Esercizi

1) $$\lim_{x \to +\infty} \underbrace{\sqrt{x}}_{+\infty} - \sqrt{\frac{x^2+1}{x}} = (*)$$

$$\sqrt{\frac{x^2+1}{x}} \to \frac{\left(1+\frac{1}{x^2}\right)}{\frac{1}{x}}, \quad \frac{1}{x^2} \to 0, \quad \left(1+\frac{1}{x^2}\right) \to 1, \quad \frac{1}{x} \to 0, \quad = +\infty$$

$x > 0$

$$\underbrace{\left(\sqrt{x} - \sqrt{\frac{x^2+1}{x}}\right)}_{} = \frac{\left(\sqrt{x} - \sqrt{\frac{x^2+1}{x}}\right)\left(\sqrt{x} + \sqrt{\frac{x^2+1}{x}}\right)}{\left(\sqrt{x} + \sqrt{\frac{x^2+1}{x}}\right)} =$$

$$\frac{x - \frac{x^2+1}{x}}{\sqrt{x} + \sqrt{\frac{x^2+1}{x}}} = \frac{\cancel{x^2} - \cancel{x^2} - 1}{x\left(\sqrt{x} + \sqrt{\frac{x^2+1}{x}}\right)}$$

$$(*) = \lim_{x \to +\infty} \frac{-1}{x\left(\sqrt{x} + \sqrt{\frac{x^2+1}{x}}\right)} = 0$$

2) $$\lim_{x \to -\infty} \underbrace{\sqrt[3]{\underbrace{x^2+1}_{\to +\infty}}}_{\to +\infty} - \underbrace{\sqrt[3]{\underbrace{2x^2}_{\to +\infty}}}_{\to +\infty} =$$

$$\frac{\left(\sqrt[3]{x^2+1} - \sqrt[3]{2x^2}\right)\left(\sqrt[3]{(x^2+1)^2} + \sqrt[3]{(x^2+1)\,2x^2} + \sqrt[3]{4x^4}\right)}{\left(\sqrt[3]{(x^2+1)^2} + \sqrt[3]{(x^2+1)\,2x^2} + \sqrt[3]{4x^4}\right.} =$$

$$= \frac{(x^2+1) - 2x^2}{\sqrt[3]{(x^2+1)^2} + \sqrt[3]{(x^2+1)\,2x^2} + \sqrt[3]{4x^4}} =$$

$$= \frac{1 - x^2}{|x|^{4/3}\left(\underbrace{\sqrt[3]{\left(1+\frac{1}{x^2}\right)^2}}_{\to 1} + \sqrt[3]{\underbrace{\left(1+\frac{1}{x^2}\right)2}_{\to 1}} + \underbrace{\sqrt[3]{4}}_{\sqrt[3]{4}}\right)} =$$

$$= \frac{\frac{1}{x^2} - 1}{\underbrace{\frac{1}{|x|^{2/3}}}_{\to 0}\ \underbrace{\left(- - - - - - - \right)}_{\to 2+\sqrt[3]{4}}} \qquad (<0)$$

>0 $\qquad$ >0

$$\lim_{x \to -\infty} \sqrt[3]{x^2+1} - \sqrt[3]{2x^2} = -\infty$$

3) $\lim_{x \to 0} \dfrac{1-\cos^3 x}{x \, tg\, x}$ (*)

$1 - \cos x$

$$\frac{1-\cos^3 x}{x\, tg\, x} = \frac{(1-\cos x)\overbrace{(1+\cos x + \cos^2 x)}^{3}}{x\, tg\, x}$$

$$(*) = 3 \lim_{x \to 0} \frac{(1-\cos x)}{x\, tg\, x} =$$

$$= 3 \lim_{x \to 0} \underbrace{\boxed{\frac{(1-\cos x)}{x^2}}}_{\frac{1}{2}} \; \frac{\cancel{x} \cdot x}{\cancel{x}\, tg\, x} = \frac{3}{2}$$

$\frac{x}{tg\, x} \to 1$

4) $$\lim_{x \to 3} \frac{2x-6}{\sin(\pi x)} =$$

$\sin \pi x =$

$$= \sin(\underbrace{\pi x - 3\pi} + \underbrace{3\pi}) = -\sin(\pi x - 3\pi)$$

$$= \lim_{x \to 3} -\frac{2}{\pi} \frac{\pi(x-3)}{\sin(\pi(x-3))}$$

$$\pi(x-3) = t$$

$$x \to 3 \Rightarrow t \to 0$$

$$\lim_{t \to 0} \frac{\sin t}{t} = 1$$

$$\lim_{x \to 3} \frac{\pi(x-3)}{\sin[\pi(x-3)]} =$$

$$\lim_{t \to 0} \frac{t}{\sin t} = \lim_{t \to 0} \frac{1}{\frac{\sin t}{t}} = 1$$

$$= -\frac{2}{\pi}$$

5) $\lim_{x \to +\infty} \underset{\downarrow\atop\infty}{x} \underbrace{\left(\frac{\pi}{2} - \arctan x\right)}_{\downarrow\atop 0} =$

$x = \operatorname{tg} t \qquad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$t \to \frac{\pi}{2}^-$

$= \lim_{t \to \frac{\pi}{2}^-} \underbrace{\operatorname{tg} t}_{\downarrow\atop +\infty} \underbrace{\left(\frac{\pi}{2} - t\right)}_{\downarrow\atop 0} =$

$= \lim_{t \to \frac{\pi}{2}^-} \frac{\boxed{\sin t} \to 1}{\cos t} \left(\frac{\pi}{2} - t\right) =$

$= \lim_{t \to \frac{\pi}{2}^-} \frac{\boxed{\frac{\pi}{2} - t} \to 0}{\boxed{\cos t} \to 0} = \lim_{t \to \frac{\pi}{2}^-} \frac{\frac{\pi}{2} - t}{\sin\left(\frac{\pi}{2} - t\right)} =$

$\frac{\pi}{2} - t = u \qquad t \to \frac{\pi}{2}^- \Rightarrow u \to 0$

$= \lim_{u \to 0} \frac{u}{\sin u} = 1$

6) $\lim_{x \to +\infty} \dfrac{x + a x^2 - 4x^2}{1 + 3x + 2x^2}$ * $=$ $a \in \mathbb{R}$

$$\frac{x + (a-4)x^2}{1 + 3x + 2x^2} =$$

$$= \frac{\underset{0}{\frac{1}{x}} + (a-4)}{\underset{0}{\frac{1}{x^2}} + \underset{0}{\frac{3}{x}} + 2}$$

$$= \frac{(a-4)}{2}$$

8) $\lim_{x \to 0} \frac{\sin(|x|^3)}{|x|^a} =$ $\qquad a \in \mathbb{R}$

$$\lim_{x\to 0} |x|^a = \begin{cases} 0 & \text{se } a > 0 \\ 1 & \text{se } a = 0 \\ +\infty & \text{se } a < 0 \end{cases}$$

- se $a = 0$ $\qquad \lim_{x\to 0} \frac{\sin |x|^3}{1} = 0$

- se $a < 0$ $\qquad \lim_{x\to 0} \underbrace{\frac{1}{|x|^a}}_{\to 0} \underbrace{\sin |x|^3}_{\to 0} = 0$

$+\infty$ $\to 0$

- se $a > 0$ $\qquad \lim_{x\to 0} \frac{\sin |x|^3}{|x|^a} = \frac{0}{0}$

$$= \lim_{x\to 0} \frac{|x|^3}{|x|^a} \frac{\sin |x|^3}{|x|^3} = \lim_{x\to 0} |x|^{3-a} \boxed{\frac{\sin |x|^3}{|x|^3}}$$

$\boxed{a > 0}$ $\qquad \to 1$

$$= \lim_{x\to 0} |x|^{3-a} = \begin{cases} 0 & \text{se } 3-a > 0 \\ 1 & \text{se } 3-a = 0 \\ +\infty & \text{se } 3-a < 0 \end{cases}$$

$$= \begin{cases} 0 & a < 3 \\ 1 & a = 3 \\ +\infty & a > 3 \end{cases}$$

Esercizi

1) $\lim_{x \to +\infty} \sqrt{x} - \sqrt{\frac{x^2+1}{x}} =$

2) $\lim_{x \to -\infty} \sqrt[3]{x^2+1} - \sqrt[3]{2x^2} =$

3) $\lim_{x \to 0} \frac{1-\cos^3 x}{x \operatorname{tg} x}$

4) $\lim_{x \to 3} \frac{2x-6}{\sin(\pi x)}$

5) $\lim_{x \to +\infty} x \left(\frac{\pi}{2} - \arctan x \right)$

6) $\lim_{x \to +\infty} \frac{\left(\frac{\pi}{2} - \arctan x^2 \right)}{\left(1 - \cos \frac{1}{x} \right)}$

7) $\lim_{x \to +\infty} \frac{x + a x^2 - 4x^2}{1 + 3x + 2x^2}$ $\qquad a \in \mathbb{R}$

8) $\lim_{x \to 0^+} \frac{x^a + x^4}{x^2} =$ $\qquad a \in \mathbb{R}$

9) $\lim_{x \to 0} \frac{\sin |x|^3}{|x|^a} =$ $\qquad a \in \mathbb{R}$

Altri limiti notevoli

- $\lim_{x \to +\infty} \frac{x^\alpha}{a^x} = 0 \qquad \forall a > 1, \ \alpha \in \mathbb{R}$
- $\lim_{x \to +\infty} \frac{|\log_a x|^\alpha}{x^\beta} = 0 \qquad \forall a > 0, \ a \neq 1 \qquad \forall \alpha \in \mathbb{R} \ \forall \beta > 0$
- $\lim_{x \to -\infty} a^x |x|^\alpha = 0 \qquad \forall a > 1, \ \alpha \in \mathbb{R}$
- $\lim_{x \to 0^+} x^\beta |\log_a x|^\alpha \qquad \forall a > 1, \ \alpha \in \mathbb{R} \quad \beta > 0$

- $\lim_{x \to +\infty} \frac{x^\alpha}{a^x}$ $\quad a^x \to +\infty$ $\qquad \lim_{x \to +\infty} x^\alpha = \begin{cases} +\infty & \alpha > 0 \\ 1 & \alpha = 0 \\ 0 & \alpha < 0 \end{cases}$

$= 0$ se $\alpha \leq 0$ $\qquad \lim\limits_{\ldots} \ldots = 0$

se $\alpha > 0$ $\qquad \frac{\infty}{\infty}$

$$\lim_{x \to -\infty} a^x |x|^\alpha \qquad a > 1 \quad \alpha > 0$$

$$= \lim_{x \to -\infty} \frac{|x|^\alpha}{a^{-x}} = \lim_{x \to -\infty} \frac{|-x|^\alpha}{a^{-x}} =$$

$x = -t$ se $x \to -\infty \Rightarrow t \to +\infty$

$$= \lim_{t \to +\infty} \frac{|t|^\alpha}{a^t} = 0$$

$$\lim_{x \to 0^+} x^\beta |\log_a x|^\alpha \qquad \forall a > 1,\ \alpha \in \mathbb{R} \quad \beta > 0$$

$0 \, (+\infty)$ $(\alpha > 0)$

$$= \lim_{x \to 0^+} \frac{|\log_a x|^\alpha}{x^{-\beta}} = \lim_{x \to 0^+} \frac{\left|-\log_a \left(\frac{1}{x}\right)\right|^\alpha}{\left(\frac{1}{x}\right)^\beta} =$$

$$= \lim_{x \to 0^+} \frac{\left|\log_a \left(\frac{1}{x}\right)\right|^\alpha}{\left(\frac{1}{x}\right)^\beta} \qquad \frac{1}{x} = t$$

$$= \lim_{t \to +\infty} \frac{|\log_a (t)|^\alpha}{t^\beta} = 0$$

Esercizi

1) $$\lim_{x \to +\infty} \frac{\log_4 x}{3^x}$$

2) $$\lim_{x \to +\infty} \frac{2x^4 + 4a^x}{x^4 + x^3 + 1} \qquad a > 0$$

3) $$\lim_{x \to 0} \sqrt{|x|} \cdot \log_{\frac{1}{4}} |\sin x|$$

4) $$\lim_{x \to +\infty} 3^{\frac{x^2}{1-x}} \left(\frac{1}{1 - \cos \frac{1}{x}} \right)$$

1) $$\lim_{x\to+\infty} \frac{\log_4 x}{3^x} = \lim_{x\to+\infty} \underbrace{\frac{\log_4 x}{x}}_{\to 0} \cdot \underbrace{\frac{x}{3^x}}_{\to 0} = 0$$

2) $$\lim_{x\to+\infty} \frac{2x^4 + 4a^x}{x^4 + x^3 + 1}$$ $\boxed{a > 0}$

$$\lim_{x\to+\infty} a^x = \begin{cases} +\infty & a > 1 \\ 1 & a = 1 \\ 0 & 0 < a < 1 \end{cases}$$

$\boxed{0 < a \leq 1}$ $$\lim_{x\to+\infty} \frac{2 + 4\frac{a^x}{x^4}}{1 + \frac{1}{x} + \frac{1}{x^4}} = 2$$

$\frac{a^x}{x^4} \to 0$, $\frac{1}{x} \to 0$, $\frac{1}{x^4} \to 0$

$$\lim_{x\to+\infty} \frac{a^x}{x^4} = 0 \quad \text{se} \quad a \leq 1$$

$\boxed{a > 1}$

$$\lim_{x\to+\infty} \frac{2x^4 + 4a^x}{x^4 + x^3 + 1} = \lim_{x\to+\infty} \frac{2\frac{x^4}{a^x} + 4}{\frac{x^4}{a^x} + \frac{x^3}{a^x} + \frac{1}{a^x}} = +\infty$$

$\frac{x^4}{a^x} \to 0$, $\frac{x^3}{a^x} \to 0$, $\frac{1}{a^x} \to 0$

3) $$\lim_{x\to 0} \sqrt{|x|} \cdot \log_{\frac{1}{4}} \underbrace{|\sin x|}_{t} =$$

$\forall a > 0$

$$\lim_{t\to 0^+} t^a \log_{\frac{1}{4}} t = 0$$

$t = |\sin t|$

$$= \lim_{x\to 0} \underbrace{\frac{\sqrt{|x|}}{|\sin x|^a}}_{1} \underbrace{|\sin x|^a \log_{\frac{1}{4}} |\sin x|}_{0}$$

$$\lim_{u \to 0} \frac{\sin u}{u} = 1$$

$$= \lim_{x \to 0} \frac{\sqrt{|x|}}{|x|^{\alpha}} \underbrace{\left| \frac{|x|^2}{\sin x} \right|}_{\to 1} \cdot \underbrace{|\sin x| \log_{\frac{1}{4}} |\sin x|}_{\to 0} = 0$$

$$\frac{\sqrt{|x|}}{|x|^{\alpha}} \to \begin{cases} +\infty & se \ \alpha > \frac{1}{2} \\ 1 & se \ \alpha = \frac{1}{2} \\ 0 & se \ 0 < \alpha < \frac{1}{2} \end{cases}$$

4) $$\lim_{x \to +\infty} 3^{\frac{x^2}{1-x}} \left(\frac{1}{1 - \cos \frac{1}{x}} \right) = (*)$$

$$\lim_{x \to +\infty} \frac{3^{\frac{x^2}{1-x}}}{3^{-x}} = \lim_{x \to +\infty} 3^{\frac{x^2}{1-x} + x} = \lim_{x \to +\infty} 3^{\frac{\cancel{x^2} + x - \cancel{x^2}}{1-x}} = 3^{-1} = \frac{1}{3}$$

$$(*) = \lim_{x \to +\infty} \underbrace{\frac{3^{\frac{x^2}{1-x}}}{3^{-x}}}_{\frac{1}{3}} \cdot 3^{-x} \left(\frac{1}{1 - \cos \frac{1}{x}} \right)$$

$$= \frac{1}{3} \lim_{x \to +\infty} 3^{-x} x^2 \underbrace{\frac{\frac{1}{x^2}}{\left(1 - \cos \left(\frac{1}{x}\right)\right)}}_{\to 2} =$$

$$\frac{1}{x} = t$$

$$\lim_{t \to 0} \frac{(1 - \cos(t))}{(t^2)} = \frac{1}{2}$$

$$= \frac{2}{3} \lim_{x \to +\infty} 3^{-x} x^2 = \frac{2}{3} \lim_{x \to +\infty} \frac{x^2}{3^x} = 0$$

Osservazione (importante!)

$$\boxed{f(x)} = g(x)^{h(x)} \qquad \text{dom} f = \{ x \in \mathbb{R} ,\ g(x) > 0 \} \cap \text{dom}\, h$$

$$= a^{\log_a \left(g(x)^{h(x)} \right)} \qquad a \neq 1$$

$$\boxed{= a^{h(x) \log_a g(x)}}$$

forme indeterminate

$$0^0 \qquad 1^{\infty} \qquad \infty^0$$

5) $$\lim_{x \to 0^+} \left(x^2\right)^x =$$

6) $$\lim_{x \to +\infty} \left(x^3\right)^{\frac{1}{x}} =$$

7) $$\lim_{x \to +\infty} \left(\sqrt{x^2+1} - \sqrt{x^2} \right)^{\left(x^{\frac{3}{2}}\right)}$$

8) $$\lim_{x \to +\infty} \cos\left(\frac{x^2+1}{x+3} \right) - \cos\left(\frac{x^2}{x+3} \right)$$

9) $$\lim_{x \to +\infty} \sin\left(\sqrt[3]{x+1} \right) + \sin\left(\sqrt[3]{x} \right)$$

1) $$\lim_{x\to 0^+} (x^2)^x = \lim_{x\to 0^+} a^{2\overbrace{\boxed{x \log_a x}}^{\to 0}} = 1$$

2) $$\lim_{x\to +\infty} (x^3)^{\frac{1}{x}} = \lim_{x\to +\infty} a^{3\frac{1}{x}\log_a x} = 1$$

3) $$\lim_{x\to +\infty} \left(\sqrt{x^2+1} - \sqrt{x^2}\right)^{(x^{\frac{3}{2}})} \quad (*)$$

$$\lim_{x\to +\infty} \left(\sqrt{x^2+1} - \sqrt{x^2}\right) =$$

$$= \lim_{x\to +\infty} \frac{\cancel{x^2}+1-\cancel{x^2}}{\sqrt{x^2+1}+\sqrt{x^2}} = 0$$

$$(*) = \lim_{x\to +\infty} a^{\underbrace{\overbrace{x^{\frac{3}{2}}}^{\to +\infty} \underbrace{\log_a \underbrace{\left(\frac{1}{\sqrt{x^2+1}+\sqrt{x^2}}\right)}_{0}}_{-\infty}}_{-\infty}} = 0$$

$a > 1$

1^∞ ??

Limiti di successioni

$$\lim_{n \to +\infty} a_n = \lim_{n \to +\infty} f(n)$$

$$\boxed{\text{se } \exists \lim_{x \to +\infty} f(x) = \ell \in \mathbb{R}^* \Rightarrow \lim_{n \to +\infty} f(n) = \ell}$$

Esempi: 1) $\displaystyle \lim_{n \to \infty} \frac{2n^2 + n - 1}{(1-4n)^2} = \frac{1}{8}$

$$\lim_{x \to +\infty} \frac{2x^2 + \cancel{x} - 1}{(1-4x)^2} = \frac{1}{8}$$

2) $\displaystyle \lim_{n \to \infty} \frac{\sin n}{n} = 0$

$$\lim_{x \to +\infty} \frac{\sin x}{x}$$

$$0 \leq \left| \frac{\sin x}{x} \right| < \frac{1}{|x|}$$

3) $\displaystyle \lim_{n \to \infty} \left(\frac{1}{2}\right)^n \cdot n^2 \sin n$

4) $\displaystyle \lim_{n \to \infty} (-1)^n \frac{1}{n}$

5) $\displaystyle \lim_{n \to \infty} (-1)^n n$

Limiti da ricordare

- $\displaystyle \lim_{n \to \infty} \frac{a^n}{n!} = 0$
- $\displaystyle \lim_{n \to \infty} \frac{n!}{n^n} = 0$
- $\displaystyle \lim_{n \to \infty} \sqrt[n]{n} = 1$
- $\displaystyle \lim_{n \to \infty} \sqrt[n]{n!} = +\infty$

Numero di Nepero

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$

1) $\left(1 + \frac{1}{n}\right)^n$ è monotona crescente

$$\Rightarrow \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = l \begin{cases} \in \mathbb{R} \\ = +\infty \end{cases}$$

2) $2 < \left(1 + \frac{1}{n}\right)^n < 3 \quad \forall n$

$$\Rightarrow \quad 2 < \lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n < 3$$

3) $\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n$ è irrazionale

$2,71 \ldots\ldots$

e numero di Nepero

e^x $\qquad$ $\log_e x = \log x$

$\log_{10}$

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$$

Altri limiti notevoli

- $$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e$$
- $$\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = e \qquad 1^\infty$$
- $$\lim_{x \to 0} \left(1 + x\right)^{\frac{1}{x}} = e$$
- $$\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$$
- $$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$
- $$\lim_{x \to 0} \frac{a^x - 1}{x} = \log a$$
- $$\lim_{x \to 0} \frac{(1+x)^2 - 1}{x} = 2$$

– $$\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x =$$ sostituzione $x = -t$

$$= \lim_{t \to +\infty} \left(1 - \frac{1}{t}\right)^{-t} = \lim_{t \to +\infty} \left[\left(1 - \frac{1}{t}\right)^t\right]^{-1} =$$

$$= \lim_{t \to +\infty} \left[\left(\frac{t-1}{t}\right)^t\right]^{-1} = \lim_{t \to +\infty} \left(\frac{t}{t-1}\right)^t =$$

$$= \lim_{t \to +\infty} \left(\frac{t-1+1}{t-1}\right)^t = \lim_{t \to +\infty} \left(1 + \frac{1}{t-1}\right)^t =$$ sostituisco $n = t - 1$

$$= \lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^{n+1} = \lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n \left(1 + \frac{1}{n}\right) =$$

$$= e$$

- $\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$ $\quad x = \frac{1}{t}$

$x \to 0^+ \qquad t \to +\infty$

$x \to 0^- \qquad t \to -\infty$

$x \to 0^+$

$$\lim_{t \to +\infty} \left(1 + \frac{1}{t}\right)^t = e$$

$x \to 0^-$

$$\lim_{t \to -\infty} \left(1 + \frac{1}{t}\right)^t = e$$

- $$\underbrace{\lim_{x \to 0} \frac{\log(1+t)}{t} = \lim_{x \to 0} \log \underbrace{(1+t)^{\frac{1}{t}}}_{e}}_{} = 1$$

- $\lim_{x \to 0} \frac{e^x - 1}{x} =$ $\quad x = \log t$

$t = e^x$

$\lim_{t \to 1} \frac{t-1}{\log t} =$ $\quad t = 1 + t - 1$

$$= \lim_{t \to 1} \frac{\overbrace{t-1}^{u}}{\log(1 + \underbrace{t-1}_{u})} = \lim_{u \to 0} \frac{u}{\log(1+u)} = 1$$

- $$\lim_{x \to 0} \frac{a^x - 1}{x} = \lim_{x \to 0} \frac{e^{x \log a} - 1}{x} =$$

$$= \lim_{x \to 0} \boxed{\frac{e^{x \log a} - 1}{x \log a}} \cdot \log a \to \log a$$

$t = x \log a \to \frac{e^t - 1}{t}$

$\to 1$

$$\lim_{x \to 0} \frac{(1+x)^{\alpha} - 1}{x} = \qquad \alpha \in \mathbb{R}$$

$$= \lim_{x \to 0} \frac{e^{\alpha \log(1+x)} - 1}{x} = \qquad \lim_{t \to 0} \frac{e^{t} - 1}{t}$$

$$= \lim_{x \to 0} \underbrace{\boxed{\frac{e^{\alpha \log(1+x)} - 1}{\alpha \log(1+x)}}}_{\to 1} \cdot \alpha \underbrace{\boxed{\frac{\log(1+x)}{x}}}_{\to 1} = \alpha$$

$t = \alpha \log(1+x)$

Esercizi

1) $\lim_{x\to+\infty}\left(\frac{x+1}{x-1}\right)^x$

2) $\lim_{x\to 0}(\cos x)^{\frac{1}{x^2}}$

3) $\lim_{x\to 0}\frac{\sqrt{1+\sin x}-1}{x}$

1) $$\lim_{x\to+\infty}\left(\frac{x+1}{x-1}\right)^x = \lim_{x\to+\infty}\left(1+\underbrace{\frac{x+1}{x-1}-1}_{\downarrow\, 0}\right)^x$$

$$= \lim_{x\to+\infty}\left(1+\boxed{\frac{2}{x-1}}\right)^x$$

t

$\frac{1}{t}$

$$= \lim_{x\to+\infty}\left[\left(1+\frac{2}{x+1}\right)^{\frac{x-1}{2}}\right]^{\underbrace{\frac{2}{x-1}\cdot x}_{\downarrow\, 2}} =$$

$t = \frac{x-1}{2}$

$$\lim_{t\to+\infty}\left(1+\frac{1}{t}\right)^t = e$$

$$= e^2$$

2) $$\lim_{x\to 0}(\cos x)^{\frac{1}{x^2}} =$$

$$= \lim_{x\to 0}\underbrace{\left[\left(1+\underbrace{\cos x-1}_{\downarrow\, 0}\right)^{\frac{1}{\cos x-1}}\right]}_{e}^{\boxed{(\cos x-1)\cdot\frac{1}{x^2}}} = e^{-\frac{1}{2}}$$

$$(\cos x-1)\cdot\frac{1}{x^2} \to -\frac{1}{2}$$

$$-\underbrace{\frac{(1-\cos x)}{x^2}}_{\frac{1}{2}}$$

$\cos x-1 = \frac{1}{t}$

3) $\lim_{x \to 0} \frac{\sqrt{1+\sin x} - 1}{x} = \lim_{x \to 0} \frac{(1+\sin x)^{\frac{1}{2}} - 1}{x} =$

$$\lim_{t \to 0} \frac{(1+t)^{\alpha} - 1}{t} = \alpha$$

$$= \lim_{x \to 0} \underbrace{\boxed{\frac{(1+\sin x)^{\frac{1}{2}} - 1}{\sin x}}}_{\sin x = t} \underbrace{\boxed{\frac{\sin x}{x}}}_{\to 1} = \frac{1}{2}$$

$$\lim_{t \to 0} \frac{(1+t)^{\frac{1}{2}} - 1}{t} = \frac{1}{2}$$

Funzioni continue

Definizione $f : X \subseteq \mathbb{R} \to \mathbb{R}$ $\boxed{x_0 \in X}$

se x_0 è di accumulazione per X diremo che f è continua in x_0 se $\lim_{x \to x_0} f(x) = f(x_0)$

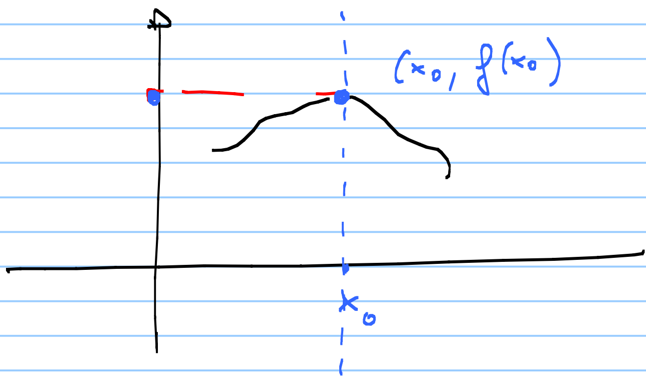
$A \subseteq X$ diremo che f è continua in A se è continua in x_0, $\forall x_0 \in A$.

Osservazione tutte le funzioni elementari sono continue nel loro dominio

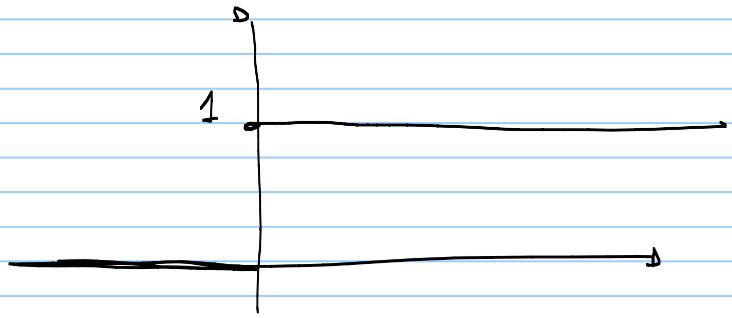
x^2, $|x|$, a^x, $\log_a x$, $\sin x$, $\cos x$, $\operatorname{tg} x$, $\arcsin x$, $\arccos x$, ...

Definizione $f : X \subseteq \mathbb{R} \to \mathbb{R}$, $x_0 \in X$, x_0 di accumulazione per X, si dice continua a dx (sx) in x_0 se $\lim_{x \to x_0^+} f(x) = f(x_0)$
$(x \to x_0^-)$

Definizione $f : X \subseteq \mathbb{R} \to \mathbb{R}$, $x_0 \in X$, x_0 di accumulaz. per X, x_0 si dice punto di discontinuità per f se f non è continua in x_0



$$f(x) = \begin{cases} 1 & \text{se } x \geq 0 \\ 0 & \text{se } x < 0 \end{cases}$$

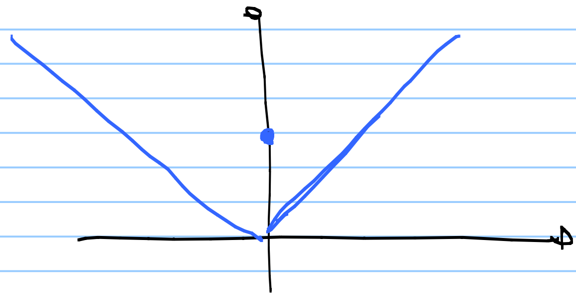


$$\left.\begin{array}{l} \lim_{x \to 0^-} f(x) = 0 \\ \lim_{x \to 0^+} f(x) = 1 \end{array}\right] \longrightarrow \nexists \lim_{x \to 0} f(x)$$

f non è continua in 0

$\lim_{x \to 0^+} f(x) = 1 = f(0)$ è continua a dx in 0

$$f(x) = \begin{cases} |x| & \text{se } x \neq 0 \\ 1 & \text{se } x = 0 \end{cases}$$



$\lim_{x \to 0} f(x) = 0 \neq f(0) = 1$

f non è continua

(né a dx né a sx)

Continuità delle funzioni monotone

$f : I \subseteq \mathbb{R} \to \mathbb{R}$ I intervallo

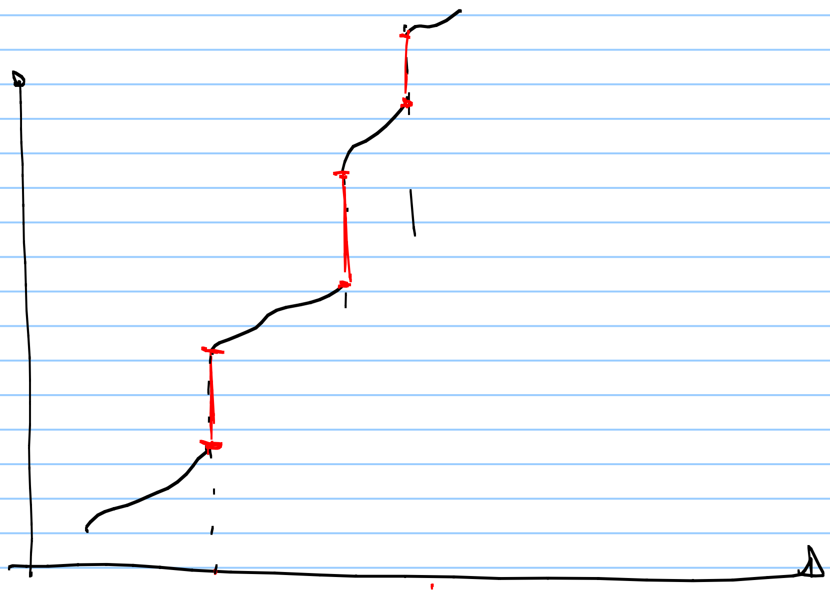
f sia monotona crescente (decrescente)

x_0 interno a I

$$\lim_{x \to x_0^-} f(x) = \sup_{\{x < x_0\}} f(x) \leq f(x_0)$$

$$\lim_{x \to x_0^+} f(x) = \inf_{\{x > x_0\}} f(x) \geq f(x_0)$$

$$\lim_{x \to x_0^-} f(x) \leq f(x_0) \leq \lim_{x \to x_0^+} f(x)$$



Questo implica che le funzioni monotone hanno al più un insieme NUMERABILE di punti di discontinuità.

Teorema (della permanenza del segno)

$f: X \subseteq \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}$, x_0 sia di acc. per X

se f è continua in x_0 e $f(x_0) > 0$ (< 0)

Allora esiste $\mathcal{U}$ intorno di x_0 t.c. $f(x) > 0$ (< 0)

$\forall x \in X \cap \mathcal{U}$

Teorema (sull' algebra delle funzioni continue)

$f, g: X \subseteq \mathbb{R} \to \mathbb{R}$ x_0 di acc. per X, $x_0 \in X$

f, g siano continue in x_0 allora

1) $c \cdot f$ è continua in x_0 , $\forall c \in \mathbb{R}$
2) $f + g$ è continua in x_0
3) $f \cdot g$ è continua in x_0
4) se $g(x_0) \neq 0$ $\Rightarrow$ $\frac{1}{g}$, $\frac{f}{g}$ sono continue in x_0