19 Novembre

Formule di Taylor e "o piccolo"

Def. $f(x) = o(g(x))$, $x \to x_0$ significa

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$$

es. $f \to 0$ $g \to 0$ $\frac{f}{g} \to 0 \iff f = o(g)$

es. $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

$1 - \cos x = o(x)$

$\cos x = 1 - o(x)$

$\cos x = 1 + o(x)$ $x \to 0$

non so cosa è ma so che va a zero più velocemente di x per $x \to 0$

oss. $o(c\,g(x)) = o(g(x))$

$-o(x) = o(x)$

$c\,o(g(x)) = o(g(x))$

Relazione tra "o piccolo" e asintotico

$$\boxed{f(x) \sim g(x) \quad x \to x_0 \iff f(x) = g(x) + o(g(x)) \quad x \to x_0}$$

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$$

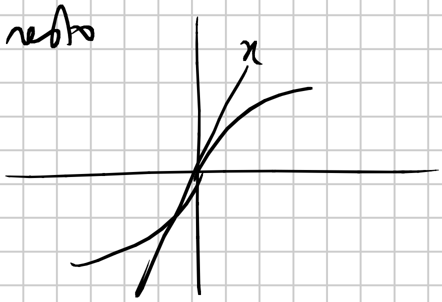
$\lim_{x \to 0} \frac{\operatorname{sen} x}{x} = 1$ $\quad \operatorname{sen} x \sim x$ $\quad x \to 0$

$$0 \leftarrow \frac{\operatorname{sen} x}{x} - 1 = \frac{\operatorname{sen} x - x}{x} \to 0 \iff$$

$\Leftrightarrow \quad \operatorname{sen} x - x = o(x) \quad x \to 0$ per def. di "o piccolo"

$$\boxed{\operatorname{sen} x = x + o(x) \quad x \to 0}$$ limite notevole

$$\operatorname{sen} x = \text{polinomio} + \text{resto}$$



es. $\displaystyle \lim_{x \to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$

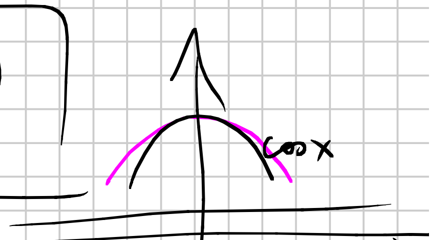
$$\lim_{x \to 0} \frac{1-\cos x}{\left(\frac{x^2}{2}\right)} = 1$$

$$\overbrace{1-\cos x}^{f} \sim \overbrace{\frac{x^2}{2}}^{g} \quad x \to 0$$

$$1-\cos x = \frac{x^2}{2} + o\left(\frac{x^2}{2}\right) \quad x \to 0$$

$$\cos x = 1 - \frac{x^2}{2} - o\left(\frac{x^2}{2}\right) \qquad -o\left(\frac{x^2}{2}\right) = o(x^2)$$

$$\boxed{\cos x = 1 - \frac{x^2}{2} + o(x^2) \quad x \to 0}$$



$$\frac{e^x - 1}{x} \to 1 \quad x \to 0 \qquad \Leftrightarrow \qquad \boxed{e^x = 1 + x + o(x)} \quad x \to 0$$

es. $\displaystyle f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$

$$\Downarrow$$

$$\lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right) = 0$$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{(x - x_0)} = 0$$

per def di "o piccolo"

$$f(x) - f(x_0) - f'(x_0)(x - x_0) = o(x - x_0) \quad x \to x_0$$

$$\boxed{f(x) = \underbrace{f(x_0) + f'(x_0)(x - x_0)}_{\text{polinomio di grado 1}} + o(x - x_0) \quad x \to x_0}$$

polinomio di grado 1 + Resto

retta tangente al grafico in x_0

Scopo $\quad f(x) =$ Polinomio di grado n $+ o(x^n)$ $\quad x \to 0$

Se prendo una $f(x)$ derivabile n volte esiste un polinomio che approssima la $f(x)$ in un intorno di un p.to x_0?

Caso particolare $\quad x_0 = 0 \quad x \to 0$

$x_0 = 0$ formula della derivata

$$f(x) = f(0) + f'(0)x + o(x) \quad x \to 0$$

Formula di Mc Laurin di grado 1

Formula di Mc Laurin di ordine n

$$f(x) = T_n(x) + o(x^n) \quad x \to 0$$

voglio trovare una approssimazione di questo tipo

Def.

f derivabile n volte in $x=0$

$$T_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k$$

è il polinomio di Mc Laurin di $f(x)$ in $x=0$

$f^{(0)}(0) \doteq f(0)$

Es. Mi calcolo il polinomio di McLaurin di $f(x) = e^x$ in $x=0$, di ordine n

$f(0) = e^0 = 1$

$f'(x) = e^x \qquad f'(0) = 1$

$f''(x) = e^x \qquad f''(0) = 1$

$f^{(n)}(x) = e^x \qquad f^{(n)}(0) = 1$

$$T_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!}$$

es. $f(x) = \operatorname{sen} x$ polinomio di Mc Laurin in $x=0$

$f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\operatorname{sen} x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

$f^{IV}(x) = \operatorname{sen} x \qquad f^{IV}(0) = 0$

$$T_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Analog. $T_n(x)$ di $\cos x$ in $x=0$

(fare voi)

$$f'(0) = f'''(0) = \dots = 0$$

$$T_n(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!}$$

$f(x)$ derivabile n volte

$$\Updownarrow$$

$$T_n(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + f'''(0)\frac{x^3}{3!} + \dots + f^{(n)}(0)\frac{x^n}{n!}$$

$$T_n(0) = f(0)$$

$$T_n'(0) = f'(0)$$

$$T_n''(0) = f''(0)$$

$$\vdots$$

$$T_n^{(n)}(0) = f^{(n)}(0)$$

$$T_n'(x) = f'(0) + f''(0)x + f'''(0)\frac{x^2}{2} + \dots$$

$\Rightarrow$ il polinomio $T_n(x)$ ha un contatto di "ordine n" con $f(x)$ in $x=0$

Teorema Formula di Mc Laurin di ordine n con resto di Peano.

f derivabile n volte in $x=0$. Allora

$$\underbrace{f(x)}_{\text{funzione da approssimare}} = \underbrace{T_n(x)}_{\text{polinomio che approssima}} + \underbrace{o(x^n)}_{\substack{\text{errore} \\ \text{(resto di Peano)}}}, \quad x \to 0$$

Dim. fino a $n=2$

? $n=1$ $\quad f(x) = f(0) + f'(0)x + o(x)$? $\quad x \to 0$

fatto prima con def. di derivata

$n=2$? $\quad f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + o(x^2)$? $\quad x \to 0$

devo dim. che

$$f(x) - f(0) - f'(0)x - f''(0)\frac{x^2}{2} = o(x^2) \quad x \to 0$$

per def di "o piccolo"

$$? \quad \frac{f(x) - f(0) - f'(0)x - f''(0)\frac{x^2}{2}}{x^2} \xrightarrow[x \to 0]{?} 0$$

Hôpital $\quad \dfrac{f'(x) - f'(0) - f''(0)x}{2x} =$

$$= \frac{1}{2}\left(\frac{f'(x) - f'(0)}{x} - f''(0)\frac{\cancel{x}}{\cancel{x}} \right) \xrightarrow[x \to 0]{} 0$$

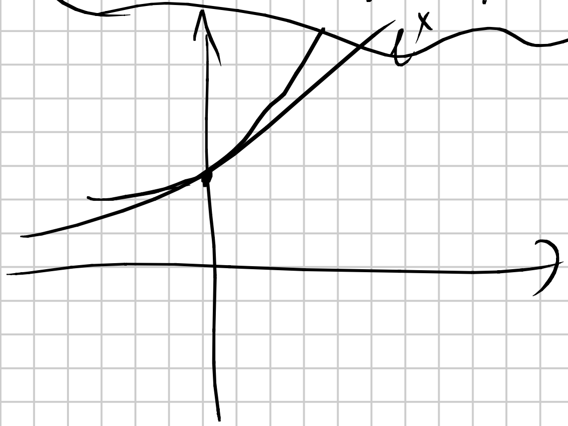
$$\frac{f'(x) - f'(0)}{x} \xrightarrow{x \to 0} f''(0)$$

Applicazione $\quad e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + o(x^n) \quad x \to 0$

$n=5$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + o(x^5)$$

$x \to 0$



$$\operatorname{sen} x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o\left(x^{2n+1}\right)$$

$$f(x) = \log(1+x) \qquad x \to 0$$

$f(0) = 0$ $\qquad f'(x) = \dfrac{1}{1+x}$

$f'(0) = 1$ $\qquad f''(x) = -\dfrac{1}{(1+x)^2}$

$f''(0) = -1$ $\qquad f'''(x) = \dfrac{2}{(1+x)^3}$

$f'''(0) = 2$

$f^{IV}(0) = -2 \cdot 3 = -3!$

$$f(x) = x - \frac{x^2}{2} + \frac{\cancel{2}}{\cancel{2} \cdot 3} x^3 + \frac{-\cancel{3!}\, x^4}{4\cancel{!}} + o(x^4)$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots (-1)^{n-1} \frac{x^n}{n} + o(x^n)$$

Analogamente si può dimostrare la formula di Taylor di ordine n con resto di Peano

$$f(x) = T_{n,x_0}(x) + o\left((x-x_0)^n\right)_{x \to x_0}$$

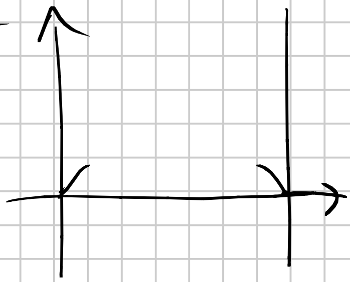
dove

$$T_{n,x_0}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

x_0 è un p.to generico.

<u>Esercizio su studio di funzione</u>

$$f(x) = 3^{-\frac{1}{|\sin x|}}$$



$$D = \{ x \neq k\pi, \ k \in \mathbb{Z} \}$$

f è periodica di periodo π $\qquad (0, \pi)$

$$f(x) > 0 \quad \forall x \in D$$

$$\lim_{x \to 0^+} f(x) = 0 = \lim_{x \to \pi^-} f(x)$$

si può prolungare per continuità

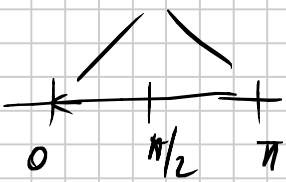
$$\tilde{f} = \begin{cases} 3^{-\frac{1}{|\sin x|}} & x \in (0, \pi) \\ 0 & x = 0, \ x = \pi \end{cases}$$

è continua.

$$f(x) = 3^{-\frac{1}{|\sin x|}} = 3^{-\frac{1}{\sin x}} = e^{-\frac{1}{\sin x}\log 3}$$

$$f'(x) = e^{-\frac{\log 3}{\sin x}} \cdot \frac{(+\log 3)}{\sin^2 x} \cos x = 0$$

$$\Leftrightarrow \cos x = 0 \Leftrightarrow x = \frac{\pi}{2}$$



$f' > 0 \quad (0, \pi/2) \qquad f' < 0 \quad \left(\frac{\pi}{2}, \pi\right)$

$$x = \frac{\pi}{2} \qquad f\left(\frac{\pi}{2}\right) = \frac{1}{3}$$

$$f'_+(0) = \lim_{x\to 0^+} f'(x)$$

$$\lim_{x\to 0^+} e^{-\frac{\log 3}{\sin x}} \frac{\log 3}{\sin^2 x} \cos x$$

$x \to 0$ $\quad$ $y = \sin x$, $y \to 0$

$$\lim_{x\to 0^+} \frac{e^{-\frac{\log 3}{\sin x}}}{\sin^2 x} = \lim_{y\to 0^+} \frac{e^{-\frac{\log 3}{y}}}{y^2}$$

$\frac{1}{y} = t$

$y \to 0^+$, $t \to +\infty$

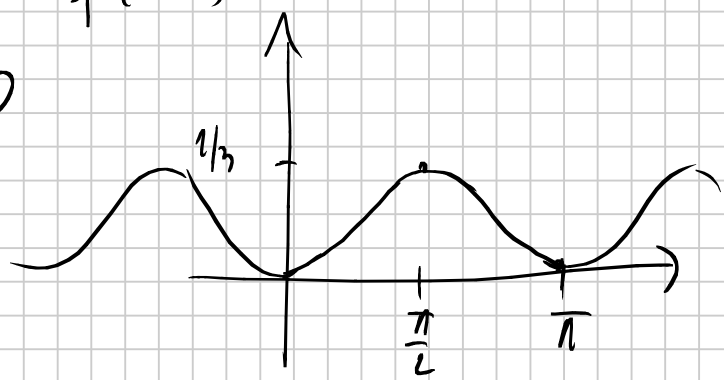
$$= \lim_{t\to +\infty} e^{-t\log 3} \cdot t^2 =$$

$$= \lim_{t\to +\infty} \frac{t^2}{\left(e^{\log 3}\right)^t} = 0$$

$$\lim_{x\to 0^+} f'(x) = 0 = f'_+(0)$$

$$\lim_{x\to \pi^-} f'(x) = 0$$

no f''



$$f(x) = \log\left(e^x + e^{-x}\right) + x$$

$D = \mathbb{R}$ $\qquad$ no simmetrie

$$\lim_{x\to +\infty} f(x) = +\infty$$

$$\lim_{x\to +\infty} \frac{f(x)}{x} = \lim_{x\to +\infty} \frac{1}{x} \log\left(e^x + e^{-x}\right) + 1$$

$$= \lim_{x\to +\infty} \frac{1}{x} \log\left(e^x\left(1 + e^{-2x}\right)\right) + 1 =$$

$$= \lim_{x\to+\infty} \frac{1}{x}\left(x + \lg(1+e^{-2x})\right) + 1$$

$$= \lim_{x\to+\infty} 2 + \underbrace{\frac{1}{x}\lg(1+e^{-2x})}_{\to 0} = 2$$

$m = 2$

$$\lim_{x\to+\infty} f(x) - 2x = \lim_{x\to+\infty} \log(e^{x}+e^{-x}) + x - 2x$$

$$= \lim_{x\to+\infty} \lg(e^{x}+e^{-x}) - x =$$

$$= \lim_{x\to\infty} \cancel{x} + \lg(1+e^{-2x}) - \cancel{x} = 0$$

$y = 2x$ è asintoto obliquo per $x\to+\infty$

$$\lim_{x\to-\infty} \log(e^{x}+e^{-x}) + x =$$

$$= \lim_{x\to-\infty} \lg\left(e^{-x}(e^{2x}+1)\right) + x$$

$$= \lim_{x\to-\infty} -\cancel{x} + \lg(1+e^{2x}) + \cancel{x} = 0$$

$\lim_{x\to-\infty} f(x) = 0$ asintoto orizzontale $x\to-\infty$

$f'(x)$

$f(x) = \log(e^{x}+e^{-x}) + x$

f è derivabile $\forall x\in\mathbb{R}$

$$f'(x) = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} + 1 = 0$$

$$\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} = -1$$

$$e^{x} - \cancel{e^{-x}} = -e^{x} - \cancel{e^{-x}}$$

$2e^x = 0$ no! $\quad f'(x) \neq 0 \quad \forall x \in \mathbb{R}$

$$f'(x) = \frac{e^x - \cancel{e^{-x}} + e^x + \cancel{e^{-x}}}{e^x + e^{-x}} = \frac{2e^x}{e^x + e^{-x}} > 0$$

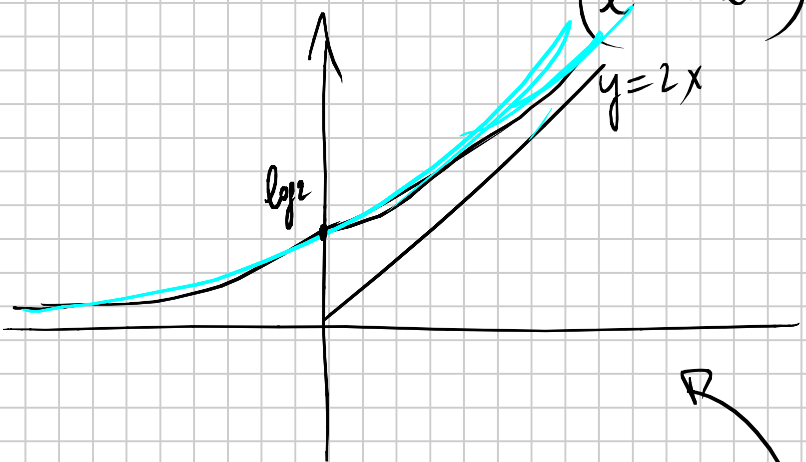
f è strett. crescente $\forall x \in \mathbb{R}$.

$$f''(x) = 2\,\frac{e^x(e^x + e^{-x}) - e^x(e^x - e^{-x})}{(e^x + e^{-x})^2} =$$

$$= \frac{2e^x}{(e^x + e^{-x})^2}\left(\cancel{e^x} + e^{-x} - \cancel{e^x} + e^{-x}\right) =$$

f strett. convessa

$$= \frac{2\,\underbrace{e^x e^{-x}}_{=1}}{(e^x + e^{-x})^2} > 0 \quad \forall x \in \mathbb{R}$$



$f(0) = \log(2)$

$$f(x) = \log(e^x + e^{-x}) + x > 0 \quad \forall x \in \mathbb{R}$$

$$\Downarrow$$

$$f(x) = \log(e^{-x}(e^{2x} + 1)) + x =$$

$$= \cancel{-x} + \log(1 + e^{2x}) + \cancel{x}$$

$$f(x) = \log(1 + e^{2x}) > 0$$

domani

$$f(x) = \arcsin\left(\frac{|x-1|}{x+3}\right)$$