Soluzioni esercizi su integrali definiti e Indefiniti

1) $\displaystyle\int_{-2}^{2} |t+1| \operatorname{arctg}|t|\,dt = \int_{-2}^{-1} -(t+1)\operatorname{arctg}(-t)\,dt + \int_{-1}^{0}(t+1)\operatorname{arctg}(-t)\,dt$

$\displaystyle + \int_{0}^{2}(t+1)\operatorname{arctg} t\,dt = \int_{-2}^{-1}(t+1)\operatorname{arctg} t\,dt - \int_{-1}^{0}(t+1)\operatorname{arctg} t\,dt +$

$\displaystyle + \int_{0}^{2}(t+1)\operatorname{arctg} t\,dt$ (nota: $\operatorname{arctg}(-t) = -\operatorname{arctg} t$)

Calcolo $\displaystyle\int (t+1)\operatorname{arctg} t\,dt = \underbrace{\int t\operatorname{arctg} t\,dt}_{\text{per parti}} + \underbrace{\int 1\cdot\operatorname{arctg} t\,dt}_{\text{per parti}} =$

$\displaystyle = \frac{t^2}{2}\operatorname{arctg} t - \int \frac{t^2}{2}\,\frac{1}{1+t^2}\,dt + t\operatorname{arctg} t + \int \frac{t}{1+t^2}\,dt =$

$\displaystyle = \frac{t^2}{2}\operatorname{arctg} t - \frac{1}{2}\int \frac{t^2+1-1}{t^2+1}\,dt + t\operatorname{arctg} t - \frac{1}{2}\int \frac{2t}{\underbrace{1+t^2}_{=y}}\,dt$

$\displaystyle = \frac{t^2}{2}\operatorname{arctg} t - \frac{1}{2}\int\left(1 - \frac{1}{t^2+1}\right)dt + t\operatorname{arctg} t - \frac{1}{2}\log(1+t^2)$

$\displaystyle = \frac{t^2}{2}\operatorname{arctg} t - \frac{1}{2}t + \frac{1}{2}\operatorname{arctg} t + t\operatorname{arctg} t - \frac{1}{2}\log(1+t^2) + K =$

$=: G(t) + K$

e con tale funzione si riescono a calcolare i 3 integrali definiti

$I = G(-1) - G(-2) - (G(0) - G(1)) + G(2) - G(0) =$

$= -2G(0) + G(-1) + G(1) - G(-2) + G(2) =$

$\displaystyle = -2(\cdot\, 0) + \frac{1}{2}\operatorname{arctg}(-1) + \frac{1}{2} + \frac{1}{2}\operatorname{arctg}(-1) -$

$\displaystyle - \operatorname{arctg}(-1) - \frac{1}{2}\log 2 + \dots\dots\dots$

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2) $\displaystyle\int_{-4}^{4}\frac{1}{x^2+4}\,dx \underset{\text{pari}}{=} 2\int_{0}^{4}\frac{1}{x^2+4} = \frac{2}{4}\int_0^4 \frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx = \frac{2}{2}\operatorname{arctg}\left(\frac{x}{2}\right)\Big|_0^4 = \operatorname{arctg} 2.$

3) $\displaystyle\int_{-1}^{1}\frac{1}{x^2-|2x|+2x+4}\,dx = \int_{-1}^{0}\frac{1}{x^2+4x+4}\,dx + \underbrace{\int_{0}^{1}\frac{1}{x^2+4}\,dx}_{\text{fatto sopra}}$

$\displaystyle\int_{-1}^{0}\frac{1}{x^2+4x+4}\,dx = \int_{-1}^{0}\frac{1}{(x+2)^2}\,dx = -\frac{1}{x+2}\Big|_{-1}^{0}$

$\displaystyle = -\frac{1}{2} - \left(-\frac{1}{1}\right) = 1 - \frac{1}{2} = \frac{1}{2}$

4) $\displaystyle\int \frac{1+x}{x^2+3x+2}\,dx =$

$x^2+3x+2 = 0$

$\Delta = 9 - 8 = 1$

$\displaystyle x = \frac{-3\pm 1}{2} = \begin{cases} -2 \\ -1 \end{cases}$

$x^2+3x+2=(x+2)(x+1)$

$$= \int \frac{\cancel{1+x}}{(x+2)\cancel{(x+1)}}\,dx = \int \frac{1}{x+2}\,dx = \log|x+2|+K.$$

5) $$\int x^3 e^{-x}\,dx \overset{\text{3 volte per PARTI}}{=} -e^{-x}\cdot x^3 + \int e^{-x}\cdot 3x^2\,dx = -e^{-x}x^3 - e^{-x}3x^2 + \int e^{-x}\cdot 6x\,dx = -e^{-x}x^3 - e^{-x}3x^2 - e^{-x}6x + \int e^{-x}\cdot 6\,dx = -e^{-x}x^3 - e^{-x}3x^2 - e^{-x}6x - 6e^{-x}$$
$$= e^{-x}\left(-x^3-3x^2-6x-6\right)+K.$$

6) $$\int \underbrace{x^3}_{\text{deriv}} \sinh x\,dx = \cosh x\, x^3 - \int 3x^2\cosh x\,dx = \cosh x\, x^3 - \sinh x\cdot 3x^2 + 3\int 2x\sinh x\,dx = \cosh x\,x^3 - 3x^2\sinh x + 6x\cosh x - 6\int \cosh x\,dx = \cosh x\,x^3 - 3x^2\sinh x + 6x\cosh x - 6\sinh x + K$$

7) $$\int \underbrace{2^x}_{\text{deriv}}\, \underbrace{\sin(3x)}_{\text{integro}} = -\frac{1}{3}\cos(3x)\cdot 2^x + \int \frac{1}{3}\underbrace{\cos(3x)}_{\text{integro}}\, 2^x\log 2\,dx =$$
$$= -\frac{2^x}{3}\cos(3x) + \frac{\log 2}{3}\left(\frac{1}{3}\sin(3x)\cdot 2^x - \int \frac{1}{3}\sin(3x)\,2^x\log 2\,dx\right)$$
$$= -\frac{2^x}{3}\cos(3x) + \frac{\log 2\cdot 2^x}{9}\sin(3x) - \frac{(\log 2)^2}{9}\int \sin(3x)\,2^x$$
$$\Rightarrow \left(1+\frac{(\log 2)^2}{9}\right)\int 2^x\sin(3x)\,dx = -\frac{2^x}{3}\cos(3x) + \frac{2^x\log 2}{9}\sin(3x)$$
$$\Rightarrow \int 2^x\sin(3x)\,dx = \frac{-\frac{2^x}{3}\cos(3x) + 2^x\frac{\log 2}{9}\sin(3x)}{1+\frac{(\log 2)^2}{9}} + K$$

8) $$\int \frac{1+3x}{x^2+x-2}\,dx =$$

$x^2+x-2=(x-1)(x+2)$

$\Delta = 1+8=9$

$x = \frac{-1\pm 3}{2} = \begin{cases} 1 \\ -2 \end{cases}$

$$= \int \frac{1+3x}{(x-1)(x+2)}\,dx$$ fratti semplici

$$\frac{1+3x}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2} = \frac{Ax+2A+Bx-B}{(\ )(\ )}$$

$A+B=3$ $\quad B=3-A$

$2A-B=1$ $\quad 2A-3+A=1$ $\quad 3A=4$ $\quad A=4/3$

$B = 3-\frac{4}{3} = \frac{5}{3}$

$$\frac{1+3x}{(x-1)(x+2)} = \frac{4}{3(x-1)} + \frac{5}{3(x+2)}$$

$$I = \frac{4}{3}\int \frac{1}{x-1}\,dx + \frac{5}{3}\int \frac{1}{x+2}\,dx = \frac{4}{3}\log|x-1| + \frac{5}{3}\log|x+2|$$

9) $\int \frac{\cos x}{1+\sin^2 x}\,dx = \int \frac{1}{1+t^2} =$ $\quad\quad \sin x = t$, $\cos x\,dx = dt$

$= \arctg(\sin x) + K$

10) $\int \frac{e^{1/x}}{x^3}\,dx = \int -e^t \cdot t\,dt =$ $\quad\quad \frac{1}{x} = t \quad -\frac{1}{x^2}dx = dt$

$= -\left(e^t t - \int e^t\,dt\right) = -e^t t + e^t =$

$= -\frac{e^{1/x}}{x} + e^{1/x} + K.$

11) $\int \underset{\text{si integra}}{x} \log^2 x\,dx \underset{\text{per parti}}{=} \frac{x^2}{2}\log^2 x - \int \frac{x^{\cancel{2}}}{\cancel{2}}\,\cancel{2}(\log x)\frac{1}{\cancel{x}}\,dx =$

$= \frac{x^2}{2}\log^2 x - \int \underset{\text{si integra}}{x} \log x\,dx = \frac{x^2}{2}\log^2 x - \left(\frac{x^2}{2}\log x - \int \frac{x^{\cancel{2}}}{2}\cdot\frac{1}{\cancel{x}}dx\right)$

$= \frac{x^2}{2}\log^2 x - \frac{x^2}{2}\log x + \frac{x^2}{4} + K.$

12) $\int_{-\pi}^{\pi} \underbrace{x^2|\sin x|}_{\text{f. pari}}\,dx = 2\int_0^{\pi} x^2 \sin x\,dx = 2\left(-x^2\cos x\right)\Big|_0^{\pi} +$

$+ 2\int_0^{\pi} 2x\cos x\,dx = -2x^2\cos x\Big|_0^{\pi} + 4x\sin x\Big|_0^{\pi} - 4\int_0^{\pi}\sin x\,dx =$

$= \left(-2x^2\cos x + 4x\sin x + 4\cos x\right)\Big|_0^{\pi} = 2\pi^2 - 4 - 4 =$

$= 2\pi^2 - 8$

13) $\int_{-\pi}^{\pi} x^2 \sin x\,dx = 0$ perché $f(x) = x^2 \sin x$ è dispari e l'intervallo di integrazione è simmetrico.

14) $\int x \arctg x\,dx \underset{\text{per parti}}{=} \frac{x^2}{2}\arctg x - \int \frac{x^2}{2}\,\frac{1}{1+x^2}\,dx =$

$= \frac{x^2}{2}\arctg x - \frac{1}{2}\int \frac{x^2+1-1}{1+x^2}\,dx = \frac{x^2}{2}\arctg x - \frac{1}{2}\int 1\,dx +$

$+ \frac{1}{2}\int \frac{1}{1+x^2}\,dx = \frac{x^2}{2}\arctg x - \frac{x}{2} + \frac{1}{2}\arctg x + K.$

15) $\int \frac{1+2x}{x^2+6x+9}\,dx = \int \frac{1+2x}{(x+3)^2}\,dx$ $\quad\quad (x+3)^2 = t$, $2(x+3)\,dx = dt$, $(2x+6)\,dx = dt$

$= \int \frac{2x+6-6+1}{(x+3)^2}\,dx = \int \frac{1}{t}\,dt - 5\int \frac{1}{(x+3)^2}\,dx =$

$= \log(x+3)^2 + \frac{5}{(x+3)} + K.$

16) $\int_{\pi/6}^{\pi/4} \frac{1}{\tg x \log(\sin x)}\,dx = \int_{\pi/6}^{\pi/4} \frac{\cos x}{\sin x \log(\sin x)}\,dx =$ $\quad\quad \sin x = t$, $\cos x\,dx = dt$

$$= \int_{1/2}^{\frac{\sqrt{2}}{2}} \frac{1}{t \log t}\, dt \underset{\log t = y}{=} \int_{\log \frac{1}{2}}^{\log\left(\frac{\sqrt{2}}{2}\right)} \frac{1}{y}\, dy = \log\left(\log \frac{\sqrt{2}}{2}\right) - \log\left(\log \frac{1}{2}\right) =$$

$$= \log\left(\frac{\log 2^{-1/2}}{\log 2^{-1}}\right) = \cdots = -\log 2$$

17) $$\int_0^1 \frac{e^t}{\sqrt{1+e^{2t}}}\, dt \underset{\substack{e^t = y \\ e^t dt = dy}}{=} \int_1^e \frac{1}{\sqrt{1+y^2}}\, dy =$$

$y = \operatorname{senh} x$
$dy = \cosh x\, dx$
$1 + y^2 = 1 + \operatorname{senh}^2 x = \cosh^2 x$

$$= \int_{\operatorname{settsenh} 1}^{\operatorname{settsenh} e} \frac{1}{\sqrt{\cosh^2 x}} \cosh x\, dx \underset{\substack{\cosh x > 0 \\ \forall x}}{=} \int_{\operatorname{settsenh} 1}^{\operatorname{settsinh} e} 1\, dx = \operatorname{settsinh} e - \operatorname{settsinh} 1$$

18) $$\int_0^1 \frac{1}{\sqrt{4x^2+1}}\, dx \underset{\substack{2x = \operatorname{senh} t \\ 2\, dx = \cosh t\, dt}}{=} \int_0^{\operatorname{settsenh} 2} \frac{1}{\cancel{\cosh t}} \frac{1}{2} \cancel{\cosh t}\, dt =$$

$$= \frac{1}{2} \operatorname{settsenh} 2.$$

19) $$\int_{-4}^{4} \sqrt{x^2 - |2x| + 2x + 4}\, dx = \int_{-4}^{0} \sqrt{x^2 + 4x + 4}\, dx + \int_0^4 \sqrt{x^2+4}\, dx$$

$$= \int_{-4}^{0} \sqrt{(x+2)^2}\, dx + \int_0^4 \sqrt{x^2+4} = \int_{-4}^{0} |x+2|\, dx + 2\int_0^4 \sqrt{\left(\frac{x}{2}\right)^2 + 1}\, dx$$

$\frac{x}{2} = \operatorname{senh} t$

$$= \int_{-4}^{-2} -(x+2)\, dx + \int_{-2}^{0} (x+2)\, dx + 2\int_0^{\operatorname{settsenh} 2} 2\cosh^2 t\, dt$$

$dx = 2\cosh t\, dt$

$$\int \cosh^2 t = \int \cosh t \cdot \cosh t = \operatorname{senh} t \cosh t - \int \operatorname{senh}^2 t\, dt =$$

$$= \operatorname{senh} t \cosh t - \int (\cosh^2 t - 1)\, dt = \operatorname{senh} t \cosh t + t - \int \cosh^2 t\, dt$$

$$\Rightarrow 2\int \cosh^2 t = \operatorname{senh} t \cosh t + t \quad \Rightarrow \int \cosh^2 t = \frac{\operatorname{senh} t \cosh t + t}{2}$$

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