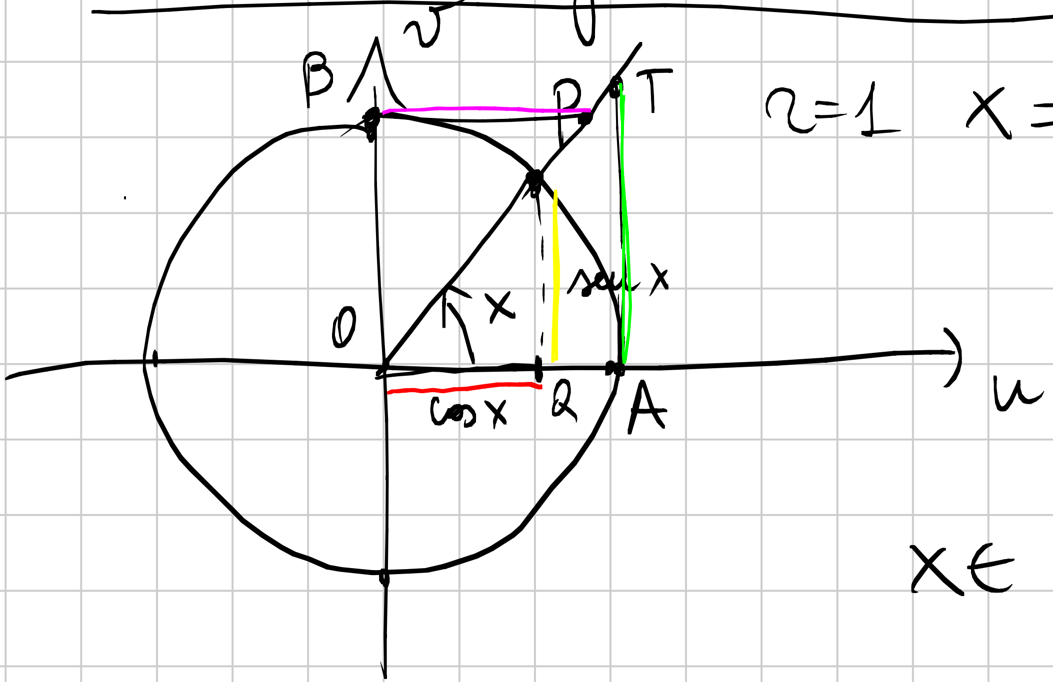
9 Ottobre 2013

Funzioni trigonometriche



$r = 1$ $\quad x =$ misura in radianti dell'angolo $A\hat{O}P$:= lunghezza arco AP

$x \in [0, 2\pi)$

$\cos x$ (coseno di x) := ascissa di P

sen x := ordinata di P
(seno di x)

tg x := lunghezza (con segno) del segmento AT
(tangente di x)

Poiché i triangoli OPQ e OAT sono simili

$$\operatorname{tg} x = \frac{\operatorname{sen} x}{\cos x}$$

$$\operatorname{cotg} x = \frac{\cos x}{\operatorname{sen} x}$$

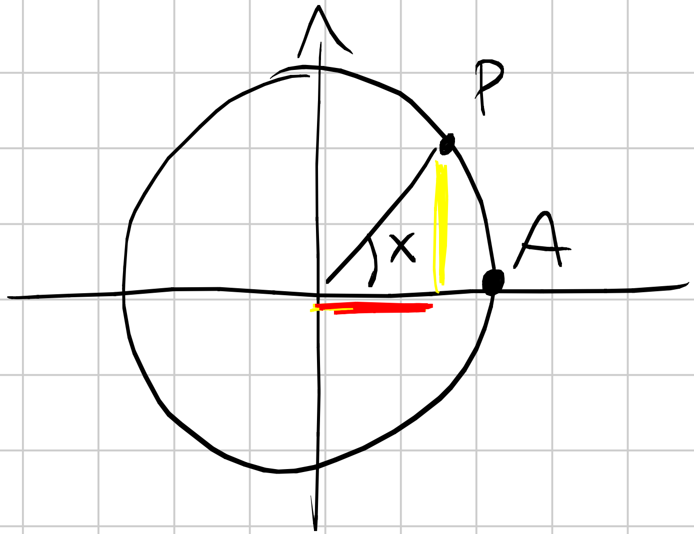
segmento viola del disegno

$P = (1,0)$

$x = 0$

$\cos 0 = 1$
$\operatorname{sen} 0 = 0$

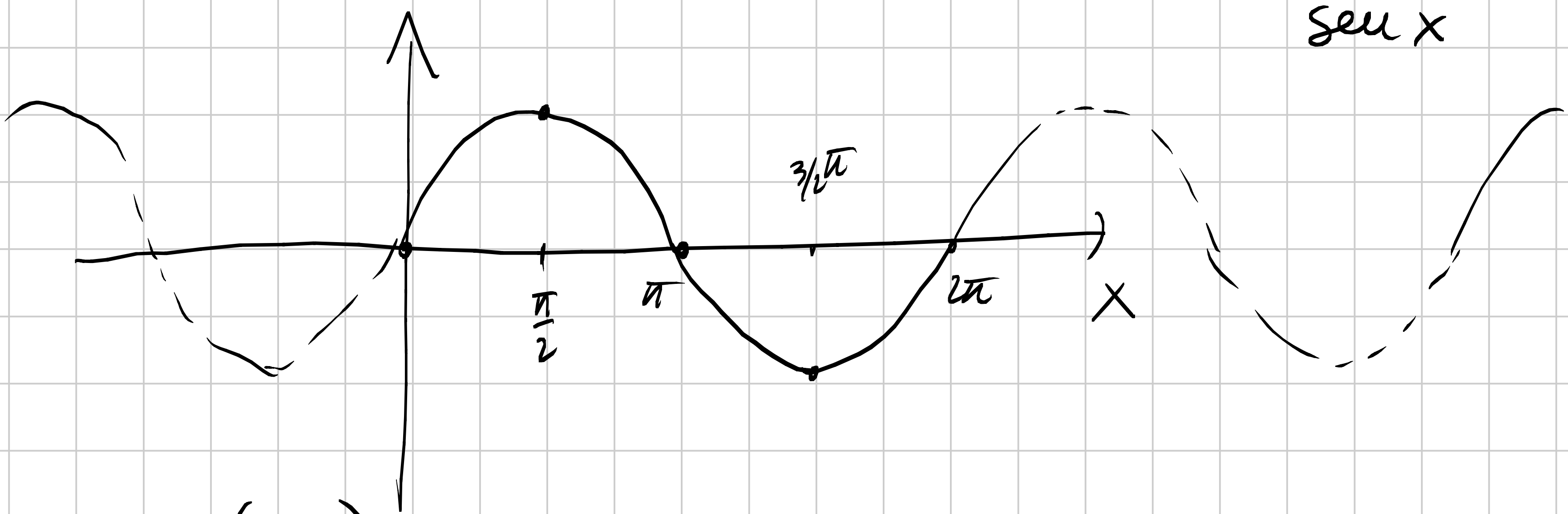
$x = 2\pi$ $\cos 0 = 1$
$\operatorname{sen} 0 = 0$



$\operatorname{sen}^2 x + \cos^2 x = 1$ $\forall x \in [0, 2\pi)$

Prendo $x \in \mathbb{R}$

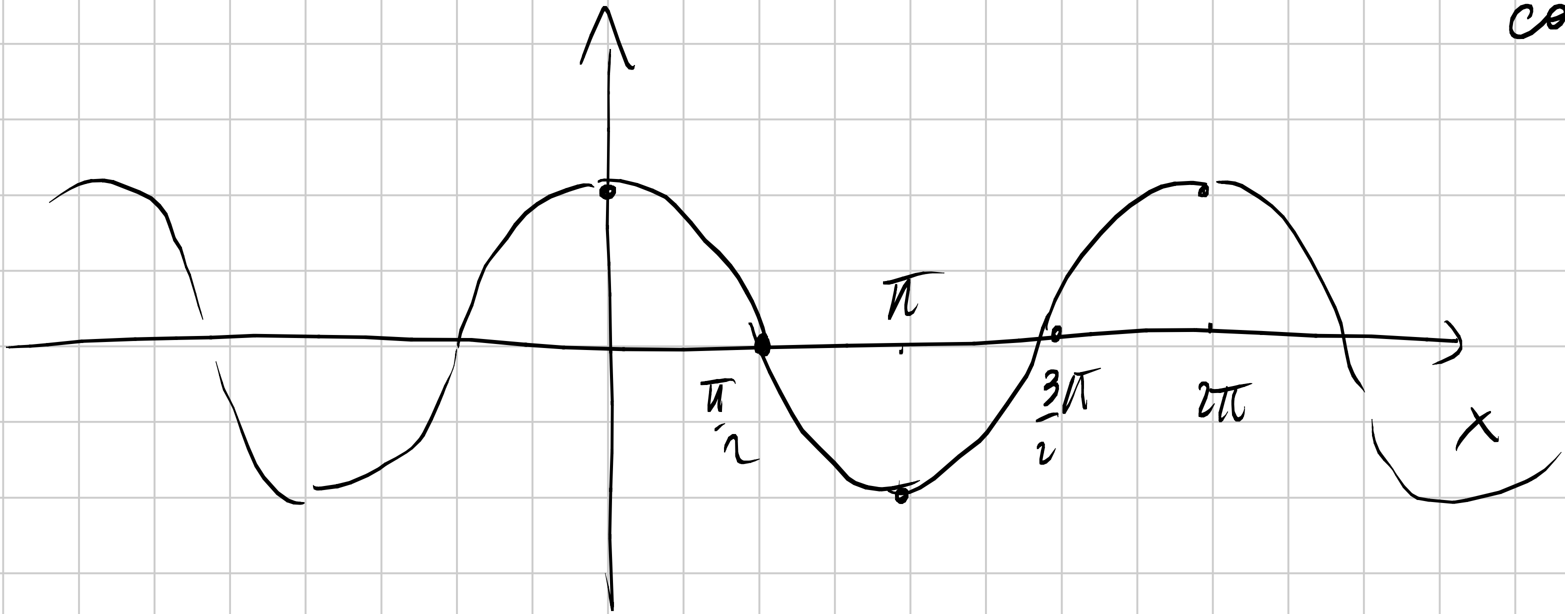
sen x, cos x periodiche di periodo 2π



$\operatorname{sen}(-x) = -\operatorname{sen} x$ dispari

$D = \mathbb{R}$ $\quad \operatorname{Im}(\operatorname{sen} x) = [-1, 1]$

cos x



$\cos(x) = \cos(-x)$ pari

$D = \mathbb{R}$ $\mathrm{Im}(\cos x) = [-1, 1]$

Periodica di periodo 2π

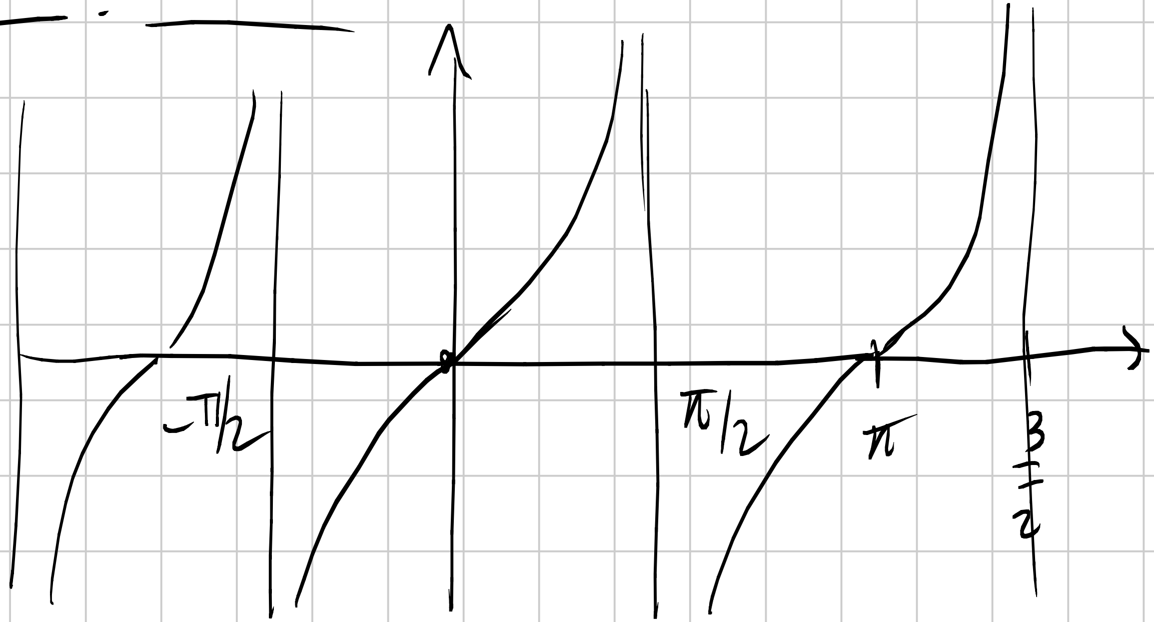
$$\operatorname{sen} x = \cos\left(x - \frac{\pi}{2}\right)$$

$$|\operatorname{sen} x| \leq 1$$

$$|\cos x| \leq 1$$

$$\operatorname{tg} x = \frac{\operatorname{sen} x}{\cos x}$$

$$D = \mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi \right\}_{k \in \mathbb{Z}}$$



Periodica di periodo π

$Tg(-x) = -tg\, x$ dispari

es. $f(x) = \operatorname{sen} 3x$

qual'è il periodo di questa funzione

$f(x+T) = f(x)$

$f(x+T) = \operatorname{sen}(3(x+T)) = \operatorname{sen}(3x+3T) \stackrel{?}{=}$

$= \operatorname{sen} 3x =$

$3T = 2\pi$

$= f(x)$

$$T = \frac{2\pi}{3}$$

In generale

$f(x) = \operatorname{sen}(\omega x)$ periodica di periodo

$$T = \frac{2\pi}{\omega}$$

Funzioni iperboliche

$$\mathrm{Sh}\,x = \mathrm{senh}\,x := \frac{e^{x} - e^{-x}}{2}$$

seno iperbolico

$$\mathrm{Ch}\,x = \cosh x := \frac{e^{x} + e^{-x}}{2}$$

coseno iperbolico

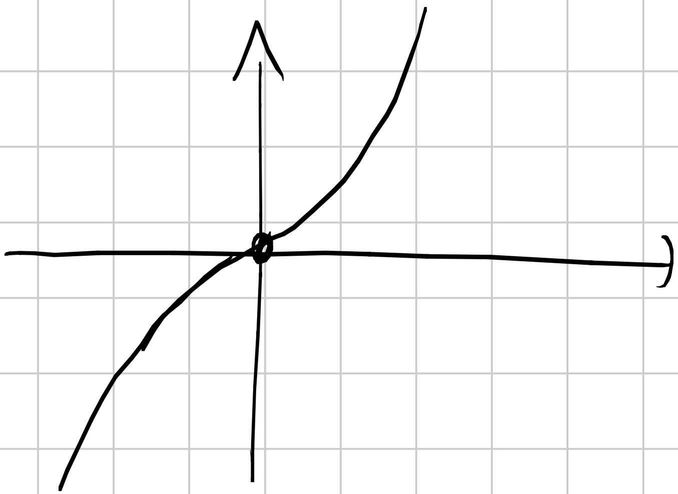
$$\mathrm{Th}\,x = \mathrm{tgh}\,x := \frac{\mathrm{senh}\,x}{\cosh x} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

$$D = \mathbb{R} \qquad \operatorname{senh} x = \frac{e^{x} - e^{-x}}{2}$$

$$\operatorname{senh}(-x) = \frac{e^{-x} - e^{x}}{2} = -\frac{e^{x} - e^{-x}}{2} = -\operatorname{senh} x$$

dispari

$\operatorname{senh} 0 = 0$



$D = \mathbb{R}$

$\operatorname{Im}(\operatorname{senh} x) = \mathbb{R}$

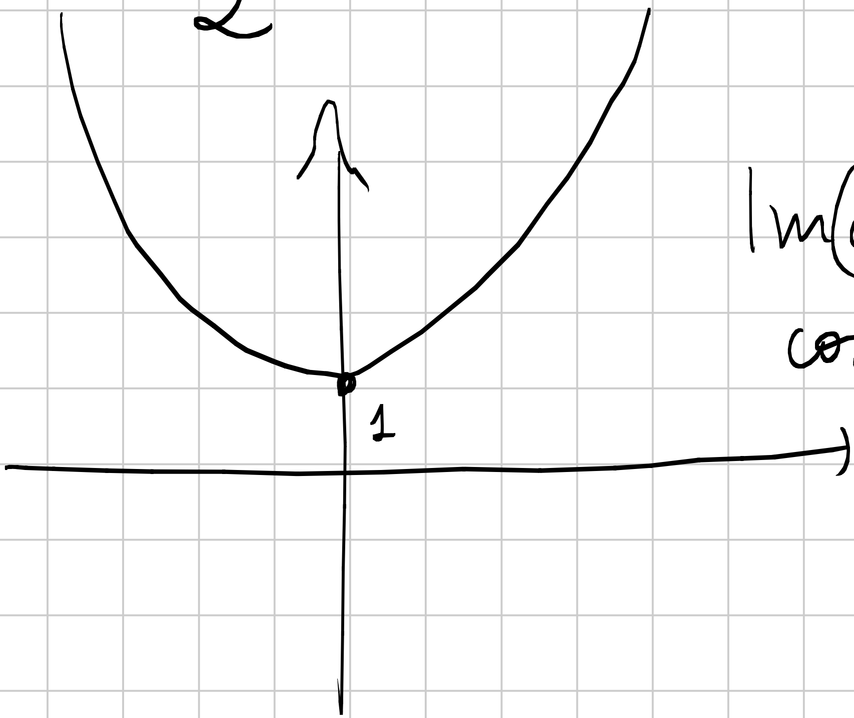
Dispari, strett. cresc.

$$\cosh x := \frac{e^{x} + e^{-x}}{2} \qquad D = \mathbb{R}$$

$$\cosh(-x) = \frac{e^{-x} + e^{x}}{2} = \cosh x$$

pari

$$\cosh 0 = 1$$



$$\operatorname{Im}(\cosh x) = [1, +\infty)$$

$$\cosh x \geq 1$$

$$\operatorname{tgh} x = \frac{\operatorname{senh} x}{\cosh x} = \frac{e^{x} - e^{-x}}{e^{x} + e^{x}}$$

$D = \mathbb{R}$

$$\operatorname{tgh}(-x) \overset{?}{=} -\operatorname{tgh}(x)$$

Fate voi!

$\operatorname{Im}(\operatorname{tgh} x) = (-1, 1)$

Dispari

strett. crescente

$$\operatorname{sen}^2 x + \cos^2 x = \underline{1}$$

Trigonometrica

$$\cosh^2 x - \operatorname{senh}^2 x = 1$$

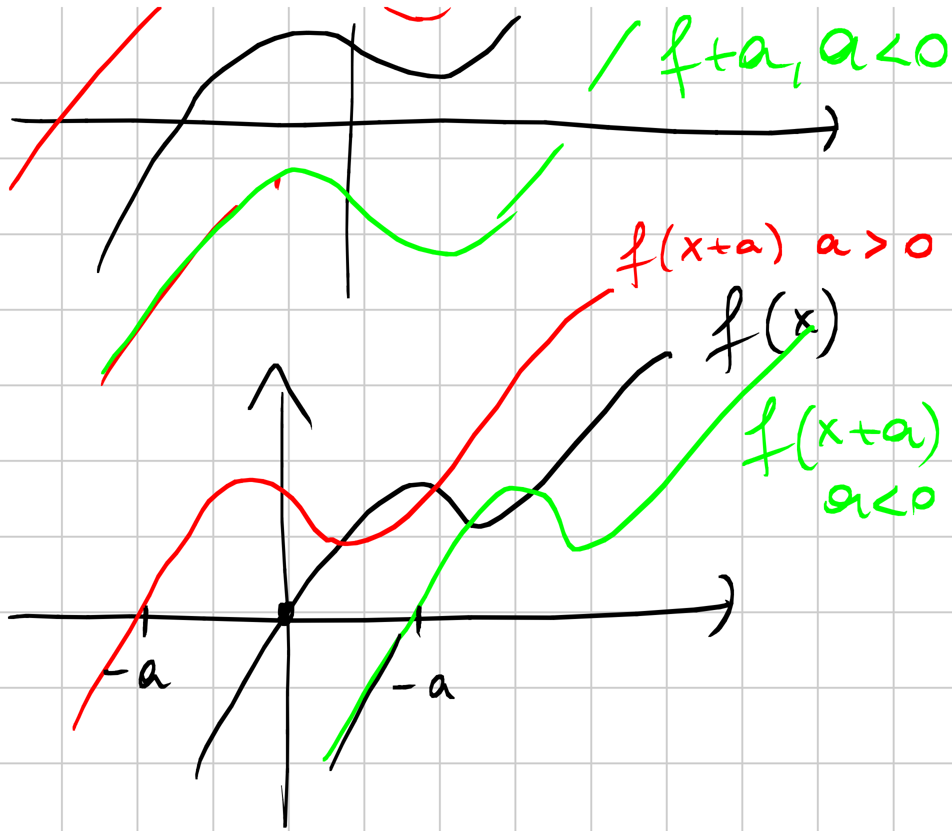
si verifica dirett. con la def. di senh e cosh (fate voi!)

Operazioni sui grafici

Si conosce il grafico di $y = f(x)$ e si vuole disegnare il grafico di

1) $y_1 = f(x) + a$

Traslazione in alto o in basso.



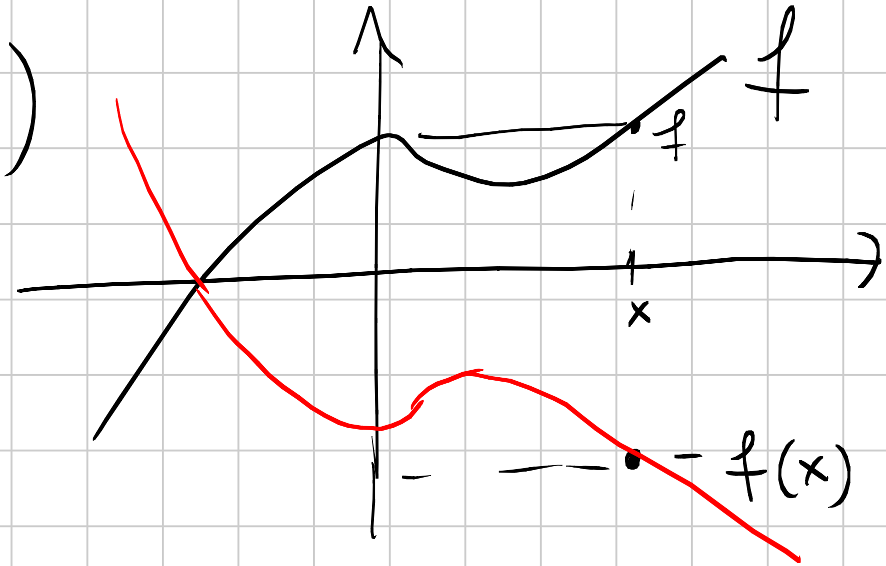
2) $y_2 = f(x+a)$

$f(x+a) = 0 \qquad x + a = 0$

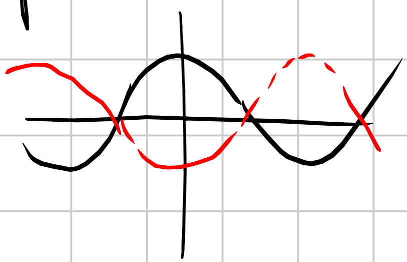
$x = -a$

Traslazione a dx $(a<0)$ o a sx $(a>0)$.

3) $y_3 = -f(x)$

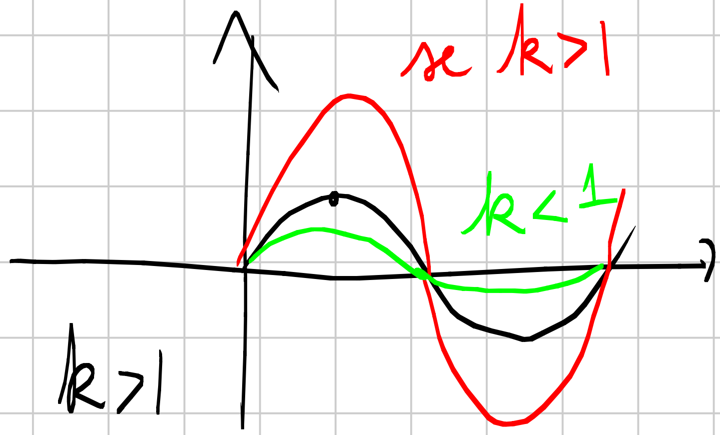


$y = \cos x$



$y = -\cos x$

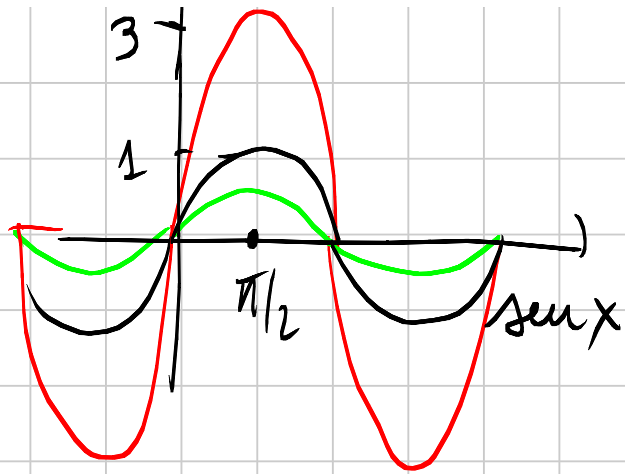
4) $y_4 = k\,f(x)$



il grafico si dilata se $k>1$
e si contrae se $k<1$

es. $3 \operatorname{sen} x$

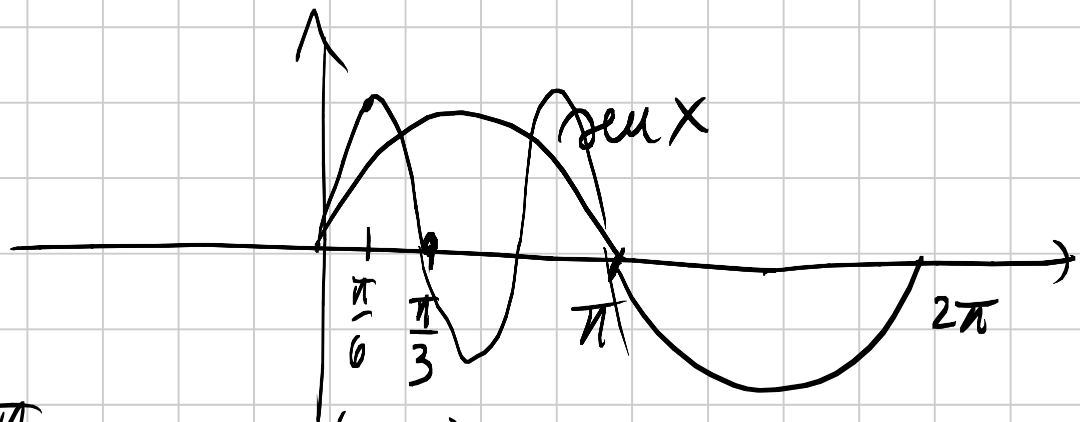
$\frac{1}{2} \operatorname{sen} x$



b) $y_5 = f(kx)$

$\operatorname{sen}(3x)$

$3x = \pi \qquad x = \frac{\pi}{3}$

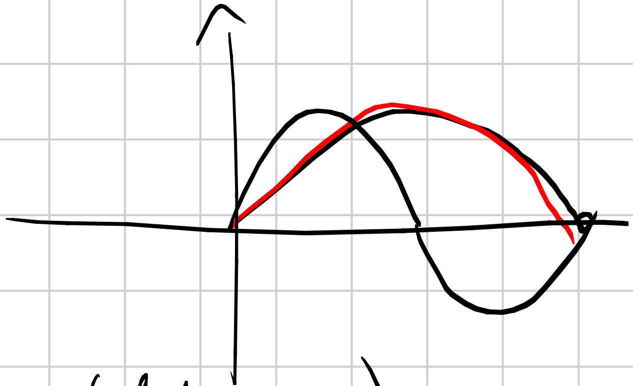


$\operatorname{sen}(3x) = 0$

$\operatorname{sen}(3x) = 1 \qquad 3x = \frac{\pi}{2} \qquad x = \frac{\pi}{6}$

si contrae (se $k>1$)

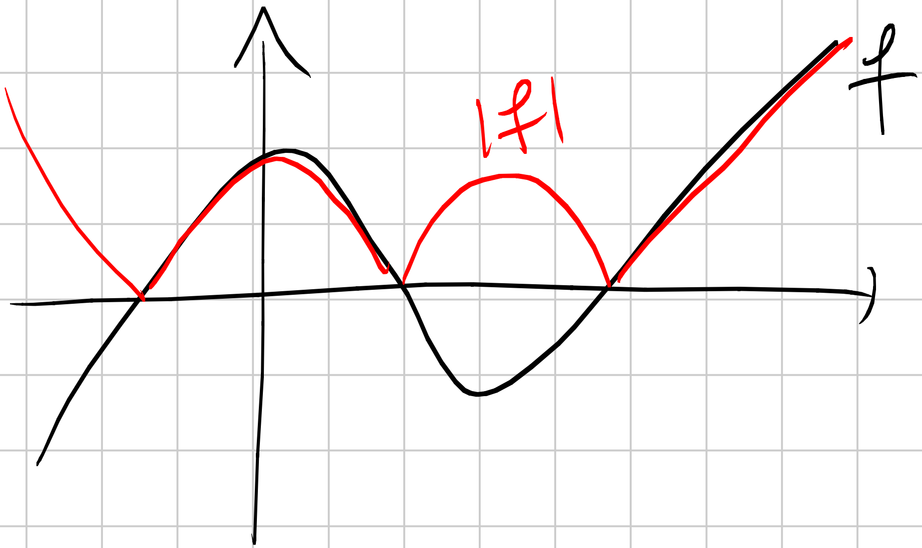
o si dilata (se $k<1$)



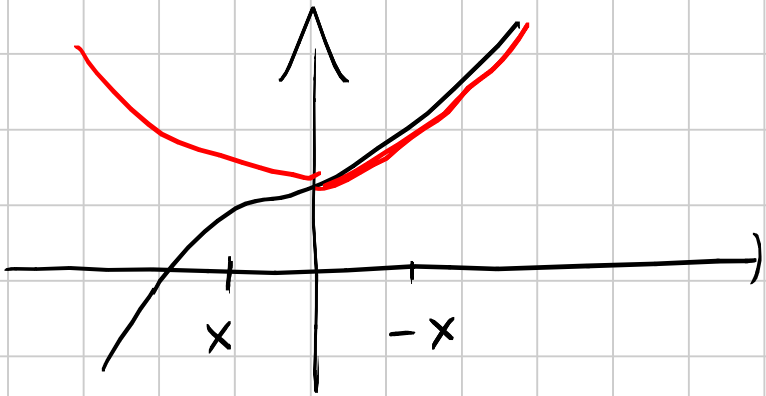
$\operatorname{sen}\left(\frac{x}{2}\right)$ (fatelo voi)

$\frac{x}{2} = \pi \qquad x = 2\pi$

6) $y_6 = |f(x)| = \begin{cases} f & \text{se } f \geq 0 \\ -f & \text{se } f < 0 \end{cases}$

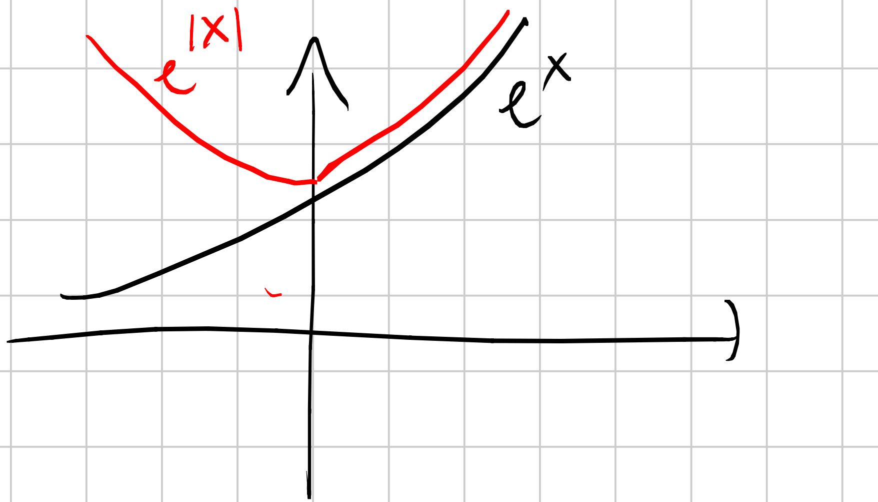


7) $y_7 = f(|x|) = \begin{cases} f(x) & x \geq 0 \\ f(-x) & x < 0 \end{cases}$



funzione pari

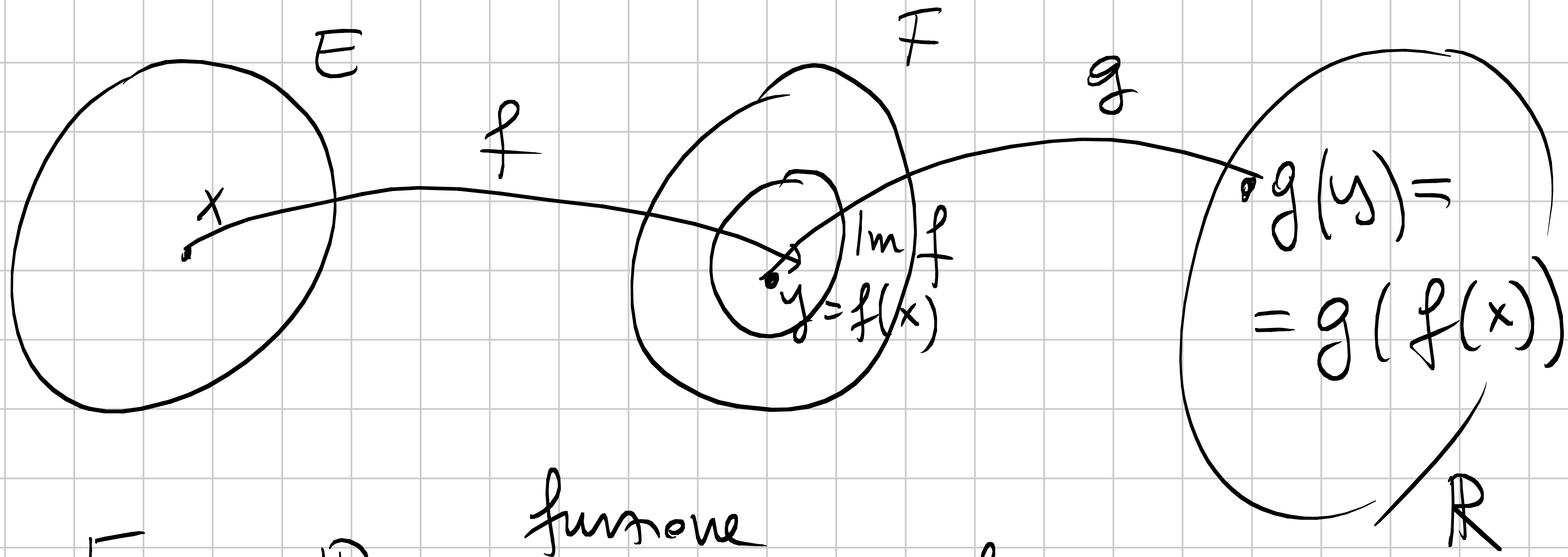
$e^{|x|}$



Funzioni composte

$f : E \to \mathbb{R}$, $g : F \to \mathbb{R}$

t.c. $\operatorname{Im} f \subseteq F$



$h : E \longrightarrow \mathbb{R}$ funzione composta di f e g

$h = g \circ f$ è definita così:

$$h(x) = (g \circ f)(x) = g(f(x))$$

OSS. $k(x) = (f \circ g)(x) = f(g(x))$

si può fare se

$\operatorname{Img} \subseteq E$

In generale

$$f \circ g \neq g \circ f$$

OSS. si possono comporre anche più di due funzioni

$$((f \circ g) \circ \tau)(x) = f(g(\tau(x)))$$

$$= f \circ (g \circ \tau)$$

si verifica che

es. $f(x) = x^2 + 1$ $\qquad$ $\operatorname{Im} f \subseteq \operatorname{Dom} g$

$g(x) = \sqrt{x}$ $\qquad$ $[1, +\infty) \subseteq [0, +\infty)$

$(g \circ f)(x)$

$x \xrightarrow[f]{} (x^2 + 1) \longrightarrow \sqrt{x^2+1} = g \circ f\,(x)$

$(f \circ g)(x)$ $\qquad$ $x \xrightarrow[g]{} \sqrt{x}$ $\qquad$ $\operatorname{Im} g \subseteq \operatorname{Dom} f$

$[0, +\infty) \subseteq \mathbb{R}$

$$x \xrightarrow[g]{} \sqrt{x} \xrightarrow[f]{} x+1 = f \circ g\,(x)$$

$$f \circ g \neq g \circ f.$$

es. $f(x) = -x^2$ $\qquad$ $\operatorname{Im} f = (-\infty, 0]$

$g(x) = \sqrt[4]{x}$ $\qquad$ $\operatorname{Dom} g = [0, +\infty)$

$\operatorname{Im} f \subseteq \operatorname{Dom} g$ NO!

$$g \circ f(x)$$

$$x \xrightarrow[f]{} -x^2 \xrightarrow[g]{} \text{? NO}$$

$f \circ g(x)$ $\qquad x \xrightarrow[g]{} \sqrt[4]{x} \xrightarrow[f]{} -\sqrt{x}$

$\mathrm{Im}\, g \subseteq \mathrm{Dom} f.$

es. 3

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 3 & x < 0 \end{cases} \qquad g(x) = \begin{cases} x^2 & x > 1 \\ -x & x \leq 1 \end{cases}$$

$f \circ g(x)$

$$x \xrightarrow{g} \begin{cases} x > 1 \longrightarrow x^2 \\ x \leq 1 \longrightarrow -x \end{cases}$$

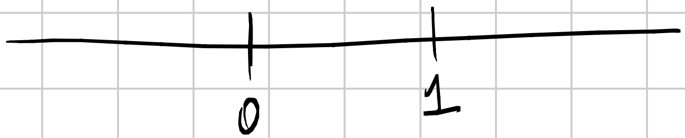
$x > 1$ $\qquad \underbrace{x^2}_{\geq 0} \xrightarrow[f]{} 1$

$$f \circ g(x) = \begin{cases} 1 & x > 1 \\ 3 & 0 < x \leq 1 \\ 1 & x \leq 0 \end{cases}$$

$x > 1$

$$f = \begin{cases} 1 & x \geq 0 \\ 3 & x < 0 \end{cases}$$

$x \leq 1$ $\qquad x \xrightarrow[g]{} -x \xrightarrow[f]{}$



$0 < x \leq 1 \implies -x < 0 \implies f = 3$

$x \leq 0 \implies -x \geq 0 \implies f = 1$

Per cose $\quad g \circ f(x) = \begin{cases} -1 & x \geq 0 \\ g & x < 0 \end{cases}$

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es. $\quad f$ crescente, g crescente $\implies g \circ f$ è crescente

f crescente, g decrescente $\implies g \circ f$ è decrescente

f decrescente, g decrescente $\implies g \circ f$ è crescente es.

$$e^{(\ldots\ldots)}$$

1) $x_1 > x_2 \Rightarrow \underbrace{f(x_1)}_{y_1} \overset{f \text{ crescente}}{\geq} \underbrace{f(x_2)}_{y_2}$

$y_1 \geq y_2 \underset{g \text{ è crescente}}{\Longrightarrow} g(y_1) \geq g(y_2)$

$$\Downarrow$$

$$g(f(x_1)) \geq g(f(x_2))$$

es. $\log(\operatorname{senh} x^3)$ composizione di f. crescenti

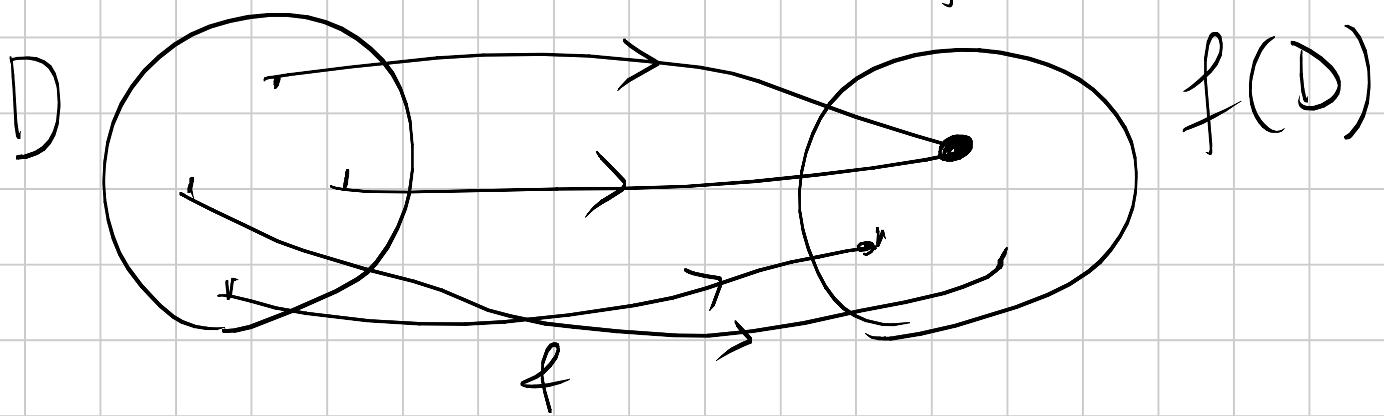
$\Rightarrow$ crescente

per x t.c.

es $\quad f(x) = \dfrac{1}{\log(x)}$ $\quad \log x > 0$

$\longrightarrow$ decrescente

Funzioni invertibili e f. inverse



$$f : D \longrightarrow f(D) \qquad , \qquad D,\ f(D) \subseteq \mathbb{R}$$
$$x \longrightarrow f(x)$$

Def. f è <u>invertibile in D</u> se $\forall\, y \in f(D)$ $\exists$ <u>uno e un solo</u> $x \in D$ t.c. $f(x) = y$

$$y \in f(D) \longrightarrow x \in D \text{ t.c. } f(x) = y$$

$$f^{-1} : f(D) \longrightarrow D$$
$$y \longrightarrow x$$

funzione inversa di f

$$x = f^{-1}(y) \iff y = f(x)$$

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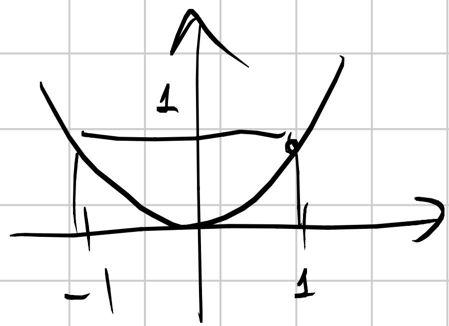
f iniettiva

- $\forall x_1, x_2 \in D, \ x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$
- $\forall x_1, x_2 \in D \ f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
- $\forall y \in f(D) \ \exists! \ x : \ f(x) = y$

<u>oss.</u> Per essere invertibile deve essere iniettiva.

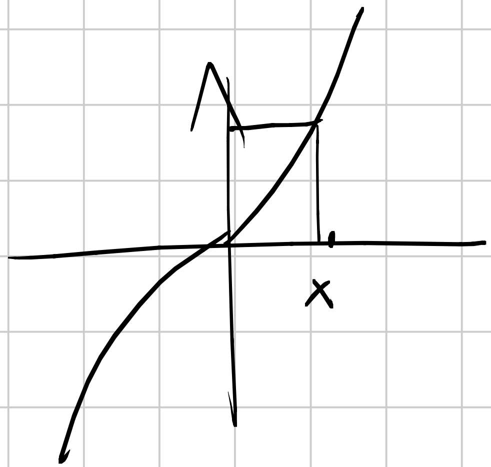
es.

$f(x) = x^2$



$x_1^2 = x_2^2 \Rightarrow x_1 = x_2$
$x_1 = -x_2$

$f(x) = x^3$



$x_1^3 = x_2^3$

$\Downarrow$

$x_1 = x_2$