Soluzioni integrali impropri

1) $\displaystyle\int_0^{+\infty} \frac{1}{e^x+1}\,dx$ $\qquad f(x)$

$f(x) \sim \frac{1}{e^x} = e^{-x}$

$\displaystyle\int_0^{+\infty} e^{-x}\,dx = \lim_{\omega\to+\infty}\left(-e^{-\omega}+1\right) = 1$ converge

• $\Rightarrow \displaystyle\int_0^{+\infty} f(x)\,dx$ converge

$e^x = t$
$e^x dx = dt$
$dx = \frac{1}{t}dt$

• $\displaystyle\int \frac{1}{e^x+1}\,dx = \int \frac{1}{(t+1)}\frac{1}{t}\,dt$

Fratti semplici

$\frac{A}{t+1} + \frac{B}{t} = \frac{At+Bt+B}{(t+1)\cdot t} = \frac{1}{t(t+1)} \quad \forall t$

$A+B=0$
$B=1 \qquad \Rightarrow A=-1$

$\displaystyle\int \frac{1}{(t+1)t}\,dt = \int -\frac{1}{t+1}\,dt + \int \frac{1}{t}\,dt = -\log|t+1| + \lg|t|$

$= \log\left(\frac{e^x}{e^x+1}\right)$

Quindi $\displaystyle\int_0^{+\infty} \frac{1}{e^x+1}\,dx = \lim_{\omega\to+\infty}\left[\log\left(\frac{e^\omega}{e^\omega+1}\right) - \log\left(\frac{1}{2}\right)\right] =$ $\quad \to 1$

$= -\log\left(\frac{1}{2}\right) = \log 2 \quad (>0\ !)$ giusto!

---

2) $\displaystyle\int_1^{+\infty} \frac{\log x + 3}{x^\alpha\left(1+7\lg x + 12\log^2 x\right)}\,dx \qquad \alpha\in\mathbb{R}$ $\qquad f(x)$

$f(x) \sim \frac{\cancel{\log} x}{x^\alpha\,(12\lg^{\cancel{2}} x)} = \frac{1}{12x^\alpha \log x} \qquad x\to+\infty$

$\displaystyle\int \frac{1}{x^\alpha \lg x}\,dx$ conv. $\Leftrightarrow \alpha>1$

Quindi $\displaystyle\int_1^{+\infty} f(x)\,dx$ conv. $\Leftrightarrow \alpha>1$

Per $\alpha=1$ calcolo

$\displaystyle\int_1^{+\infty} \frac{\lg x+3}{x\left(1+7\lg x+12\lg^2 x\right)}\,dx$

Primitiva

$\lg x = t$
$\frac{1}{x}dx = dt$

$\displaystyle\int \frac{\log x+3}{x\left(1+7\lg x+12\lg^2 x\right)}\,dx = \int \frac{t+3}{12t^2+7t+1}\,dt$

$12t^2+7t+1=0 \qquad \Delta = 49-48 = 1$

$t = \frac{-7\pm1}{24} = \begin{cases} -\frac{1}{3} \\ -1/4 \end{cases}$

$$12t^2+7t+1 = 12\left(t+\frac{1}{3}\right)\left(t+\frac{1}{4}\right) = (3t+1)(4t+1)$$

Quindi

$$\int \frac{t+3}{12t^2+7t+1}\,dt = \int \frac{t+3}{(3t+1)(4t+1)}\,dt$$ Fratti semplici

$$\frac{t+3}{(3t+1)(4t+1)} = \frac{A}{3t+1} + \frac{B}{4t+1} =$$

$$= \frac{4At+A+3Bt+B}{(3t+1)(4t+1)} \Rightarrow \begin{cases} 4A+3B=1 \\ A+B=3 \end{cases}$$

$B=3-A \qquad 4A+3(3-A)=1 \qquad 4A+9-3A=1$

$A=-8 \qquad B=3+8=11$

Quindi

$$\int \frac{t+3}{(3t+1)(4t+1)}\,dt = \int \frac{-8}{3t+1}\,dt + \int \frac{11}{4t+1}\,dt =$$

$$= -\frac{8}{3}\log|3t+1| + \frac{11}{4}\log|4t+1| =$$

$$= -\frac{8}{3}\log|3\log x+1| + \frac{11}{4}\log|4\log x+1|$$

$$\int_1^{+\infty} (\quad) = \lim_{\omega\to+\infty}\left(-\frac{8}{3}\log|3\log\omega+1| + \frac{11}{4}\log|4\log\omega+1|\right)$$

$$= \lim_{\omega\to+\infty} \log\left(\frac{(4\log\omega+1)^{11/4}}{(3\log\omega+1)^{8/3}}\right)$$

Poiché $\frac{11}{4} > \frac{8}{3}$ $\quad \frac{(4\log\omega+1)^{11/4}}{(3\log\omega+1)^{8/3}} \to +\infty$

$$\Rightarrow \lim_{\omega\to+\infty} \log(\quad) = +\infty$$

$\Rightarrow$ l'integrale diverge. (come ci si aspettava dal punto precedente).

---

3) $$\int_0^{\pi/2} \frac{\operatorname{sen} x}{\log(1+\sqrt{x})(e^{x^\alpha}-1)}\,dx$$

$f(x)$ può essere illimitata per $x\to 0^+$.

se $x\to 0^+$

$\operatorname{sen} x \sim x$

$\log(1+\sqrt{x}) \sim \sqrt{x}$

$e^{x^\alpha}-1 \sim x^\alpha$

$$f(x) \sim \frac{x}{\sqrt{x}\cdot x^\alpha} = \frac{1}{x^{\alpha-\frac{1}{2}}}$$

$$\int_0^{\pi/2} f(x)\,dx \text{ conv.} \Leftrightarrow \alpha-\frac{1}{2} < 1 \Leftrightarrow \alpha < \frac{3}{2}$$

---

4) $$\int_3^{7/2} \frac{\operatorname{sen}[(x-3)^\alpha]\,(x-4)}{(x-3)^2\log[(x-3)^2]}\,dx =$$

$x-3=t$

$$= \int_0^{1/2} \frac{\operatorname{sen}(t^\alpha)(t-1)}{t^2 \log(t^2)}\, dt$$

$$f(t) = \frac{\operatorname{sen}(t^\alpha)(t-1)}{t^2 \log(t^2)} \underset{t\to 0}{\sim} \frac{t^\alpha (-1)}{2t^2 \log t} = -\frac{1}{2t^{2-\alpha}\log t}$$

per il confronto asintotico

$\int_0^{1/2} f(t)\,dt$ converge sse $\int_0^{1/2} \frac{1}{t^{2-\alpha}\log t}\,dt$ converge

cioè sse $2-\alpha<1$

$\iff \boxed{\alpha > 1}$

5) $\int_0^1 \frac{x \arctg x}{x^\alpha}\,dx$ $\qquad f(x) \underset{x\to 0}{\sim} \frac{x\cdot x}{x^\alpha} = \frac{1}{x^{\alpha-2}}$

converge sse $\alpha-2<1 \iff \alpha<3$

$\int_1^{+\infty} \frac{x \arctg x}{x^\alpha}\,dx$ $\qquad f(x) \underset{x\to+\infty}{\sim} \frac{x \frac{\pi}{2}}{x^\alpha} = \frac{\pi}{2}\frac{1}{x^{\alpha-1}}$

converge sse $\alpha-1>1 \iff \alpha>2$

$$\int_0^1 x \arctg x\,dx \underset{\text{per parti}}{=} \frac{x^2}{2}\arctg x \Big/_0^1 - \int_0^1 \frac{x^2}{2}\frac{1}{1+x^2}\,dx =$$

$$= \frac{\pi}{8} - \frac{1}{2}\int_0^1 \frac{x^2+1-1}{x^2+1}\,dx = \frac{\pi}{8} - \frac{1}{2}\cdot 1 + \frac{1}{2}\int_0^1 \frac{1}{x^2+1}\,dx =$$

$$= \frac{\pi}{8} - \frac{1}{2} + \frac{1}{2}\arctg x\Big/_0^1 =$$

$$= \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8} = \frac{\pi}{4} - \frac{1}{2}$$

Altri esercizi (assegnati a lezione)

• $\int_1^{+\infty} \frac{e^{1/x}-1}{\log x}\,dx$ $\qquad f(x) \sim \frac{1}{x\log x}$ diverge

• $\int_1^{+\infty} \frac{\operatorname{sen} 1/x}{x^\alpha + x^2}\,dx$ $\qquad f(x) \sim \frac{1}{x^{\alpha+1}+x^3} \sim \begin{cases} \frac{1}{x^{\alpha+1}} & \alpha+1>3 \ (\alpha>2) \\ \frac{1}{x^3} & \alpha+1<3 \ (\alpha<2) \\ \frac{1}{2x^3} & \alpha+1=3 \ (\alpha=2) \end{cases}$

Quindi

$\alpha>2 \Rightarrow$ converge se $\alpha+1>1 \Rightarrow \alpha>0$. Converge $\forall \alpha>2$

$\alpha<2 \Rightarrow$ converge

$\alpha=2 \Rightarrow$ converge

$\Rightarrow \boxed{\text{R.} = \text{converge } \forall \alpha \in \mathbb{R}}$

• $\int_1^{+\infty} \frac{x^{2\alpha}}{x^2+1}\,dx$ $\qquad$ 1) det. α t.c. $f\to 0$ per $x\to+\infty$

$f(x) \sim \frac{1}{x^{2-2\alpha}} \to 0$ sse $2-2\alpha>0$

cioè $\boxed{\alpha < 1}$

b) det α t.c. l'integrale converge

$f(x) \sim \frac{1}{x^{2-2\alpha}}$ conv. sse $2-2\alpha > 1$

$2\alpha < 1$

sse $\boxed{\alpha < \frac{1}{2}}$

$\alpha = 1/2$

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx \qquad \frac{1}{2}\int_1^{\infty} \frac{2x}{x^2+1} = \frac{1}{2}\log\left(x^2+1\right)$$

$$\int_1^{+\infty} \frac{x}{x^2+1}\,dx = \lim_{x\to+\infty} \frac{1}{2}\log\left(x^2+1\right) - \frac{1}{2}\log(2) = +\infty$$

diverge per $\alpha = \frac{1}{2}$

- $\displaystyle\int_0^1 \frac{\operatorname{sen}(x^\alpha)}{x^2\cos x}\,dx$ $\qquad f(x) \underset{x\to 0}{\sim} \frac{x^\alpha}{x^2\cdot 1} = \frac{1}{x^{2-\alpha}}$

conv. sse $2-\alpha < 1$

sse $\alpha > 1$.