Soluzioni esercizi limiti con sviluppi di Taylor

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1) $\lim_{x\to 0^+} \dfrac{2^x - \operatorname{sen}(\alpha x) - 1 + x^3}{1-\cos(\sqrt{x}) - \frac{1}{2}\log(x+1)} =$ $\qquad 2^x = e^{x\log 2}$ se $x\to 0^+$ $x\log 2 \to 0^+$

$= \lim_{x\to 0^+} \dfrac{1 + x\log 2 + x^2\frac{\log^2 2}{2} + o(x^2) - \alpha x + \frac{1}{6}\alpha^3 x^3 + o(\alpha^3 x^3) - 1 + x^3}{1 - \left(1 - \frac{x}{2} + \frac{x^2}{4!} + o(x^2)\right) - \frac{1}{2}\left(x - \frac{x^2}{2} + o(x^2)\right)}$

$= \lim_{x\to 0^+} \dfrac{x(\log 2 - \alpha) + x^2\frac{\log^2 2}{2} + o(x^2)}{x^2\left(\frac{5}{24}\right) + o(x^2)} =$

$= \begin{cases} \to +\infty & \alpha < \log 2 \\ \to -\infty & \alpha > \log 2 \\ \log^2 2 \cdot \left(\frac{12}{5}\right) & \alpha = \log 2 \end{cases}$ #

2) a) $x\operatorname{sen}x - x^2 = x\left(x - \frac{x^3}{3!} + o(x^3)\right) - x^2 =$

$= -\frac{x^4}{3!} + o(x^4)$ è un infinitesimo di ordine 4.

b) $\lim_{x\to 0^+} \dfrac{x\operatorname{sen}x - x^2}{(1-\cos x)x} = \lim_{x\to 0^+} \dfrac{-\frac{x^4}{6} + o(x^4)}{\left(\frac{x^2}{2} + o(x^2)\right)x} =$

$= \lim_{x\to 0^+} \dfrac{-\frac{x^4}{6} + o(x^4)}{\frac{x^3}{2} + o(x^3)} = \lim_{x\to 0^+} -\frac{x}{3} = 0$

3) $\lim_{x\to 0^+} \dfrac{\operatorname{arctg}(x+x^2) - x}{x^\alpha - \operatorname{sen}(x^2)} =$ $\qquad \alpha \in \mathbb{R}$

$= \lim_{x\to 0^+} \dfrac{(x+x^2) - \frac{(x+x^2)^3}{3} + o\left((x+x^2)^3\right) - x}{x^\alpha - x^2 - \frac{x^6}{6} + o(x^6)} =$

$= \lim_{x\to 0^+} \dfrac{x^2 - \frac{x^3}{3} + o(x^3)}{x^\alpha - x^2 + \frac{x^6}{6} + o(x^6)}$

a) se $\alpha < 2$ $= \lim_{x\to 0^+} \dfrac{x^2 + o(x^2)}{x^\alpha + o(x^\alpha)} = \lim_{x\to 0^+} x^{2-\alpha} \to 0$

b) se $\alpha = 2$ $= \lim_{x\to 0^+} \dfrac{x^2 + o(x^2)}{+\frac{x^6}{6} + o(x^6)} = \lim_{x\to 0^+} \frac{6}{x^4} = +\infty$

c) se $\alpha > 2$ $= \lim_{x\to 0^+} \dfrac{x^2 + o(x^2)}{-x^2 + o(x^2)} = -1$

4) $$f(x) = \begin{cases} \dfrac{x + \cos x - e^{x^2/2}}{2x}, & x > 0 \\ 2ae^x - 3bx, & x \le 0 \end{cases}$$

f continua e derivabile $\forall x \neq 0$. Studiamola in $x = 0$

$f(0) = 2a$ $\quad \lim_{x \to 0^+} \dfrac{x + \cos x - e^{x^2/2}}{2x} \overset{(H)}{=} \lim_{x \to 0^+} \dfrac{1 - \operatorname{sen} x - e^{\frac{x^2}{2}} \cdot x}{2} = \dfrac{1}{2}$

(si può anche fare con Taylor: provare!)

f è continua in $x = 0 \iff 2a = \frac{1}{2} \Rightarrow \boxed{a = \frac{1}{4}}, \ \forall b \in \mathbb{R}$

$$f'(x) = \begin{cases} \dfrac{1}{2}\dfrac{(1 - \operatorname{sen} x - e^{x^2/2} \cdot x)x - (x + \cos x - e^{x^2/2})}{x^2} & x > 0 \\ \dfrac{1}{2}e^x - 3b & x < 0 \end{cases}$$
$(a = \frac{1}{4})$

$\lim_{x \to 0^-} f'(x) = \frac{1}{2} - 3b$

$$\lim_{x \to 0^+} \frac{x\left(1 - x + \frac{x^3}{6} + o(x^3) - x - \frac{x^3}{6} + o(x^3)\right) - \left(x + 1 - \frac{x^2}{2} - 1 - \frac{x^2}{2} + o(x^2)\right)}{2x^2}$$

$$= \lim_{x \to 0^+} \frac{x - x^2 - x^2 + o(x^2) - x + x^2}{2x^2} = -\frac{1}{2}$$

Quindi $f'_+(0) = -\frac{1}{2} = f'_-(0) = \frac{1}{2} - 3b$

$\iff -\frac{1}{2} = \frac{1}{2} - 3b \qquad 3b = 1 \qquad b = 1/3$

Quindi f è derivabile in $x = 0 \iff a = 1/4$ e $b = 1/3$

(controllare i conti!)

5) $$\lim_n \frac{1 + \operatorname{tg}\left(\frac{1}{n^3}\right) - e^{\frac{1}{n^3}}}{\left(e^{\frac{1}{n^2}} - 1\right)\frac{1}{n^\alpha}} = \lim_n \frac{1 + \frac{1}{n^3} + \frac{1}{3}\frac{1}{n^9} - 1 - \frac{1}{n^3} - \frac{1}{2n^6} + o\left(\frac{1}{n^6}\right)}{\frac{1}{n^2} \cdot \frac{1}{n^\alpha}} =$$

$$= \lim_n \frac{-\frac{1}{2n^6} + o\left(\frac{1}{n^6}\right)}{\frac{1}{n^{\alpha+2}}} = \lim_n -\frac{n^{\alpha+2}}{2n^6} = \lim_n -\frac{1}{2n^{4-\alpha}}$$

$$= \begin{cases} 0 & \alpha < 4 \\ -\frac{1}{2} & \alpha = 4 \\ -\infty & \alpha > 4 \end{cases}$$

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