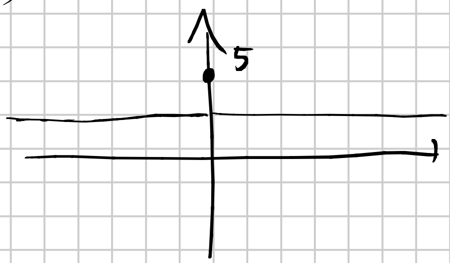
Lezione del 29 Ottobre

$\lim\limits_{x \to c} f(x) = l$ <
- con gli intorni di c e l
- $\Updownarrow$
- con le successioni

es. $\lim\limits_{x \to 0} \operatorname{sen} x = 0 = \operatorname{sen} 0$

$\lim\limits_{x \to 0} x^2 = 0 = f(0)$ $\quad f(x) = x^2$

$$f(x) = \begin{cases} 1 & x \neq 0 \\ 5 & x = 0 \end{cases}$$

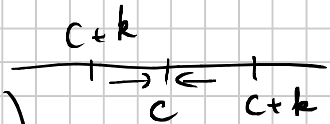


$\lim\limits_{x \to 0} f(x) = 1 \neq f(0) = 5$

Def. $f: I \to \mathbb{R}$, I intervallo, $c \in I$

f è continua in c se esiste

$\lim\limits_{x \to c} f(x) = f(c)$



$\left(\text{oppure } \lim\limits_{k \to 0} f(c+k) = f(c) \right)$

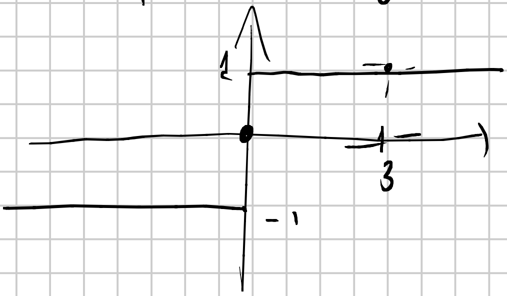
Def. Se f è continua $\forall c \in I \Rightarrow$
f è continua in I

oss. Si parla di continuità in c solo se f è definita in c

$f(x) = \frac{1}{x}$ $\quad \mathbb{R} \setminus \{0\}$

$f(x) = \frac{\operatorname{sen} x}{x}$ $\quad \mathbb{R} \setminus \{0\}$

es. $f(x) = \operatorname{sgn} x = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$



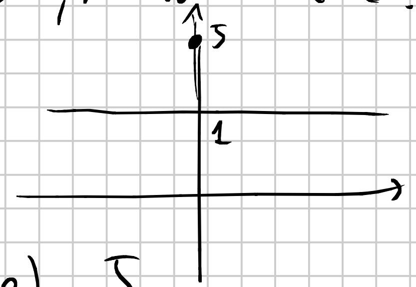
$\nexists \lim_{x \to 0} f(x) \Rightarrow f$ non è continua in $x = 0$

$f(x)$ è continua in $x = 3$? $f(3) = 1$

$\lim_{x \to 3} f(x) = 1 = f(3)$

$f(x)$ è continua $\forall x \neq 0$, ma no in $x = 0$.

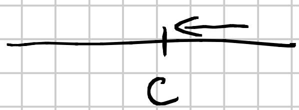
$f(x) = \begin{cases} 1 & x \neq 0 \\ 5 & x = 0 \end{cases}$



$\lim_{x \to 0} f(x) \ \exists = 1 \neq f(0) = 5$

Definizione di limite destro e sinistro

$\lim_{x \to c^+} f(x) = l$ limite destro



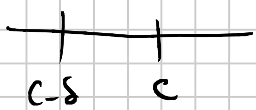
$(c, c+\delta)$

$x > c$

$\Updownarrow$

$\forall \varepsilon > 0 \ \exists \delta : |f(x) - l| < \varepsilon \quad \forall x \quad c < x < c + \delta$

$\lim_{x \to c^-} f(x) = l$ limite sinistro



$(c-\delta, c)$

$\Downarrow$

$\forall \varepsilon > 0 \ \exists \delta : |f(x) - l| < \varepsilon \quad \forall x : c - \delta < x < c$

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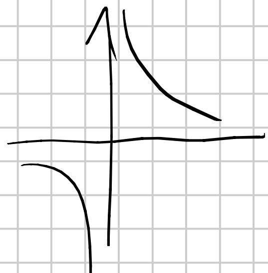
es. $f(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$

$\lim_{x\to 0^-} \operatorname{sgn} x = -1$ $\qquad$ $\lim_{x\to 0^+} \operatorname{sgn} x = 1$

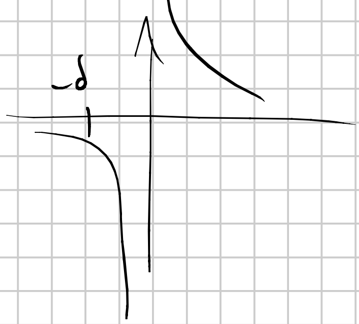
Prop. $\lim_{x\to c} f(x) = l \iff \lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = l$

es. $\lim_{x\to 0} \operatorname{sgn} x \;\nexists$



es. $\lim_{x\to 0^+} \frac{1}{x} = +\infty$

$\lim_{x\to 0^-} \frac{1}{x} = -\infty$ $\quad\Rightarrow\quad \nexists \lim_{x\to 0} \frac{1}{x}$



verifica $\lim_{x\to 0^-} \frac{1}{x} = -\infty$

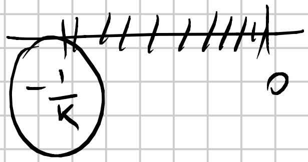
$\forall k>0 \ \exists\, \delta>0 :$

$\frac{1}{x} < -k \quad \forall x \quad -\delta < x < 0$

$x<0$

$-k<0$

$x > -\frac{1}{k}$ $\qquad$ $\frac{1}{k} = \delta$



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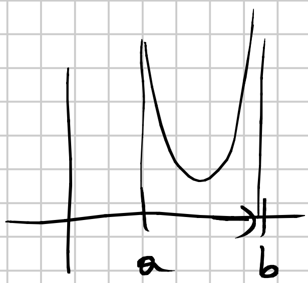
oss. f definita in (a,b)

$\lim_{x\to a^+} f(x)$

$\lim_{x\to b^-} f(x)$

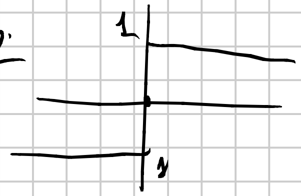


f definita in $x=c$

Def. Se $\lim_{x\to c^\pm} f(x)$ esistono finiti e sono diversi si dice che $f(x)$ ha una discontinuità di salto in $x=c$

e salto $= \lim_{x\to c^+} f(x) - \lim_{x\to c^-} f(x)$

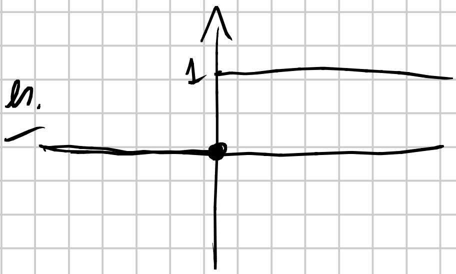
es.



$\operatorname{sgn} x$ ha una discontinuità di salto in $x=0$

$S = 2$

es.



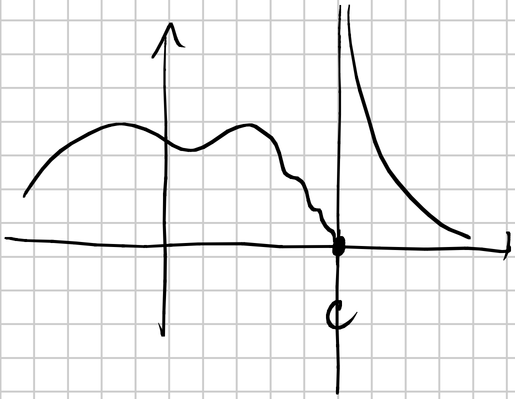
$$f(x) = \begin{cases} 1 & x>0 \\ 0 & x \leq 0 \end{cases}$$

$$\lim_{x \to 0^-} f(x) = 0 = f(0)$$

$$\lim_{x \to 0^+} f(x) = 1$$

f <u>non</u> è continua in $x=0$ ma è <u>continua da sx</u>

es.



$$\lim_{x \to c^-} f(x) = 0 = f(c)$$

$$\lim_{x \to c^+} f(x) = +\infty$$

è <u>continua</u> da sx ma da dx ha <u>un asintoto verticale</u>

<u>Teo.</u> f, g continue in c

$f \pm g$, $f \cdot g$, $\frac{f}{g}$ (se $g(c) \neq 0$)

sono continue

<u>Dim.</u> dall'algebra dei limiti.

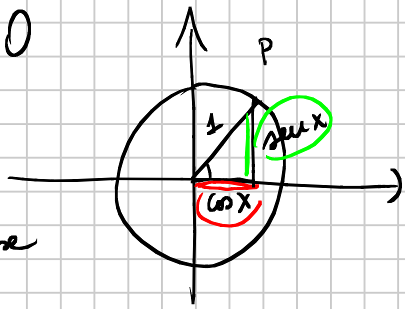
<u>Teo.</u> Le funzioni elementari: potenze, esponenziali, logaritmi, f. trigonometriche sono continue nel loro insieme di definizione.

<u>Dim</u> solo che $\operatorname{sen} x$ è continua $\forall x \in \mathbb{R}$

$\lim_{x \to 0} \operatorname{sen} x = 0 = \operatorname{sen} 0$ ho già dim. che sen x è continua in $x=0$

dim. che

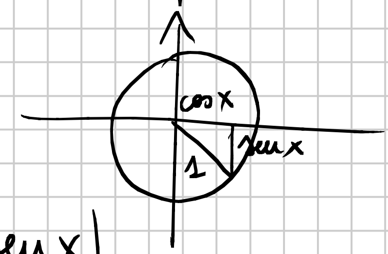
$$\lim_{x\to 0} \cos x = 1 = \cos 0$$



1° quadrante $\boxed{\operatorname{sen} x + \cos x \geq 1}$

cateti — ipotenusa

4° quadrante $\boxed{|\operatorname{sen} x| + \cos x \geq 1}$



$$0 \leq 1 - \cos x \leq |\operatorname{sen} x|$$

$$\downarrow \qquad \qquad \downarrow_{x\to 0} \qquad \qquad \downarrow_{x\to 0}$$

$$0 \qquad \qquad 0 \qquad \qquad 0$$

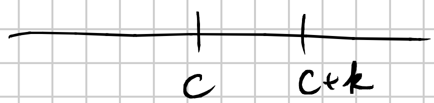
(fatto la settimana scorsa)

Quindi ora so che $\lim_{x\to 0} \operatorname{sen} x = 0$

$\lim_{x\to 0} \cos x = 1$

Dim. che sen x è continua anche in $x = c \neq 0$

$$\lim_{x\to c} \operatorname{sen} x = \operatorname{sen} c$$



oppure

$$\lim_{k\to 0} \operatorname{sen}(c+k) = \operatorname{sen} c$$

dim che

$$\operatorname{sen}(c+k) - \operatorname{sen} c \to 0 \quad \text{se } k \to 0$$

$\Longrightarrow$

$$\operatorname{sen}(c+k) - \operatorname{sen} c = \operatorname{sen} c \cos k + \cos c \operatorname{sen} k -$$

$$- \operatorname{sen} c = \operatorname{sen} c (\cos k - 1) + \cos c \operatorname{sen} k$$

$$|f(x)| \leq \underset{\downarrow 0}{g(x)} \qquad \text{allora} \quad f(x) \to 0$$

$$|\operatorname{sen}(c+k) - \operatorname{sen} c| = |\operatorname{sen} c\,(\cos k - 1) + \cos c \operatorname{sen} k| \le |\operatorname{sen} c\,\underbrace{(\cos k-1)}_{\substack{\downarrow k\to 0 \\ 0}}| + |\underbrace{\cos c \operatorname{sen} k}_{\downarrow 0}|$$

$k \to 0$

$|\operatorname{sen}(c+k) - \operatorname{sen} c| \to 0$ se $k \to 0$

cioè $\lim_{k\to 0} \operatorname{sen}(c+k) = \operatorname{sen} c$

cioè $\operatorname{sen} x$ è continua in $x = c$ $\forall c \in \mathbb{R}$

#.

$f(x) = x^3 + \lg x + \frac{e^x}{3^x} + \operatorname{tg} x$ dove è definita è continua

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<u>Teorema</u> (del cambio di variabile nel limite)

<u>es.</u> $\lim_{x\to+\infty} 3^{1/x} \overset{?}{=} \lim_{t\to 0} 3^t = 1$

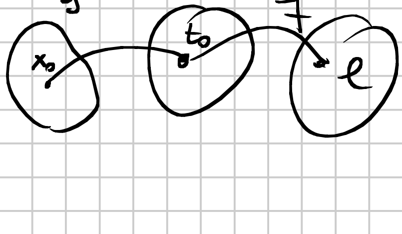
$\frac{1}{x} = t$ $\quad x \to +\infty$, $t \to 0$

$\lim_{x\to 0^+} \operatorname{sen}\left(\frac{1}{x}\right) \overset{?}{=} \lim_{t\to+\infty} \operatorname{sen} t \;\nexists$

$\frac{1}{x} = t$ $\quad x \to 0^+$, $t \to +\infty$

<u>Teo.</u> (cambio di variabile nel limite)

f, g due funzioni t.c. è definita $f \circ g$ e supponiamo che



1) $\lim_{x\to x_0} g(x) = t_0$

2) $\lim_{t\to t_0} f(t) = l$

3) $g(x) \neq t_0$ in un intorno di x_0

Allora $\lim_{x \to x_0} f(g(x)) = \lim_{t \to t_0} f(t) = l$

Dim. devo dim. che $\forall x_n \to x_0$

$f(g(x_n)) \to l$ (def. di limite con le successioni)

$x_n \to x_0$ per l'ipotesi 1) $g(x_n) \to t_0$

per l'ipotesi 2) $f(g(x_n)) \to l$ #

Conseguenze

g continua in x_0, $g(x_0) = t_0$

f continua in t_0

$\Rightarrow f \circ g$ è continua in x_0

$$f(x) = \operatorname{arctg}\left(\operatorname{senh}\left(\frac{1}{x^2-1} e^x\right) + e^{\operatorname{tg} x}\right)$$

, è continua dove è definita

Limiti notevoli

- $\boxed{\lim_{x \to \pm\infty} \left(1 + \frac{1}{x}\right)^x = e}$

per le successioni $\left(1+\frac{1}{n}\right)^n \to e$

$\forall a_n \to \pm\infty$

$\lim_n \left(1 + \frac{1}{a_n}\right)^{a_n} = e \Leftrightarrow \lim_{x \to \pm\infty} \left(1+\frac{1}{x}\right)^x = e$

def. di limite con le successioni

- $\lim_{x \to \pm\infty} \left(1 + \frac{\alpha}{x}\right)^x$

$\alpha \in \mathbb{R}$

$\frac{\alpha}{x} = \frac{1}{t}$ $x = \alpha t$

$x \to \pm\infty$ $t \to \pm\infty$

$$= \lim_{t \to \pm\infty} \left(1 + \frac{1}{t}\right)^{\alpha t} =$$

a seconda del segno di α

$$= \lim_{t \to \pm\infty} \left(\left(1 + \frac{1}{t}\right)^{t}\right)^{\alpha} = e^{\alpha}$$

- $$\boxed{\lim_{y \to 0} \frac{\log(1+y)}{y} = 1}$$

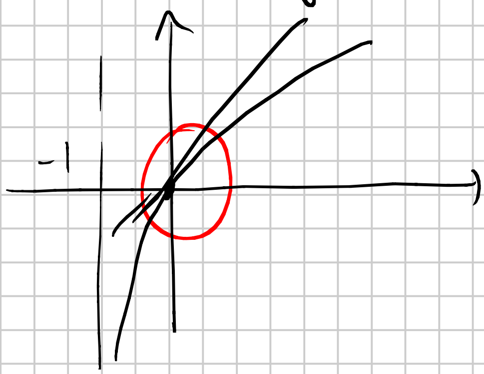
$x = \frac{1}{y}$ $\quad$ $y \to 0^+$ $\quad$ $x \to +\infty$ $\quad$ $y = \frac{1}{x}$

$y \to 0^-$ $\quad$ $x \to -\infty$

$$\lim_{y \to 0} \frac{\log(1+y)}{y} = \lim_{x \to \pm\infty} x \log\left(1 + \frac{1}{x}\right) =$$

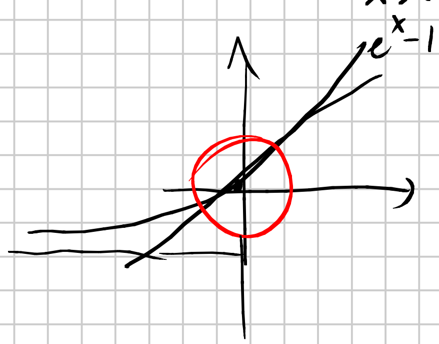
$$= \lim_{x \to \pm\infty} \log \underbrace{\left(1 + \frac{1}{x}\right)^{x}}_{\to e} = 1$$

$$\lim_{y \to 0} \frac{\log(1+y)}{y} = 1 \qquad \log(1+y) \underset{y \to 0}{\sim} y$$



$\log(1+y)$ $\quad$ $y > -1$

- $$\lim_{x \to 0} \frac{e^x - 1}{x} = 1 \iff e^x - 1 \underset{x \to 0}{\sim} x$$



$y = e^x - 1$ $\quad$ $x \to 0$

$e^x = y + 1$ $\quad$ $y \to 0$

$x = \log(y+1)$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{y \to 0} \frac{y}{\log(y+1)} = 1$$

Es. $\lim_{x\to 0} \frac{3^x - 1}{x} = \lim_{x\to 0} \frac{e^{\log(3^x)} - 1}{x}$ $\quad \frac{e^x - 1}{x} \to 1 \quad x\to 0$

$= \lim_{x\to 0} \frac{e^{x \log 3} - 1}{x \log 3} \cdot \log 3 = \log 3$

$\frac{e^{x \log 3} - 1}{x \log 3} \to 1$

$x \log 3 = y \quad x \to 0 \quad y \to 0$

Es. $\lim_{x\to 0} \frac{a^x - 1}{x} = \log a$ stesso conto.

Es. $\left(f(x)\right)^{g(x)} = e^{\log\left(f(x)\right)^{g(x)}} = e^{g(x) \log f(x)}$

$g(x) \log f(x)$

Es. $\lim_{x\to +\infty} \left(\frac{x+3}{x+4}\right)^{\frac{x^2+7}{2x+1}}$

$\frac{x+3}{x+4} = \frac{\cancel{x}(1+\frac{3}{x})}{\cancel{x}(1+\frac{4}{x})} \to 1$

$\frac{x^2+7}{2x+1} = \frac{x^{\cancel{2}}(1+\frac{7}{x})}{\cancel{x}(2+\frac{1}{x})} \to +\infty$

1^∞

$\left(\frac{x+3}{x+4}\right)^{\frac{x^2+7}{2x+1}} = e^{\boxed{\frac{x^2+7}{2x+1} \log\left(\frac{x+3}{x+4}\right)}}$

$\frac{x^2+7}{2x+1} \log\left(\frac{x+3}{x+4}\right) =$

$\frac{x+3+1-1}{x+4} = \frac{x+4-1}{x+4} = 1 - \frac{1}{x+4}$

$= \frac{x^{\cancel{2}}(1+\frac{7}{x^2})}{\cancel{x}(2+\frac{1}{x})} \log\left(1 - \frac{1}{x+4}\right)$

y

$\log(1+y) \quad x\to +\infty \quad y \to 0$

$= x \frac{\left(1+\frac{7}{x^2}\right)}{\left(2+\frac{1}{x}\right)} \cdot \frac{\log\left(1-\frac{1}{x+4}\right)}{\left(-\frac{1}{x+4}\right)} \left(-\frac{1}{x+4}\right) \to -\frac{1}{2}$

$\frac{1}{2}$ 1

$$1 \leftarrow \frac{\log(1+y)}{y} \qquad y\to 0 \qquad y = -\frac{1}{x+4}$$

$x \to +\infty$

$$x\left(-\frac{1}{x+4}\right) = \frac{-\cancel{x}}{\cancel{x}\left(1+\frac{4}{x}\right)} \to -1$$

OSS. I limiti di successione che ho messo su MOODLE si possono fare anche usando i limiti notevoli fatti ora ....

$$\lim_{n\to+\infty} \alpha n - \sqrt{n^2+3} \qquad \alpha \in \mathbb{R}$$

$\alpha = 0 \qquad -\sqrt{n^2+3} \to -\infty$

$\alpha < 0 \qquad \lim_n (\quad) = -\infty$

$\alpha > 0 \qquad \infty - \infty$

$$\lim_{n\to+\infty} \left(\alpha n - \sqrt{n^2+3}\right)\frac{\left(\alpha n + \sqrt{n^2+3}\right)}{\left(\alpha n + \sqrt{n^2+3}\right)} =$$

$$= \lim_n \frac{\alpha^2 n^2 - n^2 - 3}{\alpha n + \sqrt{n^2+3}} = \lim_n \frac{n^2(\alpha^2-1)-3}{\alpha n + \sqrt{n^2+3}}$$

1) $\alpha^2 - 1 = 0 \qquad \alpha = \pm 1 \qquad (\alpha > 0) \Rightarrow \alpha = 1$

$$= \lim_n \frac{-3}{n+\sqrt{n^2+3}} = 0$$

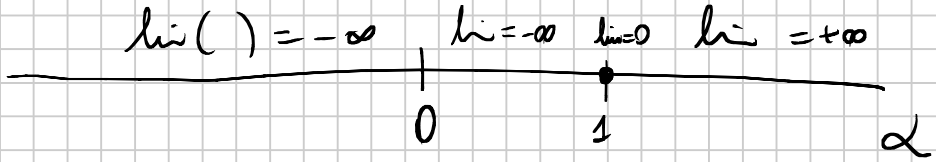
2) $\alpha^2 \neq 1 \qquad \alpha \neq 1 \quad (\alpha > 0)$

$$\lim_n \frac{n^2(\alpha^2-1)-3}{\alpha n + \sqrt{n^2+3}} = \lim_n \frac{n^{\cancel{2}}\left((\alpha^2-1) - \frac{3}{n^2}\right)}{\cancel{n}\left(\alpha + \sqrt{1+\frac{3}{n^2}}\right)}$$

se $\alpha^2 - 1 > 0$ ($\alpha > 1$) $\qquad \lim_n (\quad) = +\infty$

se $\alpha^2 - 1 < 0$ $\quad \lim_n (\quad) = -\infty$

$0 < \alpha < 1$



es. $\lim_n \underbrace{\frac{e^n}{2^{n!}}}_{a_n}$

$\frac{a_{n+1}}{a_n} \to \ell$ $\quad \ell < 1 \;\; a_n \to 0$ $\quad \ell > 1 \;\; a_n \to +\infty$

$$\frac{e^{\cancel{n}+1}}{2^{(n+1)!}} \frac{2^{n!}}{\cancel{e^n}} = e \frac{2^{n!}}{2^{n!(n+1)}}$$

$$= e \frac{2^{n!}}{\left(2^{n+1}\right)^{n!}} = e \left(\frac{2}{2^{n+1}}\right)^{n!} = e \frac{1}{\left(2^n\right)^{n!}}$$

$$= e \frac{1}{2^{n \cdot n!}} \to 0$$

$$\ell = 0 < 1$$

$$\lim_n \frac{e^n}{2^{n!}} = 0$$

criterio del rapporto!