20 Novembre 2013

domani

11–12.30 in Viale Morgagni (VM4)
studi di funzione

ricevimento studenti (mio)
è anticipato alle 8.30 (nel mio studio)

Ieri

Formula di <u>MacLaurin</u> di ordine <u>n</u> con resto di <u>Peano</u>

$f(x)$ derivabile n volte in $x=0$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \dots \frac{f^{(n)}(0)}{n!}x^n + o(x^n) \quad x \to 0$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + o(x^n)$$

$$\operatorname{sen} x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+1})$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n})$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots (-1)^{n+1}\frac{x^n}{n} + o(x^n)$$

<u>Proprietà del simbolo "o piccolo"</u>

$$f(x) = o(g(x)) \underset{x \to x_0}{} \iff \frac{f(x)}{g(x)} \to 0 \quad x \to x_0$$

In particolare se $f, g \to 0$

f è infinitesimo di ordine superiore a g $\quad x \to x_0$

$$f(x) = o(x^n) \underset{x \to 0}{} \longrightarrow \frac{f(x)}{x^n} \to 0 \quad x \to 0$$

$$\frac{o(x^n)}{x^n} \to 0 \quad x \to 0$$

es. $\quad x^2 = o(x) \quad x \to 0$

$$\frac{x^2}{\cancel{x}} \xrightarrow{?} 0 \quad x \to 0$$

$$x^5 = o(x)$$

- $o(x) + o(x) = o(x)$
- $o(x) - o(x) = o(x)$
- $o(kx) = o(x)$
- $x \to 0 \quad o(x) + o(x^2) = o(x)$
- $x\, o(x) = o(x^2)$ Dim ?

per def. di o deve essere $\dfrac{\cancel{x}\, o(x)}{x^{\cancel{2}}} \to 0 \quad x \to 0$

$$\frac{o(x)}{x} \to 0 \quad x \to 0 \quad \text{per definizione}$$

- $o(x) \cdot o(x) = o(x^2)$ verifico

$$\frac{o(x) \cdot o(x)}{x^2} \xrightarrow{?} 0$$

$$\underbrace{\frac{o(x)}{x}}_{\to 0} \cdot \underbrace{\frac{o(x)}{x}}_{\to 0} \to 0 \quad x \to 0$$

bisogna specificare a cosa tende x

se $x \to +\infty$ $\quad x = o(x^2)$

$$\frac{\cancel{x}}{x^{\cancel{2}}} \to 0$$

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es. 1

$$\lim_{x\to 0} \frac{x + \operatorname{sen} x + \lg(1+x)}{e^x - 1 + x^2}$$

$$\frac{x + \operatorname{sen} x + \lg(1+x)}{e^x - 1 + x^2} = \frac{x + x + o(x) + x + o(x)}{\cancel{1} + x + o(x) - \cancel{1} + \underbrace{x^2}_{o(x)}}$$

$$= \frac{3x + o(x)}{x + o(x)} = \frac{\cancel{x}\left(3 + \frac{o(x)}{x}\right)}{\cancel{x}\left(1 + \frac{o(x)}{x}\right)} \xrightarrow[x\to 0]{} 3$$

per def. di "o piccolo" $\quad \frac{o(x)}{x} \to 0 \quad x \to 0$

Strategia

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{x^\alpha + o(x^\alpha)}{x^\beta + o(x^\beta)} =$$

$$\frac{x^\alpha\left(1 + \frac{o(x^\alpha)}{x^\alpha}\right)}{x^\beta\left(1 + \frac{o(x^\beta)}{x^\beta}\right)}$$

$$= \lim_{x\to 0} \frac{x^\alpha}{x^\beta}$$

$\to 0$

oss. Gli sviluppi di Mac Laurin sono utili perché c'è una uguaglianza

$$f(x) = T_n(x) + o(x^n)$$

$$f(x) \sim g(x) \qquad f(x) + h(x) \not\sim g(x) + k(x)$$

$$h(x) \sim k(x)$$

es. Dallo sviluppo di $\operatorname{sen} x$, ci si ricava lo sviluppo di

- $\operatorname{sen}(\sqrt{x}) = \sqrt{x} - \frac{1}{6} x^{3/2} + \frac{1}{5!} x^{5/2} + o(x^{5/2})$

$(x \to 0, \sqrt{x} \to 0)$ $\sqrt{x} = y$ $\operatorname{sen} y$

- $x \to 0, \; x^2 \to 0$

$e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{3!} + o(x^6)$

- $e^{(x^2+1)} = e\left(e^{x^2}\right) = e\left(1 + x^2 + \frac{x^4}{2} + \dots o(x^6)\right)$

- $\cos(1 + x^3)$ $x \to 0$ $1 + x^3 \not\to 0$

non si può applicare lo sviluppo di McLaurin di $\cos x$ $x \to 0$

- sviluppo di McLaurin di $x \to 0$

$(\operatorname{sen} x)^2 = \left(x - \frac{x^3}{6} + o(x^3)\right)^2 =$

$= x^2 - \frac{x^4}{3} + \frac{x^6}{36} + o(x^6) +$

$+ 2x\, o(x^3) - \frac{x^3}{3} o(x^3) =$

$= x^2 - \frac{x^4}{3} + o(x^4)$

es. $\lim_{x \to 0} \frac{\cos 2\sqrt{x} - e^{-2x}}{\operatorname{sen} x - x}$

D. $\operatorname{sen} x - x = \underbrace{\cancel{x} + o(x)}_{\text{McLaurin al 1° ordine}} - \cancel{x} = o(x)$

$$\operatorname{sen} x - x = \cancel{x} - \frac{x^3}{6} + o(x^3) - \cancel{x} \Rightarrow$$

$$\operatorname{sen} x - x = -\frac{x^3}{6} + o(x^3)$$

è infinitesimo di ordine 3

$$\operatorname{sen} x - x = \underbrace{\cancel{x} - \frac{x^3}{6} + \frac{x^5}{5!}}_{\text{p. McLaurin fino a ordine 5}} + o(x^5) - \cancel{x}$$

$$= -\frac{x^3}{6} + \frac{x^5}{5!} + o(x^5)$$

$$= -\frac{x^3}{6} + o(x^3)$$

N. $\cos(2\sqrt{x}) - e^{-2x}$

$$\cos(2\sqrt{x}) = 1 - \frac{1}{2}(2\sqrt{x})^2 + o\left((\sqrt{x})^2\right)$$

$$= 1 - 2x + o(x)$$

$$e^{-2x} = 1 + (-2x) + o(x) =$$

$$= 1 - 2x + o(x)$$

$$\cos(2\sqrt{x}) - e^{-2x} = \cancel{1} - \cancel{2}x + o(x) - \cancel{1} + \cancel{2}x + o(x)$$

$$= o(x) = ?$$

ordine + alto

$$\cos(2\sqrt{x}) = 1 - \frac{1}{2}(2\sqrt{x})^2 + \frac{1}{4!}(2\sqrt{x})^4 + o\left((\sqrt{x})^4\right)$$

$$= 1 - 2x + \frac{1}{\cancel{2}\cdot 3\cdot\cancel{4}}\,{}^{2}\cancel{4}\cdot\cancel{4}\,x^2 + o(x^2)$$

$$= 1 - 2x + \frac{2}{3}x^2 + o(x^2)$$

$$e^{-2x} = 1 + (-2x) + \frac{(-2x)^2}{2} + o(x^2)$$

$$= 1 - 2x + 2x^2 + o(x^2)$$

$$\cos 2\sqrt{x} - e^{-2x} = \cancel{1} - \cancel{2x} + \frac{2}{3}x^2 + o(x^2) -$$

$$- \cancel{1} + \cancel{2x} - 2x^2 + o(x^2) =$$

$$= -\frac{4}{3}x^2 + o(x^2)$$

$$\cos 2\sqrt{x} - e^{-2x} = -\frac{4}{3}x^2 + o(x^2)$$

$\downarrow$

infuleremo di ordine 2

$$\lim_{x \to 0^+} \frac{\cos 2\sqrt{x} - e^{-2x}}{\operatorname{sen} x - x} = \lim_{x \to 0} \frac{-\frac{4}{3}x^2 + o(x^2)}{-\frac{1}{6}x^3 + o(x^3)}$$

$$= \lim_{x \to 0^+} \frac{+\frac{4}{3}\cancel{x^2}}{+\frac{1}{6}x^{\cancel{3}}} = +\infty$$

<u>oss.</u> se si ha $f(x) + g(x)$ trovare polinomi di Mc Laurin per $f(x)$ e $g(x)$ dello stesso grado.

$$\lim_{x \to 0} \frac{\operatorname{sen} x - \operatorname{senh} x}{e^x - x - \cosh x} =$$

<u>N.</u> $\operatorname{sen} x - \operatorname{senh} x = \cancel{x} - \frac{x^3}{6} + o(x^3) - \cancel{x} - \frac{x^3}{6}$

$+ o(x^3) = -\frac{x^3}{3} + o(x^3)$ infinitesimo di ordine 3

$\underline{D.}$ $e^x - x - \cosh x = \cancel{1} + \cancel{x} + \cancel{\frac{x^2}{2}} + o(x^2)$
$- \cancel{x} - \cancel{1} - \cancel{\frac{x^2}{2}} + o(x^2) = o(x^2)$?

$e^x - x - \cosh x = \cancel{1} + \cancel{x} + \cancel{\frac{x^2}{2}} + \frac{x^3}{6} + o(x^3) - \cancel{x}$
$- \left(\cancel{1} + \cancel{\frac{x^2}{2}} + \frac{x^4}{4!} + o(x^4)\right) =$

$x^4 + o(x^3)$

$= \frac{x^3}{6} + o(x^3)$ infinitesimo di ordine 3

$$\lim_{x\to 0} \frac{\operatorname{sen} x - \operatorname{senh} x}{e^x - x - \cosh x} = \lim_{x\to 0} \frac{-\frac{x^3}{3} + o(x^3)}{\frac{x^3}{6} + o(x^3)} =$$

$$= \lim_{x\to 0} \frac{-\frac{\cancel{x^3}}{3}}{\frac{\cancel{x^3}}{6}} = -2$$

$$\lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha$$
$$\Updownarrow$$
$$(1+x)^\alpha - 1 = \alpha x + o(x)$$
$$(1+x)^\alpha = 1 + \alpha x + o(x) \quad x\to 0$$

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parametro $a \in \mathbb{R}$

$$\lim_{x\to 0^+} \frac{27x^5 + \log(1+x^7)}{\sqrt{1+x^8} - 1 + a \operatorname{sen}^5 x} =$$

$\underline{N.}$ $27x^5 + x^7 + o(x^7) = 27x^5 + o(x^5)$

D. $\sqrt{1+x^8} = 1 + \frac{1}{2}x^8 + o(x^8)$

$$\left((1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + o(x^2) \right)$$

$$(\operatorname{sen} x)^5 = \left(x - \frac{x^3}{6} + o(x^3)\right)^5 =$$

$$= x^5 + o(x^5)$$

D. $\sqrt{1+x^8} - 1 + a \operatorname{sen}^5 x = \cancel{1} + \frac{1}{2}x^8 + o(x^8)$

$- \cancel{1} + a x^5 + a\, o(x^5)$

$= a x^5 + o(x^5)$

$$\lim_{x\to 0^+} \frac{27x^5 + \log(1+x^7)}{\sqrt{1+x^8} - 1 + a \operatorname{sen}^5 x} = \lim_{x\to 0^+} \frac{27x^5 + o(x^5)}{a x^5 + o(x^5)}$$

se $a \neq 0$ $= \lim_{x\to 0^+} \frac{27\cancel{x^5}}{a\cancel{x^5}} = \frac{27}{a}$

se $a = 0$ $\lim_{x\to 0^+} \frac{27x^5 + o(x^5)}{o(x^5)} = +\infty$

es. $\lim_{x\to 0^+} \frac{27x^5 + \log(1+e^{-1/x})}{x^5}$

$\log(1 + e^{-1/x})$ $\xrightarrow[x\to 0^+]{} 0$ $= e^{-1/x} + o(e^{-1/x})$

$\hookrightarrow y$

N. $27x^5 + e^{-\frac{1}{x}} + o(e^{-1/x})$

$e^{-\frac{1}{x}}$ è infinitesima superiore rispetto a qualsiasi potenza di x.

Infatti

? $e^{-\frac{1}{x}} = o(x^5)$ $\quad x \to 0^+$

$\frac{e^{-\frac{1}{x}}}{x^5} \xrightarrow{?} 0$ $\quad \frac{1}{x} = y$ $\quad x \to 0^+$, $y \to +\infty$

$\longrightarrow e^{-y} \cdot y^5 = \frac{y^5}{e^y} \xrightarrow[y \to +\infty]{} 0$

vero!

<u>in generale</u>

$$\boxed{e^{-1/x} = o(x^\alpha) \quad x \to 0^+, \ \forall \alpha > 0}$$

$$\boxed{e^{-\frac{1}{x^2}} = o(x^\alpha) \quad x \to 0^+, \ \forall \alpha > 0}$$

$$\lim_{x \to 0^+} \frac{27x^5 + e^{-1/x} + o(e^{-1/x})}{x^5} =$$

$$= \lim_{x \to 0^+} \frac{27x^5 + o(x^5)}{x^5} =$$

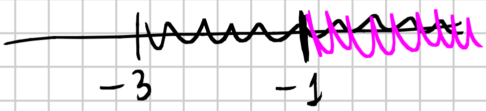
$$= 27$$

Es. studio di funzione

$$f(x) = \arcsin\left(\frac{|x-1|}{x+3}\right)$$

$$D = \left\{ x \neq -3, \quad \left|\frac{x-1}{x+3}\right| \leq 1 \right\}$$

$$\begin{cases} 1) \quad \dfrac{x-1}{x+3} \leq 1 \\ 2) \quad \dfrac{x-1}{x+3} \geq -1 \end{cases}$$



1) $\frac{x-1}{x+3} - 1 \leq 0 \qquad \frac{\cancel{x}-1-\cancel{x}-3}{x+3} \leq 0$

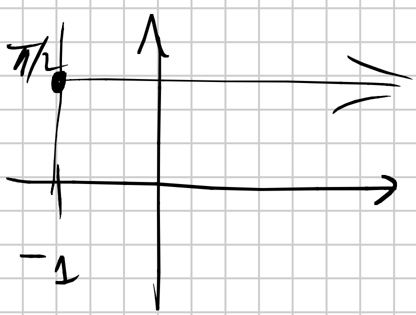
$-\frac{4}{x+3} \leq 0 \iff x+3 > 0 \qquad x > -3$

2) $\frac{x-1}{x+3} + 1 \geq 0 \qquad \frac{x-1+x+3}{x+3} \geq 0 \qquad \frac{2x+2}{x+3} \geq 0$

$\frac{2(x+1)}{x+3} \geq 0 \Rightarrow x+1 \geq 0 \qquad x \geq -1$

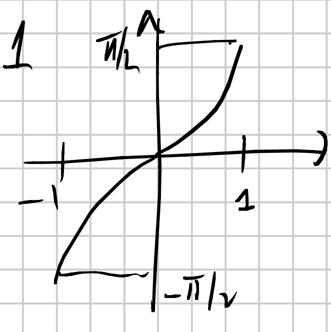
tenendo conto di 1)

$D = \{ x \in \mathbb{R}, \ x \geq -1 \}$



no simmetrie

$$\lim_{x \to +\infty} \arcsin\left(\frac{x-1}{x+3}\right) = \arcsin 1 = \frac{\pi}{2}$$



$$f(-1) = \arcsin(1) = \frac{\pi}{2}$$

$$f(x)=\begin{cases}\arcsen\left(\frac{x-1}{x+3}\right) & \text{se } x\geq 1\\ \arccos\left(\frac{1-x}{x+3}\right) & \text{se } x<1\end{cases}$$

$x\geq 1$

$$f'(x)=\frac{1}{\sqrt{1-\left(\frac{x-1}{x+3}\right)^2}}\cdot\frac{(\cancel{x}+3)-(\cancel{x}-1)}{(x+3)^2}=$$

$$=\frac{4}{\sqrt{\frac{(x+3)^2-(x-1)^2}{(x+3)^2}}\quad(x+3)^2}\qquad x\geq -1$$

$$=\frac{4\ \cancel{(x+3)}}{\sqrt{\cancel{x^2}+9+6x-\cancel{x^2}-1+2x}\quad(x+3)^{\cancel{2}}}$$

$$=\frac{4}{(x+3)\sqrt{8x+8}}=\frac{\cancel{4}^{\,2}}{(x+3)\cancel{2}\sqrt{2}\sqrt{x+1}}=f'(x)\qquad x\geq 1$$

$f'(x)>0\quad\forall\, x\geq 1$

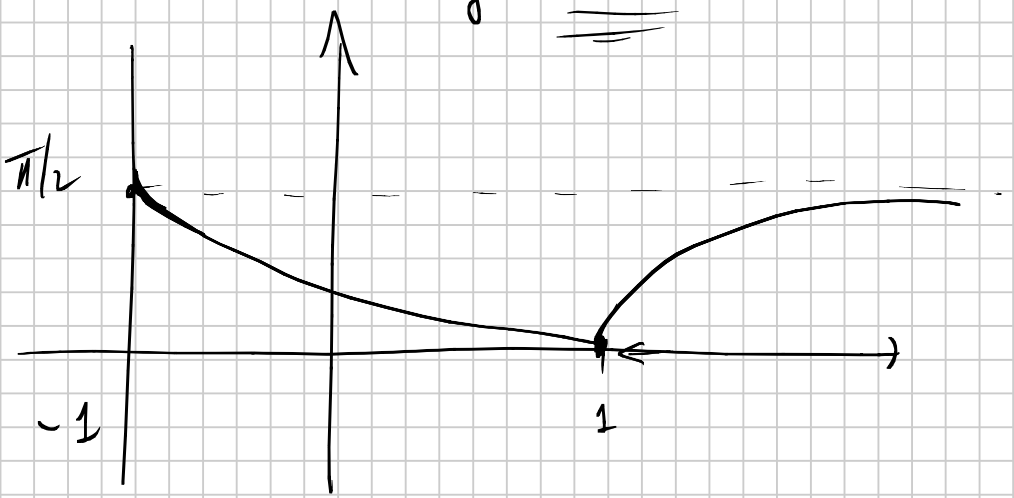
f è strett. crescente per ogni $x\geq 1$

$x\geq 1$
$$f''(x)=\frac{2}{\sqrt{2}}(-1)\frac{\left(\sqrt{x+1}+(x+3)\cdot\frac{1}{2\sqrt{x+1}}\right)}{\left((x+3)\sqrt{x+1}\right)^2}$$

$$=-\frac{2}{\sqrt{2}}\ \frac{2(x+1)+(x+3)}{(\quad)^2\ 2\sqrt{x+1}}=$$

$$=-\frac{2}{\sqrt{2}}\ \frac{3x+5}{(\quad)^2\,2\sqrt{x+1}}<0\qquad x\geq 1$$

f è strett. concava per $x \geq 1$



$f(1) = 0$

$f'_{+}(1) = \frac{1}{8}$

$x < 1 \qquad f(x) = \arcsin\left(\frac{1-x}{x+3}\right) =$

$\arcsin(-y) = -\arcsin y$

$= -\underbrace{\arcsin\left(\frac{x-1}{x+3}\right)}_{f(x) \quad x \geq 1}$

$x < 1$

$f'(x) = -\frac{2}{\sqrt{2}\,(x+3)\sqrt{1+x}} < 0$ per $x < 1$

cioè f è strett decrescente

$f''(x) = -\left(\text{la derivata } 2^a \text{ trovata per } x > 1\right) > 0$

$x < 1 \qquad f$ è strett. convessa

$f'_{-}(1) = -1/8$

$x = 1$ la f non è derivabile

$\Rightarrow x=1$ p.to angoloso

Per caso

$$f'_+(-1) = -\infty$$

come si attacca in $x=-1$

cioè si attacca con la tangente verticale