Soluzioni esercizi limiti 2

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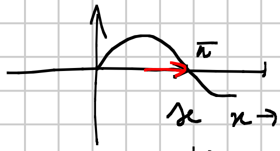
1) $$\lim_{x\to\pi^-} \frac{\sqrt{1+\operatorname{sen}x}-\sqrt{1-\operatorname{sen}x}}{1-\cos^2 x} =$$ $\left(\text{f.i. } \frac{0}{0}\right)$

$$= \lim_{x\to\pi^-} \frac{\left(\sqrt{1+\operatorname{sen}x}-\sqrt{1-\operatorname{sen}x}\right)}{1-\cos^2 x}\frac{\left(\sqrt{1+\operatorname{sen}x}+\sqrt{1-\operatorname{sen}x}\right)}{\left(\sqrt{1+\operatorname{sen}x}+\sqrt{1-\operatorname{sen}x}\right)} =$$

$$= \lim_{x\to\pi^-} \frac{(\cancel{1}+\operatorname{sen}x)-(\cancel{1}-\operatorname{sen}x)}{\operatorname{sen}^2 x\left(\sqrt{1+\operatorname{sen}x}+\sqrt{1-\operatorname{sen}x}\right)} =$$

$$= \lim_{x\to\pi^-} \frac{2\cancel{\operatorname{sen}}x}{\operatorname{sen}^{\cancel{2}} x\underbrace{\left(\sqrt{1+\operatorname{sen}x}+\sqrt{1-\operatorname{sen}x}\right)}_{\to 2}}$$



se $x\to\pi^-$ $\operatorname{sen}x\to 0$ e $\operatorname{sen}x>0$ in un intorno sx di π

quindi $\displaystyle\lim_{x\to\pi^-}\frac{1}{\operatorname{sen}x} = +\infty$

Quindi $$\lim_{x\to\pi^-} \frac{\sqrt{1+\operatorname{sen}x}-\sqrt{1-\operatorname{sen}x}}{1-\cos^2 x} = +\infty$$

2) $$\lim_{x\to-\infty}\left(\sqrt{3x^2-x}-\sqrt{3x^2+x+1}\right) =$$ $\left(\text{f.i. } +\infty-(+\infty)\right)$

$$= \lim_{x\to-\infty}\left(\sqrt{3x^2-x}-\sqrt{3x^2+x+1}\right)\frac{\left(\sqrt{3x^2-x}+\sqrt{3x^2+x+1}\right)}{\sqrt{3x^2-x}+\sqrt{3x^2+x+1}} =$$

$$= \lim_{x\to-\infty}\frac{(\cancel{3x^2}-x)-(\cancel{3x^2}+x+1)}{\sqrt{x^2\left(3-\frac{1}{x}\right)}+\sqrt{x^2\left(3+\frac{1}{x}+\frac{1}{x^2}\right)}} =$$

$$= \lim_{x\to-\infty}\frac{-2x-1}{|x|\left(\sqrt{3-\frac{1}{x}}+\sqrt{3+\frac{1}{x}+\frac{1}{x^2}}\right)} =$$

$$= \lim_{x\to-\infty}\frac{\cancel{-x}\left(2+\underbrace{\frac{1}{x}}_{\to 0}\right)}{\cancel{-x}\left(\sqrt{3-\underbrace{\frac{1}{x}}_{\to 0}}+\sqrt{3+\underbrace{\frac{1}{x}}_{\to 0}+\underbrace{\frac{1}{x^2}}_{\to 0}}\right)} =$$

(poiché $x\to-\infty$ posso considerare $x<0$ e quindi $|x|=-x$)

$$= \frac{\cancel{2}}{\cancel{2}\sqrt{3}} = \frac{1}{\sqrt{3}}$$

3) $$\lim_{x\to+\infty}\frac{1+3\operatorname{sen}x-x\operatorname{sen}2x}{x^2-1} =$$

Numeratore = $\nexists$ limite
Denom. $\to+\infty$

$$= \lim_{x\to+\infty} \frac{\cancel{x}\left(\frac{1}{x} + 3\frac{\operatorname{sen} x}{x} - \operatorname{sen} 2x\right)}{x^{\cancel{2}}\left(1-\frac{1}{x^2}\right)} =$$

$\frac{1}{x}\to 0$, $\frac{\operatorname{sen}x}{x}\to 0$, $\frac{1}{x^2}\to 0$, $\operatorname{sen}2x$: $\nexists$ limite ma limitata

$= 0$ perché prodotto di una f. infinitesima $\left(\frac{1}{x}\right)$ per una limitata.

$$4)\quad \lim_{x\to+\infty} \frac{\log(\log x)}{1+\log x} = \lim_{y\to+\infty} \frac{\log(y)}{1+y} = \qquad (\text{f.i. } \tfrac{\infty}{\infty})$$

$y = \log x$

$$= \lim_{y\to+\infty} \frac{\log y}{y\left(\frac{1}{y}+1\right)} = 0$$

$\frac{\log y}{y}\to 0$ per scala degli infiniti, $\frac{1}{y}\to 0$

$$5)\quad \lim_{x\to0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \lim_{x\to0} \frac{\left(\sqrt{1+x}-\sqrt{1-x}\right)}{x}\frac{\left(\sqrt{1+x}+\sqrt{1-x}\right)}{\sqrt{1+x}+\sqrt{1-x}} = \qquad (\text{f.i. } \tfrac{0}{0})$$

$$= \lim_{x\to0} \frac{(\cancel{1}+x)-(\cancel{1}-x)}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)} = \lim_{x\to0} \frac{2\cancel{x}}{\cancel{x}\left(\sqrt{1+x}+\sqrt{1-x}\right)} = 1$$

$$6)\quad \lim_{x\to0} \frac{\operatorname{sen}(\operatorname{tg} x)}{\operatorname{sen} x} = \lim \frac{\operatorname{sen}(\operatorname{tg} x)}{\operatorname{tg} x}\frac{\operatorname{tg} x}{\operatorname{sen} x} \qquad (\text{f.i. } \tfrac{0}{0})$$

$\frac{\operatorname{sen}(\operatorname{tg}x)}{\operatorname{tg}x} \to 1$ (limite notevole $\frac{\operatorname{sen} y}{y} \xrightarrow[y\to0]{} 1$ con $y = \operatorname{tg} x$)

$$= \lim_{x\to0} \frac{\operatorname{sen}(\operatorname{tg} x)}{\operatorname{tg} x}\,\frac{1}{\cos x} = 1.$$

$\frac{\operatorname{sen}(\operatorname{tg}x)}{\operatorname{tg}x}\to 1$, $\frac{1}{\cos x}\to 1$

$$7)\quad \lim_{x\to+\infty} 3^{x+1} - 3^{\sqrt{x^2+1}} = \qquad (\text{f.i. } +\infty-(+\infty))$$

$$= \lim_{x\to+\infty} 3^{x+1}\left(1 - 3^{\sqrt{x^2+1}-(x+1)}\right)$$

$$\lim_{x\to+\infty} \sqrt{x^2+1}-(x+1) = \lim_{x\to+\infty} \frac{(x^2+1)+x+1}{\sqrt{x^2+1}+(x+1)} =$$

$$= \lim_{x\to+\infty} \frac{x^{\cancel{2}}\left(1+\frac{2}{x^2}+\frac{1}{x}\right)}{\cancel{x}\left(\sqrt{1+\frac{1}{x^2}}+1+\frac{1}{x}\right)} = +\infty$$

$\frac{2}{x^2}\to0$, $\frac{1}{x}\to0$, $\frac{1}{x^2}\to0$, $\frac{1}{x}\to0$

Quindi

$$\lim_{x\to+\infty} \underbrace{3^{x+1}}_{+\infty}\underbrace{\left(1-3^{\sqrt{x^2+1}-(x+1)}\right)}_{-\infty} = -\infty$$

8) $\displaystyle\lim_{x\to+\infty} \frac{a^x + 4^x + \sinh x}{7^x + 2^x \sin(e^x)}$ al variare del parametro $a>0$

$$\lim_{x\to+\infty} a^x = \begin{cases} +\infty & a>1 \\ 1 & a=1 \\ 0 & 0<a<1 \end{cases} \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

$a>1$ $\displaystyle\lim_{x\to+\infty} \frac{2a^x + 2\,4^x + e^x - e^{-x}}{2\left(7^x + 2^x \sin(e^x)\right)} =$

se $a>4$ NUMERATORE $a^x\left(2 + 2\left(\frac{4}{a}\right)^x + \left(\frac{e}{a}\right)^x - \frac{e^{-x}}{a^x}\right)$

se $1<a<4$ " $4^x\left(2\left(\frac{a}{4}\right)^x + 2 + \left(\frac{e}{4}\right)^x - \frac{e^{-x}}{4^x}\right)$

se $a=4$ " $4^x\left(4 + \left(\frac{e}{4}\right)^x - \frac{e^{-x}}{4^x}\right)$

Quindi

$a>4$ $$\lim_{x\to+\infty} \frac{(\quad)}{(\quad)} = \lim_{x\to+\infty} \frac{a^x\left(2 + 2\underbrace{\left(\frac{4}{a}\right)^x}_{\to 0} + \underbrace{\left(\frac{e}{a}\right)^x}_{\to 0} - \underbrace{\frac{1}{(ea)^x}}_{\to 0}\right)}{2\cdot 7^x\left(1 + \underbrace{\left(\frac{2}{7}\right)^x \sin(e^x)}_{\to 0}\right)} =$$

(infinitesima per limitata)

$$= \begin{cases} +\infty & a>7 \\ 1 & a=7 \\ 0 & 4<a<7 \end{cases}$$

$a=4$ $$\lim_{x\to+\infty} \frac{(\quad)}{(\quad)} = \lim_{x\to+\infty} \frac{4^x\left(4 + \underbrace{\left(\frac{e}{4}\right)^x}_{\to 0} - \underbrace{\frac{1}{(e4)^x}}_{\to 0}\right)}{2\cdot 7^x\left(1 + \underbrace{\left(\frac{2}{7}\right)^x \sin(e^x)}_{\to 0}\right)} = 0$$

$1<a<4$ $$\lim_{x\to+\infty} \frac{(\quad)}{(\quad)} = \lim_{x\to+\infty} \frac{4^x\left(2 + 2\underbrace{\left(\frac{a}{4}\right)^x}_{\to 0} + \underbrace{\left(\frac{e}{4}\right)^x}_{\to 0} - \underbrace{\frac{1}{(4e)^x}}_{\to 0}\right)}{2\cdot 7^x\left(1 + \underbrace{\left(\frac{2}{7}\right)^x \sin(e^x)}_{\to 0}\right)}$$

$a=1$ $$\lim_{x\to+\infty} \frac{2 + 2\,4^x + \overset{=0}{e^x} - e^{-x}}{7^x + 2^x \sin(e^x)} =$$

$$= \lim_{x\to+\infty} \frac{4^x\left(\underbrace{\frac{2}{4^x}}_{\to 0} + 2 + \underbrace{\left(\frac{e}{4}\right)^x}_{\to 0} - \underbrace{\frac{1}{(4e)^x}}_{\to 0}\right)}{7^x\left(1 + \underbrace{\left(\frac{2}{7}\right)^x \sin(e^x)}_{\to 0}\right)} = 0$$

$a<1$ $$\lim_{x\to+\infty} \frac{2a^x + 2\cdot 4^x + e^x - e^{-x}}{7^x(\ldots\ldots)} =$$

$$= \lim_{x\to+\infty} \frac{4^x(\ldots\ldots)}{7^x(\ldots\ldots)} = 0$$ (come il caso $a=1$)

Quindi

$$\lim_{x\to+\infty} \frac{(\quad)}{(\quad)} = \begin{cases} 0 & 0<a<7 \\ 1 & a=7 \\ +\infty & a>7 \end{cases}$$

es. 6 (con "o piccolo")

$\operatorname{sen} y = y + o(y) \quad y \to 0$

$\operatorname{tg} y = y + o(y) \quad y \to 0$

$$\frac{\operatorname{sen}(\operatorname{tg} x)}{\operatorname{sen} x} = \frac{\operatorname{tg} x + o(\operatorname{tg} x)}{x + o(x)} = \frac{x + o(x)}{x + o(x)} \xrightarrow[x \to 0]{} 1$$