ESERCIZI primi 2 CAP. – Cenni di soluzione

1. $E = \{ (-1)^n + 4 , n \in \mathbb{N} \}$ $\exists \min E = 3$, $\max E = 5$

2. $E = \left\{ 2 + (-1)^n \cdot \frac{1}{n+1} , n \in \mathbb{N} \right\}$: n pari : $2 + \frac{1}{n+1}$ decresc.

n dispari : $2 - \frac{1}{n+1}$ cresc.

sui pari : assume max per $n = 0$: 3

inf per $\lim_n \left(2 + \frac{1}{n+1} \right) = 2$

sui dispari : min per $n = 1$: $\frac{3}{2}$

sup per $\lim_n \left(2 - \frac{1}{n+1} \right) = 2$

$\Rightarrow \exists \min E = \frac{3}{2}$ e $\max E = 3$ #

3. $E = \{ n^2 (\cos(n\pi) - 1) : n \in \mathbb{N} \}$

Sol. NB $\cos(n\pi)$ $\begin{cases} \text{n pari} : 1 \\ \text{n dispari} : -1 \end{cases}$

$\Rightarrow \boxed{\cos(n\pi) = (-1)^n}$!!

$n^2 ((-1)^n - 1) = \begin{cases} \text{n pari} : 0 \quad \forall n \\ \text{n dispari} : -2n^2 \to -\infty \end{cases}$

Quindi $\begin{cases} \sup E = \max E = 0 \\ \inf E = -\infty \end{cases}$

4. $E = \left\{ \frac{xy}{x^2+y^2} : x, y \in \mathbb{R}, x < y \right\}$

Sol. Per la disug. di Cauchy - Schwarz :

$|xy| \le \frac{x^2+y^2}{2} \Rightarrow \frac{|xy|}{x^2+y^2} = \left| \frac{xy}{x^2+y^2} \right| \le \frac{1}{2}$

$0 <$

Quindi $-\frac{1}{2} \leq \frac{xy}{x^2+y^2} \leq \frac{1}{2}$ $\quad \forall x, y, \; x<y.$

Congettura $\quad \inf E = -\frac{1}{2}$ (min?) $\quad \sup E = \frac{1}{2}$ (max?)

Vediamo se $\inf E = \min E = -\frac{1}{2}$: abbiamo verificare se $\exists$ $(x<y)$ tale che (deve $\in E$!)

$$\frac{xy}{x^2+y^2} = -\frac{1}{2} \iff -2xy = x^2+y^2 \iff (x+y)^2 = 0$$

cioè $x = -y$ sì, $\exists$, basta prendere $x=-1$ e $y=+1$ $(x<y!)$

- Quindi $m = -\frac{1}{2}$ è il min E.
- $\frac{1}{2}$ invece è sup E ma non max E : infatti

$$\frac{xy}{x^2+y^2} = \frac{1}{2} \iff 2xy = x^2+y^2 \iff (x-y)^2 = 0, \text{ cioè}$$

$x = y$ non appartiene ad E perché non è mai $x<y$!

FACOLTATIVO:

Resta da dim che $\frac{1}{2} = \sup E$, cioè che:

$\forall \varepsilon > 0 \quad \exists x, y$ con $x<y$ tale che $\frac{xy}{x^2+y^2} > \frac{1}{2} - \varepsilon$

Pongo $a = \frac{1}{2} - \varepsilon$, per cui posso assumere $\frac{1}{4} < a < \frac{1}{2}$

Pongo $y = 1$, devo verificare allora che $\exists x \neq 1$ per cui, $\forall a$ come sopra:

$$\frac{x}{x^2+1} > a \qquad x > ax^2 + a \qquad ax^2 - x + a < 0$$

$$x_{1,2} = \frac{1 \pm \sqrt{1-4a^2}}{2a} \qquad (\Delta > 0 \text{ se } 1-4a^2 > 0 \quad \tfrac{1}{4} > a^2 \text{ sì})$$

Quindi ci sono sol. $\frac{1-\sqrt{1-4a^2}}{2a} < x < \frac{1+\sqrt{1-4a^2}}{2a}$

? $\frac{1+\sqrt{1-4a^2}}{2a} > 1 \iff 1 + \sqrt{1-4a^2} > 2a$

sì

$\sqrt{1-4a^2} > 2a - 1$ SEMPRE!

$2a - 1 < 0$

$a < \frac{1}{2}$

Quindi $\exists x > 1$ tra le soluzioni $\forall a < \frac{1}{2}$!!

1. $3\sin^2 x + \cos^2 x < 2 + \cos x$

$3(1-\cos^2 x) + \cos^2 x < 2 + \cos x$

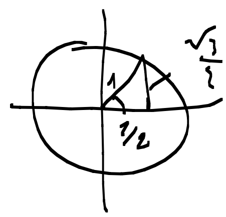
$3 - 3\cos^2 x + \cos^2 x < 2 + \cos x$

$2\cos^2 x + \cos x - 1 > 0$

$t = \cos x$: $2t^2 + t - 1 > 0$

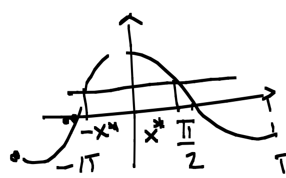
$$t_{1,2} = \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1 \pm \sqrt{9}}{4} \begin{cases} \frac{1}{2} \\ -1 \end{cases}$$

$t < -1 \quad \cup \quad t > \frac{1}{2}$



$\Longrightarrow \quad \cos x < -1 \quad \cup \quad \cos x > \frac{1}{2}$

$\varnothing$



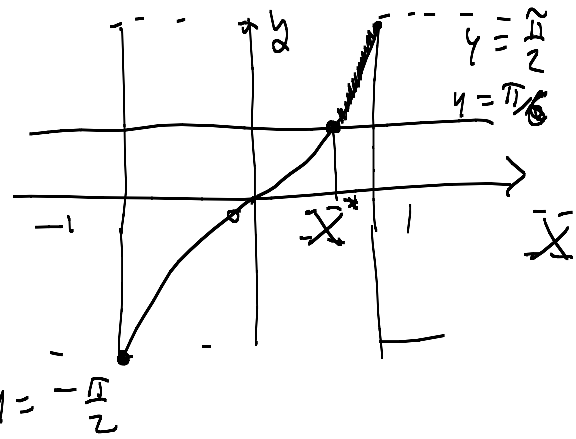
x^* t.c. $\cos x^* = \frac{1}{2}$

$x^* = \frac{\pi}{3}$

$$\boxed{-\frac{\pi}{3} + 2k\pi < x < \frac{\pi}{3} + 2k\pi, \ k \in \mathbb{Z}}$$

#

2. $\arcsin\left(\frac{x}{x^2-1}\right) > \frac{\pi}{6}$



(1) Graficamente:

$$\begin{cases} y = \arcsin(X) \\ y = \frac{\pi}{6} \end{cases}$$

ponendo $\left(X = \frac{x}{x^2-1}\right)$

$\Longrightarrow \quad X^* < X = \frac{x}{x^2-1} \leq 1$ dove X^* è la soluzione di $\arcsin(X^*) = \frac{\pi}{6} \Rightarrow X^* = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

Quindi $\begin{cases} \frac{1}{2} < \frac{x}{x^2-1} \leq 1 \\ x \neq \pm 1 \end{cases}$

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(2) Analiticamente :

$\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ dove arcsin ha come inversa $\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1,1]$

funzione strettamente crescente, quindi possiamo comporre la disequazione (senza cambiare il verso) :

$$\sin\left(\arcsin\left(\frac{x}{x^2-1}\right)\right) > \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$\Rightarrow \begin{cases} \frac{x}{x^2-1} > \frac{1}{2} \\ -1 \leq \frac{x}{x^2-1} \leq 1 \\ x \neq \pm 1 \end{cases}$$ ← dominio della funzione

e si ottiene lo stesso diseq! (Fare i conti...) #

3. $\sqrt{2-x} + \sqrt{x+4} \leq 6$

Sol. $\begin{cases} 2-x \geq 0 \\ x+4 \geq 0 \\ (\sqrt{2-x} + \sqrt{x+4})^2 \leq 36 \end{cases}$ (posso perché $\sqrt{\ } + \sqrt{\ } \geq 0$!)

$\begin{cases} x \leq 2 \\ x \geq -4 \\ 2-x+x+4+2\sqrt{(2-x)(x+4)} \leq 36 \end{cases}$

$\begin{cases} -4 \leq x \leq 2 \\ \sqrt{(2-x)(x+4)} \leq \frac{30}{2} \end{cases}$ (NB $\sqrt{(2-x)(x+4)}$ è sempre def. nel dominio delle radici di partenza!)

$$\begin{cases} -4 \le x \le 2 \\ (2-x)(x+4) \le 225 \iff x^2 + 2x + 217 \geq 0 \end{cases}$$

$\frac{\Delta}{4} = 1 - 217 < 0 \Rightarrow$ sempre verif.

R. $x \in [-4, 2]$ ✱

4. $\sqrt{\frac{9-x}{x+1}} > x - 3$

Sol.

$$\begin{cases} \frac{9-x}{x+1} \geq 0 \\ x \neq -1 \\ x - 3 < 0 \end{cases} \cup \begin{cases} \frac{9-x}{x+1} \geq 0 \\ x \neq -1 \\ x - 3 \geq 0 \\ \frac{9-x}{x+1} > (x-3)^2 \end{cases}$$

$$\begin{cases} -1 < x \le 9 \\ x < 3 \end{cases} \cup \begin{cases} -1 < x \le 9 \\ x \geq 3 \\ (\underline{\text{N.B.}}: x+1 > 0) : \text{posso molt. per } x+1 ! \\ 9 - x > (x-3)^2 (x+1) \end{cases}$$

$\boxed{-1 < x < 3}$ ... $\boxed{3 \le x < 4}$

$\boxed{\underline{R}: -1 < x < 4}$

5. $|x+3| \le \alpha \quad \alpha \in \mathbb{R}$

Sol. • $\underline{\alpha < 0}$ $\boxed{\emptyset}$

• $\underline{\alpha = 0}$ $|x+3| \le 0 \iff x + 3 = 0$, cioè $\boxed{x = -3.}$

• $\alpha > 0 \iff -\alpha \le x+3 \le \alpha$, $\boxed{-(\alpha+3) \le x \le \alpha - 3}$ #

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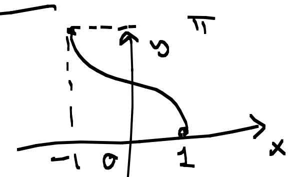
1. $f(x) = \arccos\left(\left|x^3 - \frac{1}{2}\right|\right)$

Dominio : $-1 \le \left|x^3 - \frac{1}{2}\right| \le 1 \iff -1 \le x^3 - \frac{1}{2} \le 1$

sempre !

$-\frac{1}{2} \le x^3 \le \frac{3}{2} \Rightarrow -\frac{1}{\sqrt[3]{2}} \le x \le \sqrt[3]{\frac{3}{2}}$



arccos : $[-1,1] \to [0,\pi]$

Segno : $f(x) \ge 0$: $\arccos\left(\left|x^3 - \frac{1}{2}\right|\right) \ge 0$

sempre (vedi grafico : $0 \le \arccos(\dots) \le \pi$!)

(nel dominio di f). #

2. $f(x) = \log|\sin(2e^x)|$

Dominio : $|\sin(2e^x)| > 0 \iff \sin(2e^x) \ne 0 \iff$

$2e^x \ne k\pi$, $k \in \mathbb{N}$, $k \ge 1$ (perché per $k \le 0$ non ha sol. !)

$\Rightarrow e^x \ne k\frac{\pi}{2} \iff \log(e^x) \ne \log\left(k\frac{\pi}{2}\right)$ cioè :

$\boxed{x \ne \log\left(k \cdot \frac{\pi}{2}\right) \quad \forall k \ge 1, k \in \mathbb{N}}$

Segno $\log|\sin(2e^x)| \ge 0$ (passo all'esponenziale, strettamente crescente)

$e^{\log|\sin(2e^x)|} = |\sin(2e^x)| \ge e^0 = 1$

Poiché $-1 \le \sin(\cdot) \le 1$, $f(x) > 0$ mai,

$f(x) = 0 \iff |\sin(2e^x)| = 1$ cioè:

$$\sin(2e^x) = \pm 1 \Rightarrow 2e^x = \frac{\pi}{2} + k\pi, \ k \in \mathbb{N}$$

(devo tenere solo i punti > 0, accettabili)

$$\Rightarrow e^x = \frac{\pi}{4} + k\frac{\pi}{2} \quad \text{cioè} \quad x = \log\left(\frac{\pi}{4} + k\frac{\pi}{2}\right), \ k \in \mathbb{N}$$

sono zeri di f. Altrove, $f < 0$. ✱

NB f NON è né periodica, né pari o dispari

3. $f(x) = \log(e^{2x} - 4e^x + 4)$

Dominio: $e^{2x} - 4e^x + 4 = (e^x - 2)^2 > 0 \iff e^x \ne 2$

cioè $\boxed{x \ne \log 2}$

Segno: $\log(e^{2x} - 4e^x + 4) \ge 0 \iff$

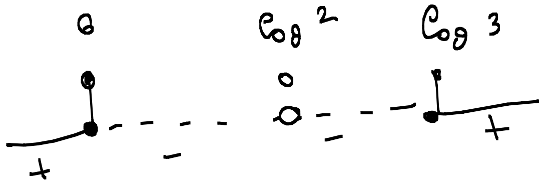
$e^{2x} - 4e^x + 4 \ge e^0 = 1$ quindi, posto $t = e^x$

$$t^2 - 4t + 3 \ge 0, \quad t_{1,2} = 2 \pm \sqrt{4-3} = \left\langle \begin{matrix} 3 \\ 1 \end{matrix} \right.$$

$$t \le 1 \ \cup \ t \ge 3$$

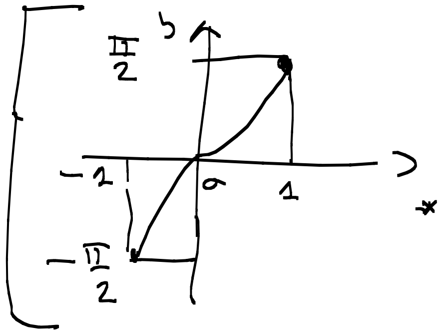
$$e^x \le 1 \ \cup \ e^x \ge 3$$

$f \ge 0$: $\boxed{x \le 0 \ \cup \ x \ge \log 3}$



(né periodica, né simm.) ✱

4. $f(x) = \arcsin\left(\dfrac{|x+2|}{x}\right)$



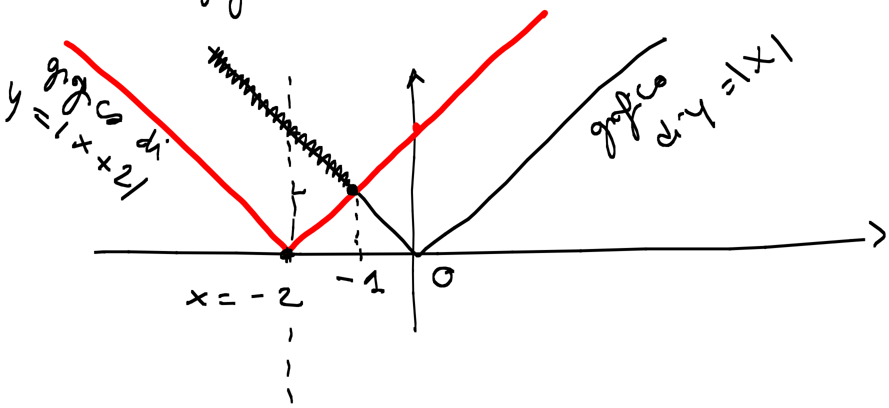
Dominio: $\begin{cases} -1 \leq \dfrac{|x+2|}{x} \leq 1 \\ x \neq 0 \end{cases}$

equivalentemente:

$$\begin{cases} x \neq 0 \\ \dfrac{|x+2|}{|x|} \leq 1 \end{cases} \iff \begin{cases} x \neq 0 \\ |x+2| \leq |x| \end{cases}$$

$(|x|>0!)$

che graficamente diventa:



R: $\boxed{x \leq -1}$

Segno $\arcsin\left(\dfrac{|x+2|}{x}\right) \geq 0 \iff \dfrac{|x+2|}{x} \geq 0$

(VEDI GRAFICO di arcsin)

poiché $|x+2| \geq 0$ sempre, $\Leftrightarrow x>0$, ma non è nel dominio.
Quindi $f(x) \leq 0$ sempre. #

5. $f(x) = \dfrac{1}{|x+1|-2}$

- Dominio: $|x+1| \neq 2 \iff x+1 \neq \pm 2$, cioè $x \neq 1, -3$

- $f(x) > 0$ (= 0 mai) $\iff |x+1| - 2 > 0 \quad |x+1| > 2 \iff$
  $x+1 < -2 \cup x+1 > 2$

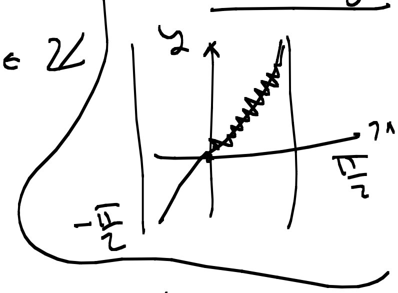
cioè $\boxed{x < -3 \cup x > 1}$ #

6. $f(x) = \sqrt[3]{\dfrac{x+2}{tg\, x}}$

Dominio (NB. $\sqrt[3]{a}$ def. x noi solo se $a \geq 0$)

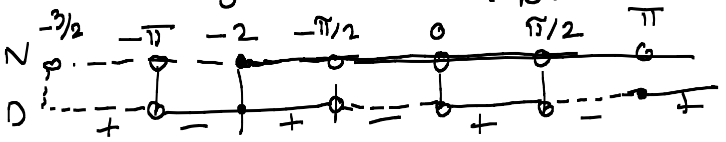
$$\begin{cases} \dfrac{x+2}{tg\, x} \geq 0 \\ x \neq \pm \dfrac{\pi}{2} + k\pi \quad \text{(dominio della } tg\text{)} \quad k \in \mathbb{Z} \\ tg\, x \neq 0 \iff x \neq k\pi, \; k \in \mathbb{Z} \end{cases}$$

GRAFICO $tg\, x$



$\dfrac{x+2}{tg\, x} \geq 0$: $\begin{cases} N \geq 0 & x \geq -2 \\ D > 0 & k\pi < x < \frac{\pi}{2} + k\pi, \; k \in \mathbb{Z} \end{cases}$

($\frac{\pi}{2} < 2$ perché $\pi < 4$)

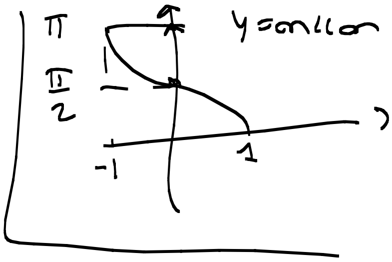


$-\pi < x < -\frac{\pi}{2}$

Quindi il dominio è
$[-2, -\frac{\pi}{2}[\;\cup\; \underbrace{]k\pi, \frac{\pi}{2} + k\pi[}_{k \geq 0} \;\cup\; \underbrace{]-\frac{\pi}{2} + k\pi, k\pi[}_{k \leq -1}$ #

Segno sempre ≥ 0. #

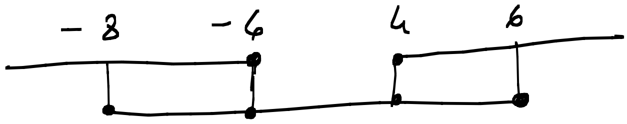
7. $f(x) = \arccos\left(|x+1| - 6\right) - \frac{\pi}{3}$



dominio : $-1 \le |x+1| - 6 \le 1$

$\begin{cases} |x+1| \ge 5 \\ |x+1| \le 7 \end{cases}$ $\begin{cases} x+1 \le -5 \ \cup\ x+1 \ge 5 \\ -7 \le x+1 \le 7 \end{cases}$

$\begin{cases} x \le -6 \ \cup\ x \ge 4 \\ -8 \le x \le 6 \end{cases}$

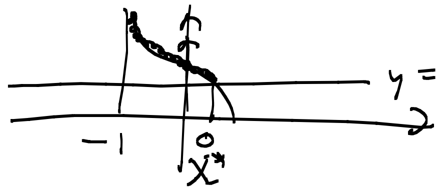


R. $x \in [-8,-6] \cup [4,6]$.

Segno: $f(x) \ge 0 \iff$

$\arccos\left(|x+1| - 6\right) \ge \frac{\pi}{3}$

(Graficamente, posto $X = |x+1| - 6$



$-1 \le X = |x+1| - 6 \le X^*$

dove X^* è l'intersezione

$\arccos(X^*) = \frac{\pi}{3} \Rightarrow \cos\left(\arccos(X^*)\right) = \cos\frac{\pi}{3} = \frac{1}{2}$

cioè $X^* = \frac{1}{2}$!

$\boxed{-1 \le |x+1| - 6 \le \frac{1}{2}}$)

Analiticamente : $\frac{\pi}{3} \in [0,\pi]$ dove arccos è l'inversa di $\cos : [0,\pi] \to [-1,1]$. Qui il coseno è strettamente decrescente

Quindi posso applicarlo alla diseg. ma la inverte :

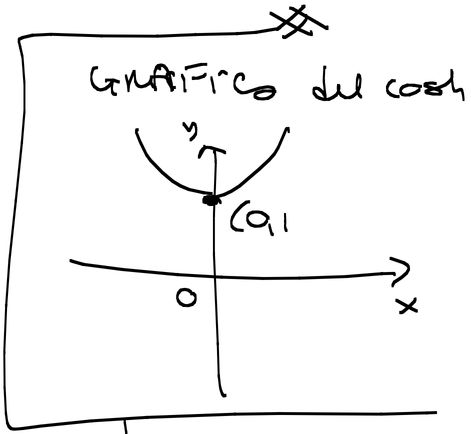
$f \ge 0$ se: $\cos\left(\arccos\left(|x+1| - 6\right)\right) \le \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

Quindi (tenendo conto anche del dominio)

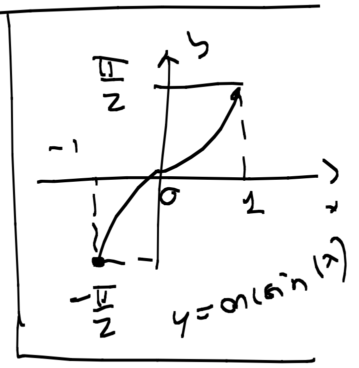
$$\begin{cases} |x+1|-6 \le \frac{1}{2} \iff |x+1| \le \frac{13}{2} \iff -\frac{13}{2} \le x+1 \le \frac{13}{2} \\ (1 \le |x+1|-6 \le 1 \leftarrow \text{GIÀ RISOLTO PRIMA!}) \end{cases}$$

...

8. $f(x) = \arcsin\left(\dfrac{1}{\cosh(\sin x)}\right)$



<u>Dominio</u>: $\begin{cases} -1 \le \dfrac{1}{\cosh(\sin x)} \le +1 \\ \cosh(\sin x) \neq 0 \end{cases}$ sempre!



$$\cosh(\sin x) \ge 1 \quad \forall x$$

$$\Rightarrow -1 < 0 < \frac{1}{\cosh(\sin x)} \le 1 \quad \forall x \in \mathbb{R}$$

Dominio: $\mathbb{R}$.

Segno $\arcsin(\ ) \ge 0 \iff \dfrac{1}{\cosh(\sin x)} \ge 0$ SEMPRE ≥ 0!

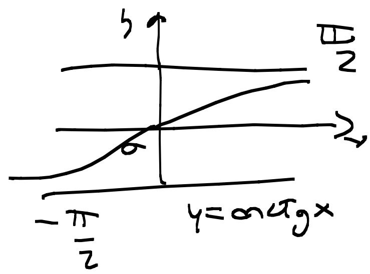
<u>NB</u> Periodica di periodo 2π e

$$\underline{f(-x)} = \arcsin\left(\frac{1}{\cosh(-\sin x)}\right) = \arcsin\left(\frac{1}{\cosh(\sin x)}\right) = \underline{f(x)}$$

$\uparrow$ $\sin x$ dispari $\qquad$ $\swarrow$ $\cosh x$ pari

è <u>PARI</u> $\Rightarrow$ basta studiarla in $[0, \pi]$.

9. $f(x) = \operatorname{arctg}\left(\sqrt{4e^{2x} - 9e^{x} + 2} - 2e^{x}\right)$



<u>Dominio</u>: arctg tutto $\mathbb{R}$,

$$4e^{2x} - 9e^{x} + 2 \ge 0$$

[Pongo $t = e^x$

$4t^2 - 9t + 2 \geq 0$, $t_{1,2} = \dfrac{9 \pm \sqrt{81-32}}{8} = \dfrac{9 \pm 7}{8} = \begin{cases} 2 \\ \frac{1}{4} \end{cases}$

L $t \leq \frac{1}{4} \cup t \geq 2$

$e^x \leq \frac{1}{4} \cup e^x \geq 2 \iff \boxed{x \leq -\log 4 \quad x \geq \log 2}$

$(= \log(1/4)!)$

Segno : $f \geq 0 \Rightarrow$

$$\sqrt{4e^{2x} - 9e^x + 2} \geq 2e^x \quad (>0!)$$

posso elevare direttamente :

$$\cancel{4e^{2x}} - 9e^x + 2 - \cancel{4e^{2x}} \geq 0$$

$$9e^x \leq 2 \qquad e^x \leq 2/9 \qquad x \leq \log(2/9)$$

($\underline{NB}$ $\quad \frac{2}{9} < \frac{1}{4} < 2 \Rightarrow \log(2/9) < \log(1/4) < \log 2$ (log cresc.))

$\boxed{R.}$ $f \geq 0$ per $x \leq \log(2/9)$ #

10. $f(x) = \log\left(4\sinh^2 x - 5\sinh x + 1\right)$

Dominio $\quad 4\sinh^2 x - 5\sinh x + 1 > 0$

pongo $t = \sinh x$

$4t^2 - 5t + 1 > 0 \iff t_{1,2} = \dfrac{5 \pm \sqrt{25-16}}{8} = \dfrac{5 \pm 3}{8} = \begin{cases} 1 \\ \frac{1}{4} \end{cases}$

$\sinh x < \frac{1}{4} \cup \sinh x > 1 \iff$

$x < \text{settsinh}\left(\frac{1}{4}\right) = x_1^*$

$x > \text{settsinh}(1) = x_2^*$

$\underline{NB}$: $0 < x_1^* < x_2^*$ (sinh crescente, >0 se $x>0$...)

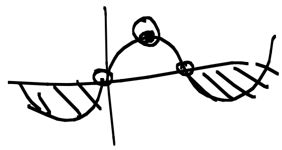
Dominio : $]-\infty, x_1^*[\cup]x_2^*, +\infty[$

Segno : $4\sinh^2 x - 5\sinh x + \cancel{1} \geq \cancel{1}$ $\quad x \leq 0 \cup \sinh x \geq \frac{5}{4}$

cioè $x \geq x_3^* = \text{settsinh}(5/4)$ ($> x_2^*$, perché sinh è crescente ...)

11. $$f(x) = \frac{\log(\sin(x))}{\sin x - 1}$$

<u>Dominio</u>: $\begin{cases} \sin x > 0 \\ \sin x \neq 1 \end{cases}$



$0 < x < \frac{\pi}{2} \cup \frac{\pi}{2} < x < \pi$ (2π periodico) $\Rightarrow$

$$\boxed{2k\pi < x < \frac{\pi}{2} + 2k\pi \cup \frac{\pi}{2} + 2k\pi < x < (2k+1)\pi, \; k \in \mathbb{Z}}$$

<u>Segno</u>: $\sin x - 1 < 0$ perché $\sin x < 1$ $\forall x$ nel dominio

$0 < \sin x < 1$ (nel dominio!)

$\Rightarrow \log(\sin x) < 0$ (vedi grafico del log!)

$\Rightarrow f(x) > 0$ $\forall x$ nel dominio! #

<u>Periodicità</u>: 2π periodica

<u>Il dominio non è simmetrico</u>: non ha senso chiedersi se è <u>pari o dispari</u> (non lo è!)