Soluzioni esercizi limiti di successioni e di funzioni, Parte III (P. Mannucci, A. Sommariva)

1. $\lim_{n\to+\infty} (-1)^n \frac{\cos^2 n}{n} = 0$ $\left(\frac{1}{n}\to 0 \text{ e } (-1)^n\cos^2 n \text{ limitata}\right)$

2. $\lim_{n\to+\infty} \frac{(-1)^{n-1}-2}{(-1)^n-2}$ $\nexists$ perché

$$\frac{(-1)^{n-1}-2}{(-1)^n-2} = \begin{cases} 3 & n \text{ pari} \\ \frac{1}{3} & n \text{ dispari} \end{cases}$$

3. $\lim_{n\to+\infty} \frac{n^n}{e^{n^2}}$ criterio della radice

$$\sqrt[n]{a_n} = \frac{n}{e^n} \xrightarrow[n]{} 0 < 1$$

$$\Rightarrow \lim_{n\to+\infty} \frac{n^n}{e^{n^2}} = 0$$

4. $$\lim_n \frac{1+n^3-n\,\mathrm{sen}\,n + n^2\,\mathrm{sen}\left(\frac{1}{n}\right)}{\log^4 n + \sqrt{n^2+1}} =$$

infinitesima × limitata

$$= \lim_n \frac{n^3\left(\frac{1}{n^3} + 1 - \frac{\mathrm{sen}\,n}{n^2} + \frac{1}{n}\,\mathrm{sen}\left(\frac{1}{n}\right)\right)}{n\left(\frac{\log^4 n}{n} + \sqrt{1+\frac{1}{n^2}}\right)} = +\infty$$

$\frac{1}{n^3}\to 0$, $\frac{\mathrm{sen}\,n}{n^2}\to 0$, $\frac{1}{n}\,\mathrm{sen}\left(\frac{1}{n}\right)\to 0$, $\frac{\log^4 n}{n}\to 0$, $\frac{1}{n^2}\to 0$

(scala degli infiniti)

5. $\lim_n \frac{4^n + a^n}{n^2 2^n + 5^n}$ $a>0$

$a>4$ $$\frac{a^n\left(\left(\frac{4}{a}\right)^n + 1\right)}{5^n\left(n^2\left(\frac{2}{5}\right)^n + 1\right)} \longrightarrow \begin{cases} +\infty & a>5 \\ 1 & a=5 \\ 0 & 4<a<5 \end{cases}$$

$\left(\frac{4}{a}\right)^n \to 0$, $n^2\left(\frac{2}{5}\right)^n \to 0$ per scala infiniti

$a=4$ $$\frac{4^n\cdot 2}{5^n\left(n^2\left(\frac{2}{5}\right)^n+1\right)} = \left(\frac{4}{5}\right)^n \cdot \frac{2}{(\quad)} \to 0$$

$a<4$ $$\frac{4^n\left(1+\left(\frac{a}{4}\right)^n\right)}{5^n(\quad)} \to 0$$

Quindi $$\lim_n (\quad) = \begin{cases} +\infty & a>5 \\ 1 & a=5 \\ 0 & 0<a<5 \end{cases}$$

6. $\lim_n \dfrac{n^2 \log\left(1+\frac{1}{n}\right) + e^n + e^{n \log n}}{n^5 + n^n}$

$n^2 \log\left(1+\frac{1}{n}\right) = n \log\left(1+\frac{1}{n}\right)^n \sim n$ se $n \to +\infty$

infatti $\log\left(1+\frac{1}{n}\right)^n \xrightarrow[n\to+\infty]{} 1$

Quindi al numeratore il pezzo "che comanda" dovrebbe essere $e^{n \log n}$

numeratore:

$\hookrightarrow e^{n\log n}\left(\underbrace{\dfrac{n \log\left(1+\frac{1}{n}\right)^n}{e^{n \log n}}}_{\to 0 \text{ infatti}} + \underbrace{\dfrac{e^n}{e^{n\log n}}}_{\to 0 \text{ (vedi sotto)}} + 1\right)$

$= \dfrac{n \log\left(1+\frac{1}{n}\right)^n}{e^{\log n^n}} = \dfrac{n \log\left(1+\frac{1}{n}\right)^n}{n^n} = \dfrac{\overbrace{\log\left(1+\frac{1}{n}\right)^n}^{\to 1}}{n^{n-1}} \xrightarrow[n\to+\infty]{} 0$

$\dfrac{e^n}{e^{n\log n}} = \dfrac{1}{e^{n\log n - n}} = \dfrac{1}{e^{n\log n\left(1-\frac{1}{\log n}\right)}} \longrightarrow 0$

Quindi $\lim_{n\to+\infty} \dfrac{(\qquad)}{(\qquad)} = \lim_{n\to+\infty} \dfrac{e^{n\log n}\left(1+\overbrace{\cdots}^{\to 0}\right)}{n^n\left(\frac{n^5}{n^n}+1\right)}$

$= \lim_{n\to+\infty} \dfrac{\cancel{n^n}\left(1+\overbrace{\cdots}^{\to 0}\right)}{\cancel{n^n}\left(\underbrace{\frac{1}{n^{n-5}}}_{\to 0}+1\right)} = 1$

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7. $\lim_n \dfrac{n^{\sqrt{n}}}{2^n} = \lim_n \dfrac{e^{\sqrt{n}\log n}}{e^{n\log 2}} = \lim_n e^{\sqrt{n}\log n - n\log 2}$

$\sqrt{n}\log n - n\log 2 = n\left(\underbrace{\frac{\log n}{\sqrt{n}}}_{\to 0} - \log 2\right) \longrightarrow -\infty$

$\to 0$ (scala degli infiniti)

$\Rightarrow \lim_n \dfrac{n^{\sqrt{n}}}{2^n} = 0$

Variante $\lim_n \dfrac{n^{\sqrt{n}}}{(\sqrt{n})^n} = \lim_n \dfrac{e^{\sqrt{n}\log n}}{e^{n\log\sqrt{n}}} =$

$= \lim_n e^{\sqrt{n}\log n - \frac{n}{2}\log n}$

$\sqrt{n}\log n - \frac{n}{2}\log n = n\log n\left(\frac{1}{\sqrt{n}} - \frac{1}{2}\right) \longrightarrow -\infty$

$\Rightarrow \lim_n \dfrac{n^{\sqrt{n}}}{(\sqrt{n})^n} = 0$

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8. $\displaystyle\lim_n \frac{n^a - \cos n}{3n^2 + \operatorname{sen}(\sqrt{n})}$ $\quad a=1$, $a=3$

$a=1$ $\quad \displaystyle\lim_n \frac{\cancel{n}\left(1 - \frac{\cos n}{n}\right)}{n^{\cancel{2}}\left(3 + \frac{\operatorname{sen}(\sqrt{n})}{n^2}\right)} = 0$ $\quad \left(\frac{\cos n}{n} \to 0,\ \frac{\operatorname{sen}(\sqrt{n})}{n^2} \to 0\right)$

$a=3$ $\quad \displaystyle\lim_n \frac{n^{3}\left(1 - \frac{\cos n}{n^3}\right)}{\cancel{n^2}\left(3 + \frac{\operatorname{sen}(\sqrt{n})}{n^2}\right)} = +\infty$

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9. $\displaystyle\lim_n \left(1 + \frac{1}{\sqrt{n}}\right)^{\sqrt[3]{n}\operatorname{sen} n} = \lim_n e^{\sqrt[3]{n}\operatorname{sen} n \log\left(1+\frac{1}{\sqrt{n}}\right)}$

$\displaystyle\sqrt[3]{n}\operatorname{sen} n \frac{\log\left(1+\frac{1}{\sqrt{n}}\right)}{\frac{1}{\sqrt{n}}}\frac{1}{\sqrt{n}} = \operatorname{sen} n \frac{1}{n^{\frac{1}{2}-\frac{1}{3}}} \frac{\log(\ )}{(\ )} =$ $\quad \left(\frac{\log(\ )}{(\ )} \to 1\right)$

$\to 1$ se $n \to +\infty$ $\left(\frac{1}{\sqrt{n}} \to 0\right)$

$\displaystyle = \underbrace{\operatorname{sen} n}_{\text{limitata}} \frac{1}{n^{\frac{1}{6}}} \left(\frac{\log\left(1+\frac{1}{\sqrt{n}}\right)}{\frac{1}{\sqrt{n}}}\right) \longrightarrow 0$ $\quad \left(\frac{1}{n^{1/6}} \to 0,\ (\ ) \to 1\right)$

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10. $\displaystyle\lim_n \left(\frac{n^2 + 3n - 2 + \operatorname{sen} n}{n^2 - 4n}\right)^{n^2 - n + 3}$

$\displaystyle\frac{n^2 - 4n + 4n + 3n - 2 + \operatorname{sen} n}{n^2 - 4n} = 1 + \frac{7n - 2 + \operatorname{sen} n}{n^2 - 4n}$

$\displaystyle\left(\quad\right)^{n^2 - n + 3} = e^{(n^2-n+3)\log\left(1 + \frac{7n-2+\operatorname{sen} n}{n^2-4n}\right)}$

$\displaystyle(n^2 - n + 3)\frac{\log\left(1 + \frac{7n - 2 + \operatorname{sen} n}{n^2 - 4n}\right)}{\frac{7n-2+\operatorname{sen} n}{n^2-4n}} \cdot \frac{7n - 2 + \operatorname{sen} n}{n^2 - 4n}$ $\quad \left(\frac{7n-2+\operatorname{sen} n}{n^2-4n} \to 0\right)$

$\to 1$ $\quad$ (limite notevole $\displaystyle\lim_{y\to 0}\frac{\log(1+y)}{y} = 1$)

$\displaystyle\frac{n^2 - n + 3}{n^2 - 4n} \cdot (7n - 2 + \operatorname{sen} n) \longrightarrow +\infty$

Quindi $\displaystyle\lim_n (\quad) = e^{(\ ) \to +\infty} \longrightarrow +\infty$

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11. $\displaystyle\lim_{x\to 0} \frac{e^{2x} - e^{-2x}}{\operatorname{tg}(3x)} \cdot \frac{3x}{3x}$ $\quad \left(\frac{3x}{\operatorname{tg}(3x)} \to 1\right)$

$$\frac{e^{2x}-e^{-2x}}{3x} = \frac{e^{-2x}\left(e^{4x}-1\right)}{3x} = e^{-2x}\frac{e^{4x}-1}{3\cdot 4x}\cdot 4$$

$\to 1$ $\left(\frac{e^y-1}{y} \underset{y\to 0}{\to} 1\right)$

Quindi in totale

$$\lim_{x\to 0}\frac{e^{2x}-e^{-2x}}{tg(3x)} = e^0\cdot 1\cdot 1\cdot\frac{4}{3} = \frac{4}{3}$$

12. $f(x)=\begin{cases}(x+1)^2 & |x|\geq 1\\ e^{-\frac{1}{1-x^2}} & |x|<1\end{cases}$

$|x|\geq 1$ $\quad$ $|x|<1$ $\quad$ $|x|\geq 1$

-1^- $\;-1^+$ $\quad 1^-$ $\;1^+$

continua $\forall x \neq \pm 1$

in $x=1$ $\quad \lim_{x\to 1^+}(x+1)^2 = 4 = f(1)$ $\quad \Rightarrow f$ non è continua in $x=1$

$\lim_{x\to 1^-} e^{-\frac{1}{1-x^2}} = 0$ $\quad$ $1-x^2 \to 0^+$

in $x=-1$ $\quad \lim_{x\to -1^+} e^{-\frac{1}{1-x^2}} = 0$ $\quad$ $1-x^2\to 0^+$ $\quad \Rightarrow f$ continua in $x=-1$

$\lim_{x\to -1^-}(x+1)^2 = 0 = f(-1)$

Es. 11 con gli "o piccoli"

$tg\,x = x + o(x) \qquad x\to 0$

$e^x = 1 + x + o(x) \qquad x\to 0$

$$\frac{e^{2x}-e^{-2x}}{tg(3x)} = \frac{\cancel{1}+2x+o(2x)-(\cancel{1}-2x+o(-2x))}{3x+o(x)}$$

$$= \frac{4x+o(x)}{3x+o(x)}$$

$$\lim_{x\to 0}\frac{e^{2x}-e^{-2x}}{tg(3x)} = \lim_{x\to 0}\frac{4\cancel{x}}{3\cancel{x}} = \frac{4}{3}$$