30/10/13

Limiti notevoli

$$\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^x = e$$

da cui derivano

- $\displaystyle\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$ $\qquad \log(1+y)\sim y \quad y\to 0$
- $\displaystyle\lim_{x\to 0}\frac{e^x-1}{x} = 1$ $\qquad e^x-1\sim x \quad x\to 0$
- $\displaystyle\lim_{x\to 0}\frac{a^x-1}{x} = \log a$
- $\displaystyle\lim_{x\to 0}\frac{(1+x)^\alpha-1}{x} = \alpha$, $\forall \alpha\in\mathbb{R}$

$$y = (1+x)^\alpha - 1 \qquad \begin{matrix} x\to 0 \\ y\to 0\end{matrix}$$

$$y+1 = (1+x)^\alpha \quad \Rightarrow \quad y = (1+x)^\alpha - 1$$

$$\log(y+1) = \alpha\log(1+x)$$

$$\begin{matrix} y\to 0 \\ x\to 0\end{matrix}$$

$$\frac{\log(1+y)}{y} = \frac{\alpha\,\log(1+x)\;x}{\left((1+x)^\alpha - 1\right)x}$$

$$\frac{\log(1+y)}{y}\xrightarrow{y\to 0} 1 \qquad \frac{\log(1+x)}{x}\to 1$$

$$\Downarrow$$

$$\lim_{x\to 0}\frac{\alpha\,x}{(1+x)^\alpha-1}\to 1$$

$$\lim_{x\to 0}\frac{(1+x)^\alpha-1}{\alpha x}\to 1$$

$$\boxed{\lim_{x\to 0}\frac{(1+x)^\alpha-1}{x}\to \alpha}$$

es. $\lim_{x\to 0} \frac{(1+x)^{\sqrt{2}}-1}{x} = \sqrt{2}$

$$\lim_{x\to 0} \frac{\sqrt[5]{1+x}-1}{x} = \frac{1}{5}$$

$$\lim_{x\to 0^+} \frac{\sqrt[5]{1+x}-1}{\sqrt{x}\,\sqrt{x}} \cdot \sqrt{x} = 0$$

$\sqrt{x}$ è un infinitesimo di ordine inferiore a $\sqrt[5]{1+x}-1$

$\to 1/5$

$$\lim_{x\to 0^+} \frac{\sqrt[5]{1+x}-1}{x^2} = \lim_{x\to 0^+} \frac{\sqrt[5]{1+x}-1}{x\cdot x} = +\infty$$

$1/5$

x^2 è infinit. di ordine sup. rispetto a $\sqrt[5]{1+x}-1$

es. $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{\sqrt[3]{1+x}-1} \cdot \frac{x}{x} \to \frac{1}{2}\cdot 3 = \frac{3}{2}$

$\frac{\log(1+x)}{x} \to 1$ $\quad x\to 0$

es. $\lim_{n} \log\left(1+\frac{1}{n}\right) n$

$n\to\infty$

$\frac{1}{n}\to 0$

$= \lim_{n} \frac{\log\left(1+\frac{1}{n}\right)}{\frac{1}{n}} \to 1$

$x = \frac{1}{n}$

es. $\lim_{n} \frac{\log(1+n)}{n}$

$n \not\to 0$!

non si può applicare il limite notevole di sopra

$\log(1+n) = \log\left(n\left(\frac{1}{n}+1\right)\right) = \log n + \underbrace{\log\left(1+\frac{1}{n}\right)}_{\to 0}$

Un altro limite notevole (e sue conseguenze)

$$\lim_{x\to 0}\frac{\operatorname{sen}x}{x}=1 \qquad (\operatorname{sen}x\sim x,\ x\to 0)$$

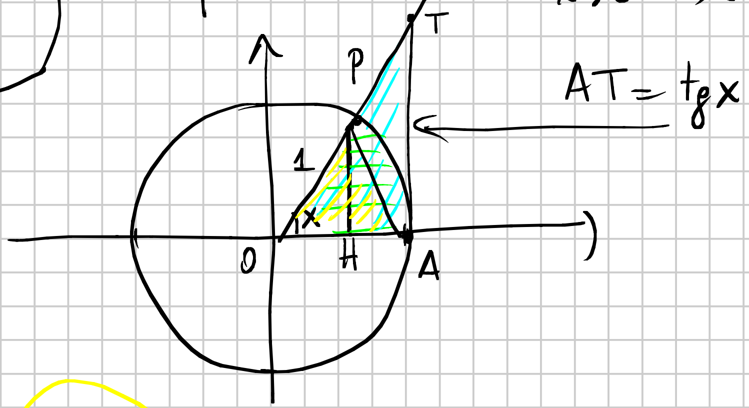
Dim. $f(x)=\dfrac{\operatorname{sen}x}{x}$ $\qquad f(-x)=\dfrac{\operatorname{sen}(-x)}{-x}=\dfrac{-\operatorname{sen}x}{-x}=\dfrac{\operatorname{sen}x}{x}=f(x)$

f è pari

mi calcolo

$$\lim_{x\to 0^+}\frac{\operatorname{sen}x}{x} \quad \text{e questo sarà} = \lim_{x\to 0^-}\frac{\operatorname{sen}x}{x}$$

$x\to 0^+$



3 aree

area triangolo $OPA = \dfrac{1\cdot PH}{2}=\dfrac{\operatorname{sen}x}{2}$

area del settore circolare $OPA = \dfrac{x\cdot 1}{2}$

area del triangolo $OAT = \dfrac{1\cdot \operatorname{tg}x}{2}$

$$\frac{\operatorname{sen}x}{\cancel{2}}\le\frac{x}{\cancel{2}}\le\frac{\operatorname{tg}x}{\cancel{2}}$$

$$\operatorname{sen}x\le x\le \operatorname{tg}x$$

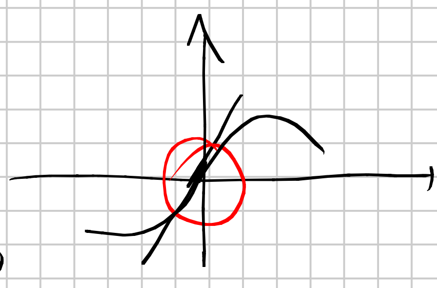
divido per $\operatorname{sen}x$

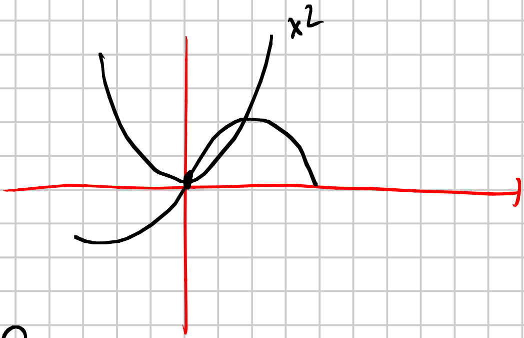
$$1\le\frac{x}{\operatorname{sen}x}\le\frac{1}{\cos x}$$

$$\underset{\substack{\downarrow x\to 0^+\\ 1}}{\cos x}\le\underset{\substack{\downarrow x\to 0^+\\ 1}}{\frac{\operatorname{sen}x}{x}}\le\underset{\substack{\downarrow x\to 0^+\\ 1}}{1}$$

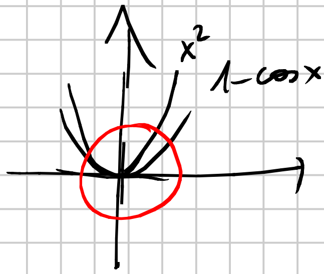
#

$$\operatorname{sen}x\sim x \quad x\to 0$$





$$\lim_{x\to 0}\frac{\operatorname{sen}x}{x}=1$$



Conseguenza

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}$$

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}\cdot\frac{1+\cos x}{1+\cos x}=\lim_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos x)}$$

$$=\lim_{x\to 0}\underbrace{\frac{\operatorname{sen}^2 x}{x^2}}_{\to 1}\cdot\frac{1}{1+\cos x}=\frac{1}{2}$$

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}$$

$1-\cos x$ è infinitesimo di ordine 2 $x\to 0$

$$\lim_{x\to 0}\frac{\operatorname{sen}x}{x}=1$$

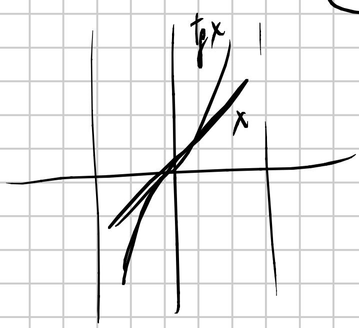
$\operatorname{sen}x$ è infinitesimo di ordine 1 $x\to 0$

$$\lim_{x\to 0^+}\frac{1-\cos x}{x^3}=\lim_{x\to 0^+}\underbrace{\frac{1-\cos x}{x^2\cdot x}}_{1/2}=+\infty$$

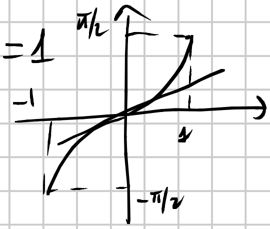
$$\lim_{x\to 0^+}\underbrace{\frac{1-\cos x}{x\,x}}_{\to\frac{1}{2}}\cdot x=0$$

$$\cdot\ \lim_{x\to 0}\frac{\operatorname{tg}x}{x}=\lim_{x\to 0}\underbrace{\frac{\operatorname{sen}x}{\cos x}\cdot\frac{1}{x}}_{\to 1}=1$$

$\operatorname{tg}x \sim x \qquad x\to 0$



- $\displaystyle\lim_{x\to 0}\frac{\operatorname{arcsen} x}{x} = \lim_{y\to 0}\frac{y}{\operatorname{sen} y} = 1$



$\operatorname{arcsen} x = y$ se $x\to 0$

$\Downarrow ?$

$x\in[-1,1]$

$x = \operatorname{sen} y$ $\quad y\to 0$ (serve aver dim. che $\operatorname{arcsen} x$ è continua in $x=0$)

la faremo fra qualche lezione.

$\operatorname{arcsen} x \sim x \quad , x\to 0$

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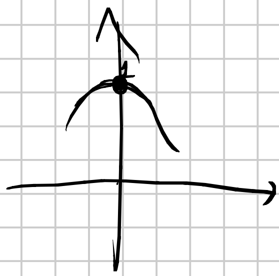
oss. $\quad f(x) = \dfrac{\operatorname{sen} x}{x} \qquad \mathbb{R}\setminus\{0\}$

ma siccome $\displaystyle\lim_{x\to 0}\frac{\operatorname{sen} x}{x} = 1$



$$\bar f = \begin{cases} \dfrac{\operatorname{sen} x}{x} & x\neq 0 \\ 1 & x = 0 \end{cases}$$

$\bar f$ è continua dove è definita?

se $x\neq 0$ $\quad \bar f = \dfrac{\operatorname{sen} x}{x}$ è continua

se $x = 0$ $\quad ! \displaystyle\lim_{x\to 0}\bar f(x) = 1 \, !$

devo verificare $\displaystyle\lim_{x\to 0}\frac{\operatorname{sen} x}{x} = 1$

f se $x\neq 0$.

$\bar f$ è continua su tutto $\mathbb{R}$

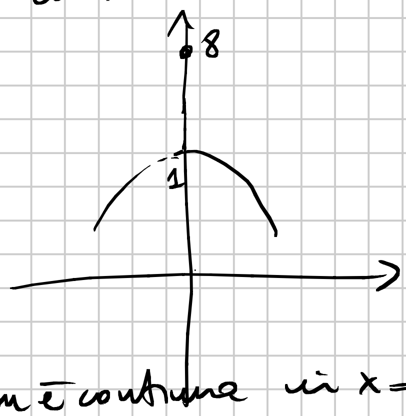
$\bar f$ prolunga per continuità la $f = \dfrac{\operatorname{sen} x}{x}$ che non è definita in $x=0$

NO // $\dfrac{\operatorname{sen} x}{x}$ è continua in $x=0$ $\qquad$ Non ha senso

$$\bar{f} \begin{cases} \frac{\operatorname{sen} x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

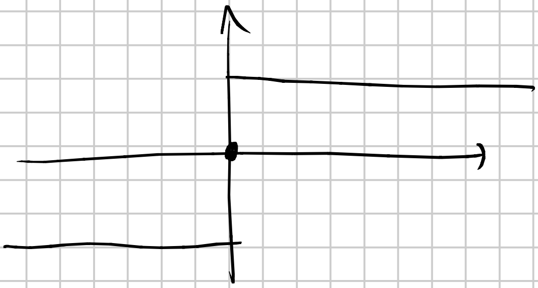
è continua in $x=0$

$$\tilde{f} = \begin{cases} \frac{\operatorname{sen} x}{x} & x \neq 0 \\ 8 & x = 0 \end{cases}$$



non è continua in $x=0$

Si può prolungare per continuità in $x=c$ una funzione t.c. $\exists$ finito $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = \ell$



$f(x) = \operatorname{sgn} x$

Gerarchie degli infiniti

$x \to +\infty$

$(\log_a x)^\alpha$ $\quad$ x^β $\quad$ b^x

$a > 1$ $\quad$ $\beta > 0$ $\quad$ $b > 1$

$\alpha > 0$

$$\lim_{x \to +\infty} \frac{(\log_a x)^\alpha}{x^\beta} = 0 \qquad \forall a > 1, \alpha > 0, \beta > 0$$

$$\lim_{x \to +\infty} \frac{x^\beta}{b^x} = 0 \qquad \forall \beta > 0, b > 1$$

es. $$\lim_{x \to +\infty} \frac{(\log x)^{1000}}{x^{\frac{1}{1000}}} = 0$$

applicazione $\lim_{x\to+\infty} \frac{\log x}{x} = 0$

$y = \frac{1}{x}$ $\quad x \to +\infty \quad y \to 0^+$

$x = \frac{1}{y}$

$\lim_{y\to 0^+} y \log \frac{1}{y} = \lim_{y\to 0^+} y(-\log y) = 0$

$$\lim_{y\to 0^+} y(-\log y) = 0$$

$$\lim_{y\to 0^+} y \log y = 0$$

In generale vale

$$\boxed{\lim_{y\to 0^+} y^\beta (-\log y)^\alpha = 0}$$

$\forall \beta > 0$
$\forall \alpha > 0$

$$\boxed{\lim_{y\to 0^+} y^\beta |\log y|^\alpha = 0}$$

$$\lim_{y\to 0^+} \sqrt[100]{y}\, |\log y|^{3200} = 0$$

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es. $\lim_{x\to 0} \frac{\operatorname{sen}(e^x - 1)}{(e^x - 1)} = \lim_{y\to 0} \frac{\operatorname{sen} y}{y} = 1$

$e^x - 1 = y$

$x \to 0 \Rightarrow y \to 0$

es. $\lim_{x\to 0} \frac{\operatorname{sen} 3x}{3x} \cdot 3 = 3$ $\quad \frac{\operatorname{sen} 3x}{3x} \to 1$

$x \to 0 \quad 3x \to 0$

$$\lim_{x\to 0^+} \frac{\operatorname{sen}(3x+1)}{x} = +\infty \qquad 3x+1 \not\to 0 \quad x\to 0$$

In generale $\frac{\operatorname{sen} x}{x} \to 1 \quad x \to 0$

$$\frac{\operatorname{sen} \varepsilon(x)}{\varepsilon(x)} \to 1 \quad \text{se} \quad \varepsilon(x) \to 0 \quad x \to 0$$

$$\frac{\operatorname{sen}\left(\overbrace{e^x - 1}^{\varepsilon(x)}\right)}{e^x - 1} \to 1$$

$$\lim_{x\to 0} \frac{\operatorname{sen}(e^x)}{e^x}$$

Funzioni infinitesime

$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0 \qquad c \in \mathbb{R}^*$

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \begin{cases} 0 & f \text{ infinitesima di ordine superiore a } g \\ \pm\infty & f \text{ infinitesima di ordine inferiore a } g \\ l \neq 0 & \text{infinitesime dello stesso ordine} \\ \nexists & \text{non sono confrontabili} \end{cases}$$

$$\frac{1-\cos x}{x^2} \to \frac{1}{2}$$

$1-\cos x$ e x^2 sono infinitesime dello stesso ordine

$$\frac{1-\cos x}{x} \to 0$$

$1-\cos x$ è infinitesima di ordine superiore a x

Def. $f, g \to 0 \quad x \to c$

$$\lim_{x\to c} \frac{f(x)}{g(x)} = 0 \iff f(x) = o(g(x)) \quad x \to c$$

"o piccolo"

es. $\frac{1-\cos x}{x} \underset{x\to 0}{\to} 0 \iff (1-\cos x) = o(x) \quad x \to 0$

es. $x^3 = o(x) \qquad x\to 0$

$\frac{x^3}{x} \to 0 \qquad x\to 0 \qquad ?$

Esercizio sui limiti di successione

es.
$$\lim_n \log\left(1+e^n\right) - n =$$

$$= \lim_n \log\left(e^n \cdot \left(\frac{1}{e^n}+1\right)\right) - n =$$

$$= \lim_n \cancel{\underbrace{\log e^n}_{=n}} + \log\left(\frac{1}{e^n}+1\right) \cancel{-n} = 0$$

es.
$$\lim_n \frac{\log(1+n^2)}{3^n} = \lim_n \frac{\log\left(n^2\left(\frac{1}{n^2}+1\right)\right)}{3^n} =$$

$$= \lim_n \frac{\log n^2 + \log\left(\frac{1}{n^2}+1\right)}{3^n} = \lim_n \left(\underbrace{\frac{2\log n}{3^n}}_{\to 0} + \underbrace{\frac{\log\left(\frac{1}{n^2}+1\right)}{3^n}}_{\to 0}\right) = 0$$

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es.
$$\lim_n \frac{n^n + 7n\log n - n \operatorname{sen} n}{(n+2)^n + n\log\frac{1}{n} + n\operatorname{sen}\frac{1}{n}} =$$

$$= \lim_n \frac{\overbrace{n^n}^{1/e^2}\left(1 + \overbrace{\frac{7n\log n}{n^n}}^{\to 0} - \overbrace{\frac{n \operatorname{sen} n}{n^n}}^{\to 0}\right)}{(n+2)^n\left(1 + \underbrace{\frac{n}{(n+2)^n}\log\frac{1}{n}}_{\to 0} + \underbrace{\frac{n}{(n+2)^n}\operatorname{sen}\frac{1}{n}}_{\to 0}\right)}$$

$$\frac{n \log n}{n^n} = \frac{\log n^n}{n^n} = \frac{\log m}{m} \to 0 \qquad n^n = m \to +\infty$$

$$\frac{n \operatorname{sen} n}{n^n} = \frac{1}{n^{n-1}} \operatorname{sen} n$$

$$\frac{n}{(n+2)^n} \log \frac{1}{n} = - \frac{n \log n}{(n+2)^n} =$$

$$= - \frac{\log n^n}{\left(n\left(1+\frac{2}{n}\right)\right)^n} = - \underbrace{\frac{\log n^n}{n^n}}_{\to 0} \cdot \underbrace{\frac{1}{\left(1+\frac{2}{n}\right)^n}}_{\left(1+\frac{2}{n}\right)^n \to e^2} \to 0$$

$$\cdot \frac{\overbrace{n \operatorname{sen} \frac{1}{n}}^{\to 1}}{(n+2)^n} \to 0$$

$$n \operatorname{sen} \frac{1}{n} \to 1 \qquad \frac{1}{n} = x \qquad n = \frac{1}{x}$$

$n \to +\infty$

$x \to 0$

$$= \frac{\operatorname{sen} x}{x}$$

$$\frac{n^n}{(n+2)^n} = \left(\frac{n}{n+2}\right)^n = \frac{1}{\left(1+\frac{2}{n}\right)^n} \to \frac{1}{e^2}$$

$$\lim \left(\cdot \; - \; - \; - \; \right) = \frac{1}{e^2}$$

<u>oss.</u> $\lim \left(\frac{n}{n+2}\right)^n$ si può fare anche così:

$$\left(\frac{n}{n+2}\right)^n = e^{n \log\left(\frac{n}{n+2}\right)}$$

$$n \log \left(\frac{n}{n+2}\right) = n \log \left(\frac{n+2-2}{n+2}\right) =$$

$$= n \log\left(1 \left(-\frac{2}{n+2}\right)_x\right)$$

$$\frac{\log(1+x)}{x} \to 1 \quad x \to 0$$

$$\left(-\frac{2}{n+2}\right)_x \to 1$$

$$n\left(-\frac{2}{n+2}\right) = \frac{-2\cancel{n}}{\cancel{n}\left(1+\frac{2}{n}\right)} \to -2$$

$$n \log\left(\frac{n}{n+2}\right) \to -2$$

$$e^{n \log\left(\frac{n}{n+2}\right)} \to e^{-2} = \frac{1}{e^2}$$

es. $\lim_n \frac{5^n - a^n}{7^n}$ $\quad a > 0$

1) $a < 1$ $\quad a^n \to 0$

$$\frac{5^n\left(1 - \frac{a^n}{5^n}\right)}{7^n} = \left(\frac{5}{7}\right)^n \left(1 - \frac{a^n}{5^n}\right)$$

$\left(\frac{5}{7}\right)^n \to 0$, $\quad a^n \to 0$, $\quad 5^n \to +\infty$, $\quad \frac{a^n}{5^n} \to 0$

2) $a > 1$ $\quad \frac{5^n - a^n}{7^n}$

$a < 5$ $\quad \frac{5^n\left(1 - \left(\frac{a}{5}\right)^n\right)}{7^n} \to 0$, $\quad \left(\frac{a}{5}\right)^n \to 0$

$a > 5$ $\quad \frac{a^n\left(\left(\frac{5}{a}\right)^n - 1\right)}{7^n} = \left(\frac{a}{7}\right)^n \left(\searrow_{-1}\right)$, $\quad \left(\frac{5}{a}\right)^n \to 0$

se $a > 7$ $\longrightarrow -\infty$

se $a < 7$ $\left(\frac{a}{7}\right)^n \to 0$ lim $= 0$

$a = 7$ $a = 5$ fare voi!