$$

Disuguaglianza Triangolare

$$|x+y| \leq |x| + |y|$$

Dim.

$$-|x| \leq x \leq |x|$$

$$-|y| \leq y \leq |y|$$

$$\Rightarrow \underbrace{-|x|-|y|}_{-b} \leq \underbrace{x+y}_{z} \leq \underbrace{|x|+|y|}_{b}$$

$$\hookrightarrow |x+y| \leq |x|+|y| \quad \#$$

<u>In generale</u> $\left|\sum_{k=1}^{n} x_k\right| \leq \sum_{k=1}^{n} |x_k|$

— . —

<u>es.</u> $|x+3| < \alpha$, $\alpha \in \mathbb{R}$ α parametro

$\alpha \leq 0$ $\nexists x$

$\alpha > 0$ $-\alpha < x+3 < \alpha \Rightarrow -\alpha-3 < x < \alpha-3$

<u>es.</u> $|x+3| > \alpha$, $\alpha \in \mathbb{R}$

$\alpha \leq 0 \qquad \forall x$

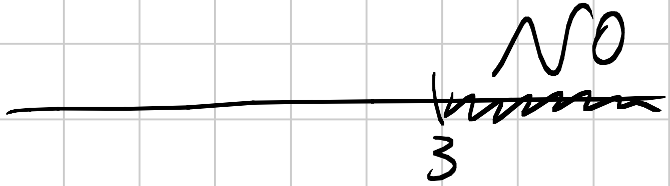
$\alpha > 0 \qquad x+3 > \alpha \quad \circ \quad x+3 < -\alpha$

$\qquad\qquad x > \alpha - 3 \quad \circ \quad x < -\alpha - 3$

<u>es.</u> $|f(x)| \leq g(x)$

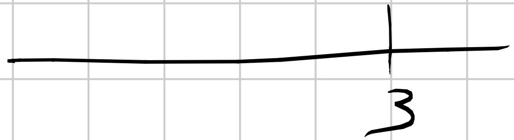
- $g(x) < 0 \Rightarrow$ mai verificata
- $g(x) \geq 0 \qquad -g(x) \leq f(x) \leq g(x)$

<u>es.</u> $|6x^2 - 13x - 15| < 3 - x$



1) $3 - x < 0$ mai $x > 3$

2) $3 - x \geq 0 \longrightarrow \boxed{x \leq 3}$



$$\begin{cases} \text{a)}\ 6x^2 - 13x - 15 < 3 - x \\ \text{b)}\ 6x^2 - 13x - 15 > -(3 - x) \end{cases}$$

a) $6x^2 - 12x - 18 < 0$ $\qquad x^2 - 2x - 3 < 0$

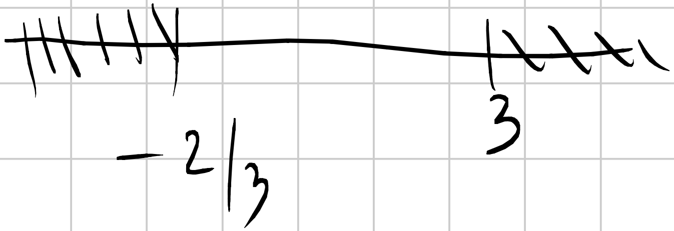
$$x = 1 \pm \sqrt{1+3} = 1 \pm 2 \begin{cases} 3 \\ -1 \end{cases}$$

$-1 < x < 3$

b) $6x^2 - 14x - 12 > 0$

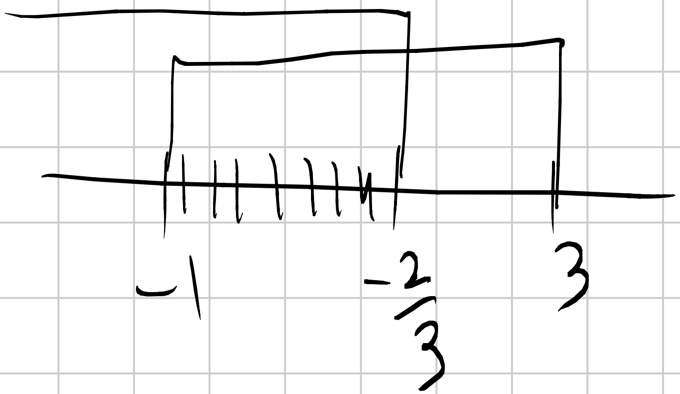
$3x^2 - 7x - 6 > 0$ ..... $x = 3$

$x = -2/3$



$x > 3$

$x < -2/3$



$-1 < x < -\frac{2}{3}$

Principio d'induzione

Si applica ai teoremi del tipo

$\forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow$ vale la proprietà $p(n)$

es. $\forall n \in \mathbb{N}\quad \underbrace{n \text{ pari} \Rightarrow n^2 \text{ è pari}}_{p(n)}$

es. $\forall n \in \mathbb{N}\quad \underbrace{2^n > n}_{p(n)}$

es $1+2+3+\cdots+n = \sum_{k=1}^{n} k = \dfrac{n(n+1)}{2}$

<u>Principio d'induzione</u> Sia $n_0 \in \mathbb{N}$ e $p(n)$ proprietà.

se si dimostra che

1) $p(n)$ è vera per $n = n_0$ (primo passo)

2) suppongo che $p(n)$ sia vera e dimostro che $p(n+1)$ è vera.

allora $\Rightarrow$ $p(n)$ è vera $\forall n \geq n_0$

$n_0 = 0$

1) $p(0)$ è vera 2) $p(1)$ è vera 2) $p(2)$ è vera

2) $\cancel{Hp}$ $p(n)$ è vera $\Rightarrow$ $\overline{Ts}$ $p(n+1)$ è vera

(ipotesi di induzione)

es. $$\sum_{k=1}^{n} k = 1+2+\dots+n \overset{?}{=} \frac{n\cdot(n+1)}{2} = p(n)$$

$\forall n \geq 1$

1) $p(n)$ è vera per $n=1$

$$1 = 1 \cdot \frac{2}{2} = 1$$ $p(1)$ è vero!

2) **Hp.** $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$ $p(n)$ è vero

Ts. $$\sum_{k=1}^{n+1} k \overset{?}{=} \frac{(n+1)\cdot(n+2)}{2}$$ $p(n+1)$ è vero

$$\underbrace{1+2+\cdots n}_{} + \boxed{n+1} = \frac{n(n+1)}{2} + n+1 =$$

$$= \frac{n(n+1)}{2}$$ per ipotesi d'induzione

$$= \frac{n(n+1) + 2n + 2}{2} = \frac{n(n+1) + 2(n+1)}{2} =$$

$$= \frac{(n+1)(n+2)}{2}$$

Es. p. 34 (Bramanti Pagani Salsa)

Per casa $\forall n \in \mathbb{N}$ $n^2 + n$ è pari

$\forall n \geq \underbrace{3}_{n_0}$ $2n + 1 < n^2$

Disuguaglianza di Bernoulli

$$\boxed{(1+x)^n \geq 1 + xn}, \quad \forall\, x \geq -1,\ x \in \mathbb{R}, \quad \forall\, n \in \mathbb{N}$$

$\hookrightarrow p(n)$

1) $n = 0$ $\qquad 1 \overset{?}{\geq} 1$ $\qquad$ vero!

2) Hp $\quad (1+x)^n \geq 1 + xn$

Ts. $\quad (1+x)^{n+1} \overset{?}{\geq} 1 + x(n+1)$

$$(1+x)^{n+1} = \underbrace{(1+x)^n}_{\geqslant 1+xn} \cdot (1+x) \geqslant (1+xn)(1+x) =$$

per ipotesi di induzione

$$= 1 + x + xn + x^2 n = 1 + x(n+1) + \underbrace{x^2 n}_{\geqslant 0}$$

$$\geqslant 1 + x(n+1)$$

ok!

Potenze e radici n-esime

$x \in \mathbb{R} \qquad x^n, \; n \in \mathbb{N}$

Teorema $y \in \mathbb{R}, \; y > 0, \; n \in \mathbb{N}, \; n \geq 1$

$\exists ! \; \underline{x > 0}, \; x \in \mathbb{R}$ t.c. $x^n = y$

Tale $x =: \sqrt[n]{y} = y^{1/n}$ (se $y = 0 \Rightarrow x = 0$)

oss. $x \overset{\text{è sempre}}{>} 0$

per def. $\sqrt{4} = 2$

$$\sqrt{x^2} = |x|$$

in generale $\sqrt[n]{y} = x$ è definita solo se $y \geq 0$

se n è dispari e $y < 0$ $\sqrt[n]{y} := -\sqrt[n]{-y}$