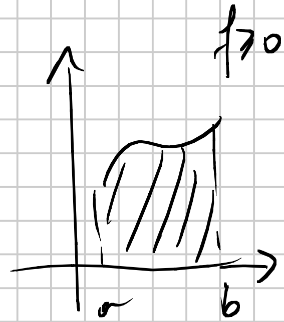
28 Novembre 2013

• $\int_a^b f(x)\,dx$ = NUMERO REALE

integrale definito



• $\int f(x)\,dx = \{\text{tutte le primitive di } f\}$

cioè

$= G(x) + K$

dove $G'(x) = f(x)$

• $F(x) = \int_{x_0}^{x} f(t)\,dt$

funzione integrale di f con $x_0 \in [a,b]$ fissato.

Teor fond. calcolo integrale: se f è continua in $[a,b]$

F è una primitiva di f e

• → $\int_a^b f(x)\,dx = G(b) - G(a)$ dove G è una primitiva di f

• → $\underbrace{\int f(x)\,dx}_{\text{tutte le primitive}} = \underbrace{\int_{x_0}^{x} f(t)\,dt}_{\text{una primitiva di } f} + K$

$$\int x^{\alpha}\, dx = \frac{x^{\alpha+1}}{\alpha+1} + K$$

$\alpha \neq -1$

$$\int \frac{1}{x}\, dx = \log|x| + K$$

$$\int \operatorname{sen} x\, dx = -\cos x + K$$

$$\int \cos x\, dx = \operatorname{sen} x + K$$

$$\int \frac{1}{1+x^2}\, dx = \operatorname{arctg} x + K$$

$$\int \operatorname{senh} x\, dx = \cosh x$$

$$f(x) \longrightarrow f'(x)$$

$$f(x) \longrightarrow \int f(x)\, dx$$

es. $f(x) = \dfrac{\operatorname{sen} x}{x}$ è elementare

$\longrightarrow f'(x) =$ viene una funzione elementare

$\longrightarrow \displaystyle\int \frac{\operatorname{sen} x}{x}\, dx$ non si riesce a esprimere attraverso funzioni elementari

$f(x) = e^{-x^2}$ $\qquad \displaystyle\int e^{-x^2}\, dx = ?$

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proprietà $\displaystyle\int f'(x)\, dx = f(x) + K$

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

Integrazione per sostituzione

Teo. $G(t)$ primitiva di $f(t)$ in I $\left(G'(t) = f(t) \ \forall t \in I\right)$

Sia $t = \varphi(x)$ funzione derivabile con derivata continua, $x \in [a,b]$ t.c. $\varphi([a,b]) \subset I$. $I = [\alpha, \beta]$

$\varphi(a) = \alpha$
$\varphi(b) = \beta$

Allora

$$\int f(t)\,dt = \int f(\varphi(x))\,\varphi'(x)\,dx$$

e

$$\int_\alpha^\beta f(t)\,dt = \int_a^b f(\varphi(x))\,\varphi'(x)\,dx$$

Dim. $G(t)$ primitiva di $f(t)$ $\quad G(t) = \int f(t)\,dt$

$t = \varphi(x)$ $\quad G'(t) = f(t)$

$G(t) = G(\varphi(x))$

$$\frac{d}{dx} G(\varphi(x)) = \underbrace{G'(\varphi(x))}_{f(\varphi(x))} \cdot \varphi'(x) = f(\varphi(x))\varphi'(x)$$

$\downarrow$

significa che

$G(\varphi(x))$ è la primitiva di $f(\varphi(x))\varphi'(x)$

$$G(t) = G(\varphi(x)) = \int f(\varphi(x))\,\varphi'(x)\,dx$$

$$\parallel \int f(t)\,dt$$

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$$\int f(t)\,dt = \int f(\varphi(x))\,\varphi'(x)\,dx$$

$t = \varphi(x)$

$1\,dt = \varphi'(x)\,dx$

es. $\int \operatorname{sen}(3x)\,dx =$

$\boxed{3x = t}$

$3\,dx = 1\,dt$

$3dx = dt$

$dx = \frac{1}{3}dt$

$= \int \operatorname{sen} t \cdot \frac{1}{3}dt =$

$= \frac{1}{3}\int \operatorname{sen} t\,dt = \frac{1}{3}\left(-\cos t + K\right) = \frac{1}{3}\left(-\cos(3x) + K\right)$

es. $\int_0^3 \frac{1}{2x+1}\,dx =$

$2x+1 = t$

$2dx = dt$

$dx = \frac{1}{2}dt$

$= \int_1^7 \frac{1}{t}\,\frac{1}{2}dt =$

$x=0 \Rightarrow t=1$

$x=3 \Rightarrow t=7$

$= \frac{1}{2}\log|t|\Big|_1^7 = \frac{1}{2}\left(\log 7 - \cancel{\log 1}\right)$

$= \frac{1}{2}\log 7$

es. $\int \operatorname{tg} x\,dx = \int \frac{\operatorname{sen} x}{\cos x}\,dx$

$\cos x = t$

$-\operatorname{sen} x\,dx = dt$

$\operatorname{sen} x\,dx = -dt$

$= \int \frac{1}{t}(-dt) =$

$$= -\int \frac{1}{t}\,dt = -\log|t| + K$$

$$= -\log|\cos x| + K$$

es. $$\int \frac{x}{x^2-1}\,dx =$$

$x^2 - 1 = t$

$2x\,dx = dt$

$$= \int \frac{1}{t}\,\frac{1}{2}\,dt = \frac{1}{2}\log|x^2-1| + K$$

es. $$\int_2^3 \frac{x}{\sqrt{x^2-1}}\,dx = \int_3^8 \frac{1}{2}\,\frac{1}{\sqrt{t}}\,dt =$$

$x^2 - 1 = t$

$2x\,dx = dt$

$x = 2 \Rightarrow t = 3$

$x = 3 \Rightarrow t = 8$

$$= \frac{1}{2}\int_3^8 t^{-1/2}\,dt =$$

$$= \frac{1}{\cancel{2}}\left(t^{1/2}\cdot\cancel{2}\right)\Big|_3^8$$

$$= \sqrt{8} - \sqrt{3}$$

si può anche fare così

$$\int_2^3 \frac{x}{\sqrt{x^2-1}}\,dx$$

$x^2 - 1 = t$

$$\int \frac{x}{\sqrt{x^2-1}}\,dx = t^{1/2} = \sqrt{x^2-1}$$

$$\int_2^3 \frac{x}{\sqrt{x^2-1}} = \sqrt{x^2-1}\Big|_2^3$$

es. $$\int \operatorname{sen}^5 x \underbrace{\cos x \, dx}_{dt}$$

$\operatorname{sen} x = t$

$\cos x dx = dt$

$$= \int t^5 \, dt = \frac{t^6}{6} + K = \frac{(\operatorname{sen} x)^6}{6} + K$$

$f(x) = t \qquad f'(x)dx = dt$

In generale

$$\boxed{\int \frac{f'(x)}{f(x)} dx = \int \frac{1}{t} dt = \log|f(x)| + K}$$

$\int \frac{1}{x^2+1} dx = \operatorname{arctg} x + K$

es $$\int \frac{1}{x^2+4} dx =$$

$$= \int \frac{1}{x^2+1+3} = \int \frac{1}{x^2+1+3} \quad ?$$

NO

$$\int \frac{1}{x^2+4} dx = \int \frac{1}{4\left(\frac{x^2}{4}+1\right)} dx =$$

$$= \frac{1}{4} \int \frac{1}{\left(\frac{x}{2}\right)^2 + 1} dx$$

$\frac{x}{2} = t$

$x = 2t$

$dx = 2dt$

$$= \frac{1}{\cancel{4}_2} \int \frac{1}{t^2+1} \cancel{2} dt =$$

$$= \frac{1}{2} \operatorname{arctg}(t) = \frac{1}{2} \operatorname{arctg}\left(\frac{x}{2}\right)$$

es. $\displaystyle\int \frac{1}{(3+5x)^6}\,dx =$

$3+5x=t$
$5\,dx=dt$

$$= \int \frac{1}{t^6}\,\frac{1}{5}\,dt = \frac{1}{5}\,\frac{1}{t^5}\left(-\frac{1}{5}\right)$$

$$= -\frac{1}{25}\,\frac{1}{(3+5x)^5} + K$$

es. $\displaystyle\int 3x e^{x^2}\,dx =$

$x^2=t$
$2x\,dx=dt$

$$= 3\int e^t\,\frac{1}{2}\,dt = \frac{3}{2}e^t = \frac{3}{2}e^{x^2}$$

$$\int e^{x^2}\,dx = \,?$$

Non è integrabile elementarmente

es. $\displaystyle\int \frac{\sqrt{x}}{2+\sqrt{x}}\,dx =$

$\sqrt{x}=t$
$x=t^2$
$dx=2t\,dt$

$$= \int \frac{t}{2+t}\,2t\,dt$$

$$= \int 2t\left(\frac{t+2-2}{2+t}\right)dt = \int 2t\left(1-\frac{2}{2+t}\right)dt$$

$$= \int\left(2t - \frac{4t}{2+t}\right)dt = \int 2t\,dt - \int \frac{4t}{2+t}\,dt$$

$$= \cancel{2}\frac{t^2}{\cancel{2}} - 4\int \frac{t+2-2}{2+t}\,dt =$$

$$= t^2 - 4\left(\int\left(1 - \frac{2}{2+t}\right)dt\right) =$$

$$= t^2 - 4\left(t - 2\int \frac{1}{2+t}\,dt\right) =$$

$2+t=y$
$dt=dy$

$$= t^2 - 4t + 8\int \frac{1}{y}\,dy =$$

$(t=\sqrt{x})$

$$= t^2 - 4t + 8\log|2+t| =$$

$$= x - 4\sqrt{x} + 8\log|2+\sqrt{x}|$$

• es. $\displaystyle\int \frac{1}{x\log x}\,dx =$

$\log x = t$
$\frac{1}{x}dx = dt$

$$= \int \frac{1}{t}\,dt = \log|\log x| + K$$

es. P.C. $\displaystyle\int \sqrt{x+2}\,dx$, $\displaystyle\int \frac{x}{\sqrt{2-3x^2}}\,dx$

$\displaystyle\int \frac{x^3}{\sqrt{1+x^4}}$, $\displaystyle\int \frac{(\log x)^7}{x}\,dx$

$\displaystyle\int \frac{\operatorname{sen}\sqrt{x}}{\sqrt{x}}\,dx$ $\displaystyle\int \frac{1}{1+e^x}\,dx$

$(e^x = t)$

$$\int \operatorname{sen}^3 x\,dx = \int \operatorname{sen} x \operatorname{sen}^2 x\,dx =$$

$$= \int \operatorname{sen} x\,(1-\cos^2 x)\,dx$$

suggerimento:

$(\cos x = t)$ ....

Integrazione per parti

Teorema f, g derivabili in $[a,b]$. Allora

$$\int f(x)\, g'(x)\, dx = f(x) \cdot g(x) - \int f'(x) g(x)\, dx$$

e

$$\int_a^b f(x)\, g'(x)\, dx = f(x) \cdot g(x) \Big|_a^b - \int_a^b f'(x) g(x)\, dx$$

Dim. $(f \cdot g)' = f'g + fg'$

$$fg' = (f \cdot g)' - f'g$$

$$\int fg'\, dx = \int (f \cdot g)'\, dx - \int f'g\, dx$$

$$= f \cdot g - \int f'g\, dx$$

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(perché $\int h' = h + k$)

Esempi

$$\underbrace{\int \underbrace{f(x)}_{\text{si deriva}}\, \underbrace{g'(x)\, dx}_{\text{si integra}}}\; = f(x)g(x) - \int f'(x) g(x)\, dx$$

$$\int \underbrace{x}_{f}\, \underbrace{\operatorname{sen} x}_{g'}\, dx = x \underbrace{(-\cos x)}_{g} - \int 1\, (-\cos x)\, dx$$

$$= -x\cos x + \int \cos x \, dx = -x\cos x + \operatorname{sen} x$$

se avessi fatto un'altra scelta

$$\int \underbrace{x}_{g'} \underbrace{\operatorname{sen} x}_{f} \, dx = \operatorname{sen} x \cdot \underbrace{\frac{x^2}{2}}_{g} - \int \cos x \cdot \frac{x^2}{2} \, dx$$

Serve anche per trovare la primitiva di alcune funzioni elementari

es $$\int \log x \, dx = \int \underbrace{1}_{g'} \cdot \underbrace{\log x}_{f} \, dx =$$

$$= \log x \cdot \underbrace{x}_{g} - \int \underbrace{\frac{1}{\cancel{x}}}_{f'} \cdot \cancel{x} \, dx = x \log x - x + K$$

es. $$\int \operatorname{arctg} x \, dx = \int \underbrace{1}_{g'} \cdot \underbrace{\operatorname{arctg} x}_{f} \, dx =$$

$$= \underbrace{x}_{g} \operatorname{arctg} x - \int x \frac{1}{1+x^2} \, dx =$$

$1+x^2 = t$

$2x\,dx = dt$

$$= x \operatorname{arctg} x - \frac{1}{2} \int \frac{1}{t} \, dt =$$

$$= x \operatorname{arctg} x - \frac{1}{2} \log\left(1+x^2\right)$$

P.C. $\int \operatorname{arcsen} x \, dx$, $\int \arccos x \, dx$

es $\displaystyle\int \underbrace{e^x}_{\substack{\text{integro}\\ g'}} \underbrace{\text{sen}\, x}_{\substack{\text{derivo}\\ f}} \, dx = \underbrace{e^x}_{g} \,\text{sen}\, x - \int e^x \cos x \, dx$

$$\int \underbrace{e^x}_{\substack{\text{integro}\\ g'}} \underbrace{\cos x}_{\substack{\text{derivo}\\ f}} \, dx = e^x \cos x - \int e^x (-\text{sen}\, x) \, dx$$

$$\int e^x \,\text{sen}\, x \, dx = e^x \,\text{sen}\, x - e^x \cos x - \int e^x \,\text{sen}\, x$$

$$2 \int e^x \,\text{sen}\, x \, dx = e^x \,\text{sen}\, x - e^x \cos x$$

$$\int e^x \,\text{sen}\, x \, dx = \frac{e^x \,\text{sen}\, x - e^x \cos x}{2} + K$$

(due integrazioni per parti successive).