A fast Nyström-Broyden solver by Chebyshev compression

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Abstract

When a Nyström-Broyden method is used to solve numerically Urysohn integral equations, the main problem is to evaluate the action of the integral operator at a low cost. Here we suitably approximate the relevant discrete integral operator of dimension n by its m-degree truncated Chebyshev series expansion (with $m \ll n$), reducing the complexity from the basic $\mathcal{O}(n^2)$ to $\mathcal{O}(mn)$. A technique to evaluate cheaply such m is presented. Several linear and nonlinear examples are considered.

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1 Polynomial compression and Nyström method

Consider the Urysohn integral equation

$$u(x) = \mathcal{K}[u](x) + g(x) = \int_a^b K(x,t,u(t))dt + g(x),\ x \in [a,b] \tag{1}$$

where $[a,b]$ is a bounded interval, $K : [a,b] \times [a,b] \times D \to \mathbb{R}$ (where $D \subseteq \mathbb{R}$) is the kernel, and u the unique solution of (1). At the level "l" of a Nyström method [3, 4, 5], the purpose is to compute u_l satisfying the approximating integral equation

$$u_l(x) = \sum_{j=1}^{n_l} w_{j,n_l} K(x, x_{j,n_l}, u_l(x_{j,n_l})) + g(x),\ \ x \in [a,b] \tag{2}$$

obtained by discretization of the right-hand side of (1) with the (convergent) numerical integration scheme

$$\int_a^b f(x)dx \approx \sum_{j=1}^{n_l} w_{j,n_l} f(x_{j,n_l}). \tag{3}$$

Fixed $z \in \mathbb{R}^{n_l}$, denote by $\mathcal{K}_l$ the discrete integral operator defined by

$$\mathcal{K}_l[z](x) := \sum_{j=1}^{n_l} w_{j,n_l} K(x, x_{j,n_l}, z_j), \;\; x \in [a,b].$$

It is well known that if for $z_l = \{(z_l)_i\} \in \mathbb{R}^{n_l}$

$$(z_l)_i = \mathcal{K}_l[z_l](x_{i,n_l}) + g(x_{i,n_l}) \tag{4}$$

$$= \sum_{j=1}^{n_l} w_{j,n_l} K(x_{i,n_l}, x_{j,n_l}, (z_l)_j) + g(x_{i,n_l}),\; i = 1, \ldots, n_l \tag{5}$$

holds, then the *Nyström interpolant* of z_l in the nodes $\{x_{i,n_l}\}$, i.e.

$$u_l(x) = \sum_{j=1}^{n_l} w_{j,n_l} K(x, x_{j,n_l}, (z_l)_j) + g(x) \tag{6}$$

satisfies (2). Concerning the error analysis of Nyström method in the linear case $K(x,t,u(t)) = H(x,t)\,u(t)$, several contributions have been given by a number of people; the final stage uses the theory of P. Anselone on collectively compact operators; see, e.g., [2, 3].
In the sequel, we set for convenience

$$f|_l := \{f(x_{i,n_l})\} \in \mathbb{R}^{n_l}$$

i.e. $f|_l$ is the vector obtained by evaluating "f" on the l-th level grid. In the implementations, at each level, a finite-dimensional fixed-point equation like (4) is solved by a Picard or a Newton-like method, until for a certain "l", the level error satisfies $\|u|_l - z_l\| \leq TOL$ (where TOL is the global tolerance). Let $z_l^{(k)} \in \mathbb{R}^{n_l}$ the k-th iteration of the nonlinear solver at a given level l and $\tilde{z}_l$, the final approximation of z_l provided by the nonlinear scheme.

The complexity of a Nyström method is essentially due to the evaluations of the semi-discrete integral operator $\mathcal{K}_l$, since

(i) in any nonlinear scheme, $z_l^{(k+1)}$ is available after $\mathcal{K}_l[z_l^{(k)}]|l$ has been computed;

(ii) when a change of grid is required, as starting point at level $l+1$ one usually considers $z_{l+1}^{(0)} = \mathcal{K}_l[\tilde{z}_l]|_{l+1} + g|_{l+1}$.

In general, the direct evaluation of $\mathcal{K}_l[z]$ at p different points $x_1, ..., x_p$ costs $\mathcal{O}(pn_l)$ flops, therefore the complexity of (i) and (ii) is respectively $\mathcal{O}\ (n_l^2)$ and $\mathcal{O}\ (n_{l+1}\, n_l)$, which entails a too large amount of computations for a basic Nyström method when n_l is large. Better performances can be achieved approximating the application of the discrete integral operator by a *fast method.* In the literature several works deal with this topic (see, e.g. [1, 6, 7, 8], and references therein), but in general they are not easily adaptable to nonlinear instances, being based on kernel approximations.

In this paper, in view of the results obtained in [9, 25, 10], in the framework of nonlinear operators, we suggest to approximate the action of $\mathcal{K}_l$, i.e. the function $\mathcal{K}_l[z]$, by *truncated Chebyshev series*, showing how to use such *compression* to implement a fast Nyström-Broyden solver. Finally, by some numerical experiments we show how this choice enhances the performance of the integral solver.

We begin by recalling some results on Chebyshev approximation [20, 23]. Fixed the Chebyshev-Lobatto nodes

$$\xi_j = \frac{2\cos(\pi j/m) - a - b}{b - a},\ j = 0, \ldots, m$$

and the weights

$$\hat{w}_0 = \hat{w}_m = \pi/2m,\ \hat{w}_j = \pi/m,\ \ j = 1, ..., m-1,$$

set

$$\alpha_k = {c_k}^{-1} \sum_{j=0}^{m} \hat{w}_j\, \mathcal{K}_l[z](\xi_j) \cos(k\pi j/m),\ \ k = 0, ..., m \tag{7}$$

where $c_0 = \pi$, $c_k = \pi/2$ for $k = 1, \ldots, m$, and T_k are the usual *first-kind Chebyshev polynomials.* The function

$$\mathcal{S}_{m,n_l}[z](x) = \sum_{k=0}^{m} \alpha_k T_k(x),\ \ x \in [a, b] \tag{8}$$

is the *m-degree truncated Chebyshev series expansion* of $\mathcal{K}_l[z]$ in $[a, b]$. Since the evaluation of $\mathcal{S}_{m,n_l}[z]$ in p points $\{x_i\}_{i=1,..,p}$ costs $\mathcal{O}(m\,(n_l + p))$ flops, in comparison with the $\mathcal{O}(n_l p)$ necessary for evaluating $\mathcal{K}_l[z]$, such approximation is useful when $\mathcal{S}_{m,n_l}[z] \approx \mathcal{K}_l[z]$ with $m \ll n_l\, p/(n_l{+}p)$. In practice, in view of the smoothing effect usually provided by integration, when $p = n_l$ and the number of nodes n_l is large (as it happens in the finer levels of Nyström method) such requirement is usually satisfied by a "*low*" degree truncated Chebyshev series, i.e. $m \ll n_l/2$ (observe that this situation often occurs at the finer levels of Nyström method).

But how to obtain at a low cost during the iterations of the nonlinear solver, a convenient $\bar{m}$ such that $S_{\bar{m},n_l}[z_l^{(k)}] \approx \mathcal{K}_l[z_l^{(k)}]$? Furthermore, what does it mean in this case "$\approx$"? To understand these points we recall what has been observed when for a fixed n_l the degree m is increased. In [9] and [25] it has been shown by various numerical examples that, when the kernel $K(x,t,y)$ is smooth in $x \in [a,b]$ for fixed $t \in [a,b]$ and $y \in D \subseteq \mathbb{R}$ then, for any $v \in D^{n_l}$, $\mathcal{K}_l[v]$ is also smooth and $\mathcal{S}_{m,n}[v] \to \mathcal{K}_l[v]$ (as $m \to \infty$), while when $K(x,t,y)$ is nonsmooth in x (as in the case of weakly singular or Green linear kernels), $\|\mathcal{S}_{m,n}[v] - \mathcal{K}_l[v]\|$ exhibits a *stalling* phenomenon for increasing m, corresponding to the size of the discretization error.

2 Convergence of the Chebyshev-Nyström-Broyden method

To show the convergence of the *compressed method* in the linear case, we modify a famous theorem, due to Anselone (cf. [2], [3], [14]).

Theorem 2.1 *Let $\mathcal{X}$ a Banach space and suppose that*

(A1) *$\mathcal{K}$, $\mathcal{K}_n : \mathcal{X} \to \mathcal{X}$ are linear operators, with $I - \mathcal{K}$ a $1-1$ map;*

(A2) *$\mathcal{K}_n v \to \mathcal{K} v$ for all $v \in \mathcal{X}$;*

(A3) *the set $\{\mathcal{K}_n \mid n \geq 1\}$ is collectively compact;*

(A4) *for every fixed n, $\|\mathcal{S}_{m,n} - \mathcal{K}_n\| \to 0$, as $m \to \infty$.*

Then,

(T1) *$\mathcal{K}$ is compact;*

(T2) *$\{\mathcal{K}_n \mid n \geq 1\}$ is uniformly bounded;*

(T3) *for any compact operator $\mathcal{M} : \mathcal{X} \to \mathcal{X}$*

$$\|(\mathcal{K} - \mathcal{K}_n)\mathcal{M}\| \to 0, \text{ as } n \to \infty; \tag{9}$$

(T4) *$\|(\mathcal{K} - \mathcal{K}_n)\mathcal{K}_n\| \to 0$, as $n \to \infty$.*

(T5) *for n sufficiently large, $(I - \mathcal{K}_n)^{-1}$ exists as a bounded operator on $\mathcal{X}$ to $\mathcal{X}$ and*

$$\|(I - \mathcal{K}_n)^{-1}\| \leq \frac{1 + \|(I - \mathcal{K})^{-1}\| \|\mathcal{K}_n\|}{1 - \|(I - \mathcal{K})^{-1}\| \|(\mathcal{K} - \mathcal{K}_n)\mathcal{K}_n\|} \tag{10}$$

If $(I - \mathcal{K})u = g$ and $(I - \mathcal{K}_n)u_n = g$ then

$$\|u - u_n\| \leq \|(I - \mathcal{K}_n)^{-1}\| \|(\mathcal{K} - \mathcal{K}_n)u\| \tag{11}$$

Moreover there exists a sequence $\{m(n)\}$ such that

(T6) *$\{\mathcal{S}_{m(n),n} \mid n \geq 1\}$ is uniformly bounded;*

(T7) *for any compact operator* $\mathcal{M} : \mathcal{X} \to \mathcal{X}$

$$\|(\mathcal{K} - \mathcal{S}_{m(n),n})\mathcal{M}\| \to 0, \textit{ as } n \to \infty;$$

(T8) $\|(\mathcal{K} - \mathcal{S}_{m(n),n})\mathcal{S}_{m(n),n}\| \to 0$, *as* $n \to \infty$*;*

(T9) *for* n *sufficiently large,* $(I - \mathcal{S}_{m(n),n})^{-1}$ *exists as a bounded operator on* $\mathcal{X}$ *to* $\mathcal{X}$ *and*

$$\|(I - \mathcal{S}_{m(n),n})^{-1}\| \leq \frac{1 + \|(I - \mathcal{K})^{-1}\|\|\mathcal{S}_{m(n),n}\|}{1 - \|(I - \mathcal{K})^{-1}\|\|(\mathcal{K} - \mathcal{S}_{m(n),n})\mathcal{S}_{m(n),n}\|}$$

If $(I - \mathcal{K})u = g$ *and* $(I - \mathcal{S}_{m(n),n})\tilde{u}_n = g$ *then*

$$\|u - \tilde{u}_n\| \leq \|(I - \mathcal{S}_{m(n),n})^{-1}\|\|(\mathcal{K} - \mathcal{S}_{m(n),n})u\|$$

Proof: Points (T1), (T2), (T3), (T4), (T5) were proved by Anselone (cf. [2], [3],[14]). By (A4) and (11), there exists a sequence $\{m(n)\}_n$ such that

$$\|\mathcal{S}_{m(n),n} - \mathcal{K}_n\| \leq \|u - u_n\|.$$

(T6) From (T2), (T5) and $\|\mathcal{S}_{m(n),n}\| \leq \|\mathcal{S}_{m(n),n} - \mathcal{K}_n\| + \|\mathcal{K}_n\|$, the uniform boundness of $\{\mathcal{S}_{m(n),n}\}$ is easily obtained.

(T7) It is a consequence of (A4), (T3) and

$$\|(\mathcal{K} - \mathcal{S}_{m(n),n})\mathcal{M}\| \leq \|(\mathcal{K} - \mathcal{K}_{m(n),n})\mathcal{M}\| + \|(\mathcal{K}_n - \mathcal{S}_{m(n),n})\mathcal{M}\|$$
$$\leq \|(\mathcal{K} - \mathcal{K}_{m(n),n})\mathcal{M}\| + \|\mathcal{K}_n - \mathcal{S}_{m(n),n}\|\|\mathcal{M}\|.$$

(T8) First, notice that

$$\|(\mathcal{K} - \mathcal{S}_{m(n),n})\mathcal{S}_{m(n),n}\| \leq \|(\mathcal{K} - \mathcal{K}_n)(\mathcal{K}_n - \mathcal{S}_{m(n),n})\|$$
$$+\|(\mathcal{K} - \mathcal{K}_n)\mathcal{K}_n\| + \|(\mathcal{K}_n - \mathcal{S}_{m(n),n})\mathcal{S}_{m(n),n}\|.$$

Since the sequences of operators $\{\mathcal{K} - \mathcal{K}_n\}$, $\{\mathcal{S}_{m(n),n}\}$ are uniformly bounded, and (A4), (T4) hold, then $\|(\mathcal{K} - \mathcal{S}_{m(n),n})\mathcal{S}_{m(n),n}\|$ converges to 0 as $n \to \infty$.

(T9) It is a direct application of a classical result on bounded operators, based on Neumann's Theorem (see [3], Thm. 4.1.1.), via (T8).

Observe that if $\mathcal{X} = \{C([a, b]), \|\cdot\|_\infty\}$ (with $[a, b]$ bounded interval), (A2) is satisfied when a convergent quadrature rule is used to define the basic Nyström method (see [14]).

Moreover, setting $K(x, t, u(t)) = L(x, t)u(t)$, $\mathcal{X} = C([a, b])$, and denoting the oscillation with *osc* , assumption (A4) holds when

$$log(h^{-1})\, osc(\mathcal{K}_n u; [a, b]; h) \to 0, \text{ for } h \to 0,$$

for any $u \in C([a,b])$ such that $\|u\| \le 1$ (cf. [23]).

If we assume the quadrature rule to be convergent, by Polya-Steklov theorem there exists a constant C_n such that

$$osc(\mathcal{K}_n u) = osc(\sum_{j=1}^{n} w_{j,n} L(x,t_j) u(t_j))$$

$$\le \sum_{j=1}^{n} |w_{j,n} u(t_j)| osc(L(x,t_j)) \le \sum_{j=1}^{n} C_n \, |u(t_j)| osc(L; [a,b]^2; h),$$

hence (A4) holds when

$$log(h^{-1}) \, osc(L; [a,b]^2; h) \to 0, \text{ for } h \to 0,$$

as in the case of any kernel $L(x,t)$ Hölder continuous on the square $[a,b]^2$.

3 Implementation details

We describe here our implementation of a Nyström-Broyden method which uses Chebyshev-like compression to evaluate the discrete Urysohn integral operators. At each level a nonlinear system is solved, of the form

$$(z_l)_i = \sum_{j=1}^{n_l} w_{j,n_l} K(x_{i,n_l}, x_{j,n_l}, (z_l)_j) + g_i, \; i = 1, \ldots, n_l \tag{12}$$

where $\{x_{i,n_l}\}_{i=1,\ldots,n_l}$ and $\{w_{i,n_l}\}_{i=1,\ldots,n_l}$ are respectively the nodes and the weights of a convergent quadrature rule, $g_i = g(x_{i,n_l})$ and $(z_l)_i$ approximates $u(x_{i,n_l})$. We start introducing and recalling some notations that will be useful in this section:

- $u : [a,b] \to \mathbb{R}$ is the solution of integral equation (1);
- $u_l : [a,b] \to \mathbb{R}$ is the solution of the semi-discrete integral equation (2);
- if $f : [a,b] \to \mathbb{R}$ and $\{x_{i,n_l}\}_{i=1,\ldots,n_l}$ is a discretization of the (bounded) interval $[a,b]$ (at level l), then set $f|_l = \{f(x_{i,n_l})\}_{i=1,\ldots,n_l}$; for example $u|_l$ is the vector obtained evaluating the solution u of the integral equation, at the points of the level l grid;
- $z_l^{(k)} \in \mathbb{R}^{n_l}$ is the k-th iteration of Broyden method, at level l;
- $\tilde{z}_l \in \mathbb{R}^{n_l}$ is the approximation of $u_l|_l$ provided by the nonlinear solver at level l ;
- let $\{x_{i,n_k}\}_{i=1,\ldots,n_k} \subseteq \{x_{i,n_l}\}_{i=1,\ldots,n_l}$ two nested discretizations of the interval $[a,b]$ and $z \in \mathbb{R}^{n_l}$ a vector satisfying $z_i \approx u(x_{i,n_l})$; we denote with $z|_k$ the restriction of z to $\{x_{i,n_k}\}_{i=1,\ldots,n_k}$;

- $TOL(l)$ and TOL are respectively the level l and the global tolerance.

Observe that to avoid misunderstandings with the symbols, we denote the vectors with "z", while with "u" we refer to functions. Here, as in similar general purpose codes, a composite Simpson's rule is used, with nodes $x_{i,n_l} = a+(i-1)h$, $h = (b-a)/(n_l - 1)$ and $n_{l+1} = 2n_l - 1$, $n_1 = 5$, i.e., changing level, the integration step is halved (this choice allows to compute fewer kernel evalutations during Nyström interpolation).

We will test our code on several problems, but not on weakly singular equations, since their discretizations by product integration require sharper strategies (the weights depend on the nodes!); a typical alternative, in the Nyström methods framework, is the Kantorovich reformulation of the problem [5, 7] but the pertinent discrete system is not included in the family of equations (12). Of course Simpson's rule can be applied to these instances, but the results are numerically unsatisfactory since the level error is in general too large. As for references on projection methods for weakly singular linear and nonlinear equations see, e.g., [4, 5, 15, 18, 19, 21, 24]. The nonlinear system (12) is solved by the well-known Broyden method [4, 16, 17]. In particular, our Matlab code uses the "*limited memory*" routine "*brsol*" (enclosed in [17], but also downloadable from the web) with the "*step*" as termination criterion, i.e. the iterative scheme stops when

$$\|{z_l}^{(k+1)} - z_l^{(k)}\|_{2,n_l} \leq TOL(l) \tag{13}$$

where

$$\|y\|_{2,n_l} = \sqrt{\frac{\sum_{i=1}^{n_l} y_i^2}{n_l}}. \tag{14}$$

About the starting iteration, for $l = 1$ we set $z_1^{(0)} \equiv 0$, while in the successive levels $z_l^{(0)}$ is obtained by Nyström interpolation (which can be done cheaply using the Chebyshev series truncations already obtained).

As in [9, 25], we choose m among the elements of a sequence $\{m_k\}$ (where $m_k = 2^k$) of polynomials degrees; more precisely, setting θ as a stalling value, e.g. $\theta = 0.4$, if for a certain $\bar{k}$

$$\frac{\|\mathcal{S}_{m_{\bar{k}+1},n_l}[z]|_l - \mathcal{S}_{m_{\bar{k}},n_l}[z]|_l\|_{2,n_l}}{\|\mathcal{S}_{m_{\bar{k}},n_l}[z]|_l\|_{2,n_l}} \leq TOL(l) \tag{15}$$

or

$$1-\theta \leq \frac{\|\mathcal{S}_{m_{\bar{k}+1},n_l}[z]|_l - \mathcal{S}_{m_{\bar{k}},n_l}[z]|_l\|_{2,n_l}}{\|\mathcal{S}_{m_{\bar{k}},n_l}[z]|_l - \mathcal{S}_{m_{\bar{k}-1},n_l}[z]|_l\|_{2,n_l}} \leq TOL(l) \tag{16}$$

then we put respectively $\bar{m} = m_{k-1}$ and $\bar{m} = m_{k-2}$ and we say that $\mathcal{S}_{\bar{m},n_l}[z] \approx \mathcal{K}_l[z]$ (observe that, differently from [9, 25], if the ratio in (16) is bigger than 1, the process is interrupted for stalling). Roughly speaking, (15) is the usual termination since two different Chebyshev truncates are sufficiently close, while in (16) the algorithm reveals that, fixed n_l, there is no benefit by increasing m.

Notice that in this last case, the sign "$\approx$" does not seem to be appropriate, because we have not reached the prescribed tolerance $TOL(l)$; actually, in view of the previous convergence analysis, (16) says that since $\|\mathcal{S}_{\bar{m},n_l}[z]|_l - \mathcal{K}_l[z]|_l\|_{2,n_l}$ is of the same size of the discretization error, it is not useful to choose a polynomial degree larger than $\bar{m}$ and in this sense $\mathcal{S}_{\bar{m},n_l}[z]|_l \approx \mathcal{K}_l[z]|_l$. Applying these arguments to solve nonlinear integral equations by Nyström method, some new problems arise. At the $k+1$-*th* iteration of Broyden method, we must evaluate the vector $K_l[z^{(k)}]|_l$; how can we estimate the relative Chebyshev degree $\bar{m}$ at a low cost? At first we chose the following strategy. Suppose $l > 1$. During, the first iteration of the nonlinear solver, a "suitable" $\bar{m}$ is computed such that $\mathcal{S}_{\bar{m},n_l}[z_l^{(0)}]|_l \approx \mathcal{K}_l[z_l^{(0)}]|_l$ and then in the successive Broyden iterations, we consider $\mathcal{S}_{\bar{m},n_l}[z_l^{(k)}]|_l$ as approximation of $\mathcal{K}_l[z_l^{(k)}]|_l$. To this purpose, if at the previous level the chosen degree was equal to $m_{\bar{k}}$, one starts computing $\mathcal{S}_{m_{\bar{k}-1},n_l}[z_l^{(0)}]|_l$, $\mathcal{S}_{m_{\bar{k}},n_l}[z_l^{(0)}]|_l$, ... until (15) or (16) holds. Usually just a few attempts are necessary to reach a convenient $\bar{m}$. Observe that choosing $m_k = 2^k$, $\bar{m}$ can be efficiently carried out since the kernel evaluations at the Chebyshev-Lobatto nodes

$$\xi_j = \frac{2\cos(\pi\, j/m_{k-1}) - a - b}{b-a},\ j = 0, ..., m_{k-1}$$

are used not only to compute $\mathcal{S}_{m_{k-1},n_l}[z_l^{(0)}]$ but also a part of $\mathcal{S}_{m_k,n_l} z_l^{(0)}$. We recall that this performance can be enhanced by a clever implementation of the *Discrete Chebyshev Transform*.

When stalling does not occur, $\bar{m}$ can be obtained by a cheaper technique. In practice, under these assumptions, if $z_l^{(0)}$ is the starting point at level l and $\tilde{z}_1$ the approximation of the solution u provided at the first level,

$$\|\mathcal{S}_{\bar{m}_{k+1},n_l}[z_l^{(0)}]|_l - \mathcal{S}_{\bar{m}_k,n_l}[z_l^{(0)}]|_l\|_{2,n_l} \approx \|\mathcal{S}_{\bar{m}_{k+1},n_1}[\tilde{z}_1]|_1 - \mathcal{S}_{\bar{m}_k,n_1}[\tilde{z}_1]|_1\|_{2,n_1} \quad (17)$$

i.e. the behaviour of Chebyshev truncates is almost invariant throughout the levels. As consequence, at the end of the first level, we build a table containing $\|\mathcal{S}_{\bar{m}_{k+1},n_1}[\tilde{z}_1]|_l - \mathcal{S}_{\bar{m}_k,n_1}[\tilde{z}_1]|_l\|_{2,n_l}$ until one of these quantities is inferior to the global tolerance TOL; then, at the beginning of level l, we choose as Chebyshev degree the first $\bar{m}$ such that

$$\|\mathcal{S}_{\bar{m},n_1}[\tilde{z}_1]|_1 - \mathcal{S}_{\bar{m}+1,n_1}[\tilde{z}_1]|_1\|_{2,n_1} \le TOL(l). \quad (18)$$

Clearly the latter approach results much inexpensive, since all the computations are done on the initial mesh.

About the lever error estimates, our investigation starts from the reliable Atkinson's analysis [4], applicable under some regularity assumptions of the kernel and of the solution u. In these cases, there exists an asymptotic costant "c" (equal to 1/15 for the Simpson's rule) satisfying

$$\|u|_{l+1} - z_{l+1}\|_{2,n_{l+1}} \approx c\,\|u|_l - z_l\|_{2,n_l}. \quad (19)$$

To implement a general purpose solver, we modified this strategy: by some numerical tests on (linear) Green kernels or on smooth integral equations with a non regular known-term g, we have seen that "c" still seems to exist, but it may strongly depend on the integral equation. Since

$$c \approx c_l = \frac{\|u|_l - \tilde{z}_l\|_{2,n_l}}{\|u|_{l-1} - \tilde{z}_{l-1}\|_{2,n_{l-1}}} \approx \tilde{c}_l = \frac{\|\tilde{z}_{l+1}|_l - \tilde{z}_l\|_{2,n_l}}{\|\tilde{z}_{l+1}|_{l-1} - \tilde{z}_{l-1}\|_{2,n_{l-1}}},\ l = 2, 3, \ldots. \tag{20}$$

we consider, as soon as $\tilde{z}_{l+1}$ is computed, $\tilde{c}_l$ as an approximation of c. Numerical experiments show that, for l sufficiently large, $\tilde{c}_l \approx \tilde{c}_{l-1}$, hence an approximation of the discretization error, at the current or at the finer levels, can be worked out by an Atkinson's type strategy since from

$$\|u|_{l+k} - \tilde{z}_{l+k}\|_{2,n_{l+k}} = c_{l+k} \cdot \ldots \cdot c_l \cdot \|u|_{l-1} - \tilde{z}_{l-1}\|_{2,n_{l-1}}$$

and

$$\|u|_{l-1} - \tilde{z}_{l-1}\|_{2,n_{l-1}} \approx \|\tilde{z}_l|_{l-1} - \tilde{z}_{l-1}\|_{2,n_{l-1}}$$

we get by the "almost invariance" of $\tilde{c}_l$

$$\|u|_{l+k} - \tilde{z}_{l+k}\|_{2,n_{l+k}} \approx \tilde{c}_{l-1}^{k+1}\|\tilde{z}_l|_{l-1} - \tilde{z}_{l-1}\|_{2,n_{l-1}} \tag{21}$$

where $k = 0, 1, \ldots$. When the approximation of the level solution $\tilde{z}_l$ is computed, we fix $TOL(l+1)$ as the error scheduled at level $l+1$.
Our code implements also a mesh independence analysis. In a large variety of cases, when n_l is sufficiently large, a constant number of iterations $\bar{k}$ is needed to compute the approximation of the level solution under a prescribed tolerance $TOL(l)$. By chance, it may happen that Broyden method does not stop since $\|{z_l}^{(\bar{k})} - {z_l}^{(\bar{k}-1)}\|_{2,n_l} > TOL(l)$ even if $\|u|_l - {z_l}^{(\bar{k})}|_l\|_{2,n_l} < TOL(l)$. To avoid this problem, we calculate (a posteriori!) what is, at level l, the first iteration ${z_l}^{(\hat{k})}$ satisfying the inequality

$$\|\tilde{z}_{l+1}|_l - {z_l}^{(\hat{k})}\|_{2,n_l} \leq TOL(l). \tag{22}$$

If

$$\|\tilde{z}_{l+1}|_l - {z_l}^{(\hat{k})}\|_{2,n_l} / \|\tilde{z}_{l+1}|_l - {z_l}^{(\hat{k}-1)}\|_{2,n_l} \approx 1$$

we set $\bar{k} = \hat{k} - 1$, otherwise $\bar{k} = \hat{k}$. When for 3 consecutive levels $\bar{k}$ is constant, we stop Broyden method either by the usual step criterion or just because $\bar{k}$ iterations have been computed.

4 Numerical examples

In this section we discuss the results obtained by applying Chebyshev-like compression to Nyström-Broyden method, to solve numerically linear and nonlinear integral equations. Such ability of solving both linear and nonlinear integral equations is not common for a wide class of fast methods based on suitable expansions of the corresponding kernel, since they are typically tailored to the linear case without being easily adaptable to more general Urysohn integral equations. We observe that these kernel-based techniques can be adapted to Hammerstein instances, by Kumar-Sloan approach [4, 18, 19].

In the next tables we use the following settings:

- n_l is the number of nodes of Simpson's rule;
- $\bar{m}$ is the order of Chebyshev truncates, chosen by our algorithm;
- CE_l and DE_l are the discretization errors $\|u_l|_l - \tilde{z}_l\|_{2,n_l}$ respectively of the basic Nyström-Broyden and of the Chebyshev-Nyström-Broyden method; ECE_l and EDE_l are the estimates provided by our algorithm;
- it_C and it is respectively the number of iterations of the Nyström-Broyden and of the Chebyshev-Nyström-Broyden method;
- $\tilde{c}_l$ is the level ratio defined in (20);
- $Flops_C$ and $Flops$ are the overall number of thousand flops (carried out until the last iteration of Broyden method at the current level) respectively of the compressed and of the basic Nyström-Broyden method.

Example 1. Consider the Green integral equation,

$$u(x) = \frac{\pi^2}{4} \int_0^1 H(x,t)\, u(t) dt + \frac{x}{2},\ x \in [0,1] \tag{23}$$

where

$$H(x,t) = \begin{cases} x\,(2-t)/2 & \text{if } 0 \le x \le t \\ t\,(2-x)/2 & \text{otherwise} \end{cases}$$

whose solution is $u(x) = \sin(\pi x/2)$.

Table I

Chebyshev and basic Nyström method applied to equation (23).

n_l	33	65	129	257	513	1025
$\bar{m}$	4	8	8	8	8	16
ECE_l	$1.4\,10^{-4}$	$3.1\,10^{-5}$	$2.1\,10^{-5}$	$3.1\,10^{-6}$	$9.4\,10^{-7}$	$1.6\,10^{-7}$
CE_l	$7.5\,10^{-4}$	$4.3\,10^{-5}$	$5.3\,10^{-6}$	$2.1\,10^{-6}$	$1.3\,10^{-6}$	$3.3\,10^{-7}$
it_C	3	4	4	3	3	2
$\tilde{c}_l$	0.09	0.21	0.23	0.58	0.53	0.28
$Flops_C$	85	189	383	719	1496	2763
EDE_l	$5.9\,10^{-4}$	$1.5\,10^{-4}$	$3.7\,10^{-5}$	$9.4\,10^{-6}$	$2.3\,10^{-6}$	$6.6\,10^{-7}$
DE_l	$4.4\,10^{-4}$	$1.1\,10^{-4}$	$2.7\,10^{-5}$	$6.9\,10^{-6}$	$1.7\,10^{-6}$	$4.8\,10^{-7}$
it	4	4	3	3	3	3
$\tilde{c}_l$	0.28	0.28	0.28	0.28	0.28	0.28
$Flops$	87	245	720	2587	9990	32138

In a previous paper [25], it has been shown that stalling usually appears when Chebyshev-like compression is applied to evaluate linear integral operators with Green kernels. This behaviour is confirmed at each level, and we can see from Table I that the numerical estimates of $\tilde{c}_l$ are almost constant in the case of Nyström method while they are less uniform for the compressed version since it may happen that some iterates $z_l^{(k)}$ are closer to $u|_l$ than to the level solution u_l. The benefit of using low order Chebyshev series truncations is evident by comparing the number of flops: the global speed-up in this case is 11.6.

Example 2. Consider the Hammerstein integral equation

$$u(x) = \int_0^1 e^{xt} \frac{1}{1+u(t)} dt + g(x), \;\; x \in [0,1] \tag{24}$$

where

$$g(x) = \begin{cases} e^x - 1 - (e^{x-1} - 1)/(x-1) & \text{if } x \neq 1 \\ e - 2 & \text{if } x = 1 \end{cases}$$

whose exact solution is $u(x) = e^{x-1}$. We expect that u can be easily approximated, since the kernel and the solution are smooth. In order to illustrate our method, we have fixed as global tolerance $TOL = 10^{-14}$ so that fine grids are considered. As expected by the convergence results on Chebyshev-compression, no stalling appears at any level.

Table II

Chebyshev and basic Nyström method applied to equation (24).

n_l	33	65	129	257	513	1025
$\bar{m}$	8	8	16	16	16	16
ECE_l	$9.2\,10^{-10}$	$5.7\,10^{-11}$	$3.6\,10^{-12}$	$2.6\,10^{-13}$	$1.3\,10^{-14}$	$8.8\,10^{-16}$
CE_l	$1.0\,10^{-9}$	$6.3\,10^{-11}$	$4.2\,10^{-12}$	$2.4\,10^{-13}$	$1.5\,10^{-14}$	$1.5\,10^{-15}$
it_C	3	4	4	4	3	3
$\tilde{c}_l$	0.05	0.06	0.06	0.06	0.05	0.06
$Flops_C$	46	87	166	430	853	1692
EDE_l	$9.9\,10^{-10}$	$6.1\,10^{-11}$	$3.9\,10^{-12}$	$2.4\,10^{-13}$	$1.5\,10^{-14}$	$8.8\,10^{-14}$
DE_l	$1.0\,10^{-9}$	$6.3\,10^{-11}$	$3.9\,10^{-12}$	$2.4\,10^{-13}$	$1.5\,10^{-14}$	$1.5\,10^{-15}$
it	4	4	3	3	3	3
$\tilde{c}_l$	0.06	0.06	0.06	0.06	0.06	0.06
$Flops$	57	178	537	1942	7506	29532

With respect to the first example, we notice that CE_l is nearly equal to DE_l, and Atkinson's estimates of the discretization error are in this case very strict. Moreover the asymptotic constant $\tilde{c}_l$ is almost the theoretical one (we recall that for the Simpson's rule $c = 1/15 \approx 0.06$). Even in this case, at the finer meshes, the improvement obtained by Chebyshev compression is evident, since a low $\bar{m}$ is sufficient to approximate the action of the integral operator, allowing a global speed-up of 17.4, while looking at the last level, we get 26.3.

Example 3. In [4], the Urysohn equation

$$u(x) = \int_0^1 \frac{1}{x+t+0.3\,u(t)}dt + g(x), \;\; x \in [0,1] \tag{25}$$

where, for $r(x) = (25\,x^2 - 50\,x - 5)^{-1/2}$,

$$g(x) = \frac{1}{1+x} - 0.5[\log(20\,x+23) - \log(10\,x+3)] -$$
$$-\frac{5\,(x-1)}{r(x)}[\operatorname{atanh}(\frac{5\,(x+3)}{r(x)}) - \operatorname{atanh}\frac{5\,(x+1)}{r(x)}]$$

has been studied to discuss an automatic program. The solution of (25) is $u(x) = 1/(1+x)$.

Table III

Chebyshev and basic Nyström method applied to equation (25).

n_l	33	65	129	257	513	1025
$\bar{m}$	16	16	32	32	32	32
ECE_l	$1.2\,10^{-7}$	$6.7\,10^{-9}$	$4.9\,10^{-10}$	$2.3\,10^{-11}$	$1.6\,10^{-12}$	$9.7\,10^{-14}$
CE_l	$1.4\,10^{-7}$	$8.4\,10^{-9}$	$4.5\,10^{-10}$	$2.6\,10^{-11}$	$1.7\,10^{-12}$	$1.1\,10^{-13}$
it_C	3	3	3	3	3	3
$\tilde{c}_l$	0.05	0.05	0.06	0.05	0.06	0.06
$Flops_C$	199	357	911	2022	4203	8522
EDE_l	$1.3\,10^{-7}$	$7.8\,10^{-9}$	$4.5\,10^{-10}$	$2.6\,10^{-11}$	$1.7\,10^{-12}$	$1.0\,10^{-13}$
DE_l	$1.4\,10^{-7}$	$7.1\,10^{-9}$	$4.3\,10^{-10}$	$2.7\,10^{-11}$	$1.7\,10^{-12}$	$1.1\,10^{-13}$
it	4	4	3	3	3	3
$\tilde{c}_l$	0.05	0.05	0.06	0.06	0.06	0.06
Flops	229	689	2063	7450	28773	112999

By the x-smoothness of the kernel in a neighboorhood of the exact solution, no stalling appears and it is evident from Table III that the behaviour of the two methods is almost the same, in terms of both errors, and iterations. The integral operator has been properly approximated by rather low degree truncates, thus it is obtained w.r.t. the basic Nyström method a speed-up of 13.2. Observe that the improvement is even larger in the last levels; changing grid, the number of flops of the compressed version doubles, while it quadruplicates in the basic case, allowing an higher speed-up in the last level (namely 19.5). Atkinson's estimates are still very precise in both cases, in agreement with the numerical experiments in [4] (even if different norms are used).

Example 4. Similarly to [22], we consider the linear equation

$$u(x) = \int_{-1}^1 \exp(xt^2)u(t)dt + g(x), \;\; x \in [-1,1] \tag{26}$$

where the known-term

$$g(x) = \sqrt{|x|} - 4 \sum_{n=0}^{+\infty} \frac{x^n}{n!(4n+3)}$$

has a discontinuous first derivative (observe that the power series on the r.h.s. is regular by uniform convergence in $[-1,1]$). The solution of the problem is $u(x) = \sqrt{|x|}$.

Table IV

Chebyshev and basic Nyström method applied to equation (26).

n_l	33	65	129	257	513	1025
$\bar{m}$	4	4	8	8	8	8
ECE_l	$1.6\,10^{-3}$	$5.9\,10^{-4}$	$2.6\,10^{-4}$	$2.0\,10^{-4}$	$7.4\,10^{-5}$	$1.2\,10^{-5}$
CE_l	$2.6\,10^{-3}$	$9.1\,10^{-4}$	$2.5\,10^{-4}$	$1.4\,10^{-4}$	$5.1\,10^{-5}$	$1.8\,10^{-5}$
it_C	3	3	2	2	2	2
$\tilde{c}_l$	0.26	0.26	0.28	0.15	0.45	0.27
$Flops_C$	49	71	118	213	404	782
EDE_l	$2.5\,10^{-3}$	$9.1\,10^{-4}$	$3.2\,10^{-4}$	$1.2\,10^{-4}$	$4.0\,10^{-5}$	$1.4\,10^{-5}$
DE_l	$2.6\,10^{-3}$	$9.2\,10^{-4}$	$3.3\,10^{-4}$	$1.1\,10^{-4}$	$4.0\,10^{-5}$	$1.4\,10^{-5}$
it	3	3	2	2	2	2
$\tilde{c}_l$	0.34	0.35	0.35	0.36	0.35	0.35
$Flops$	56	152	413	1426	5418	21150

By the x-smoothness of the kernel, no stalling appears and it is evident that the behaviour of the two methods is almost the same, in terms of both errors and iterations. As in the case of the linear Green kernel, the behaviour of the compressed version presents some anomalies on $\tilde{c}_l$ due to the fact that Chebyshev method may compute iterates $z_l^{(k)}$ that are closer to $u|_l$ than to the level solution u_l. We observe that the presence of a non regular known term g has an effect on the quality of the quadrature, but not on the performance of Chebyshev compression (only the integral operator is approximated). In this case, the global speed-up is 27.0, while relatively to the last level it reaches 41.6.

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References

[1] B. Alpert, G. Beylkin, R. Coifman and V. Rokhlin, Wavelet-like bases for the fast solution of second-kind integral equations, *SIAM J. Sci. Comput.* **14** (1993), 159-184.

[2] P. Anselone, *Collectively compact operator approximation theory and applications to integral equations.*, Prentice-Hall, New Yersey, 1971.

[3] K.E. Atkinson, *The Numerical Solution of Integral Equations of the Second Kind*, Cambridge Monographs on Applied and Computational Mathematics 4, Cambridge University Press, Cambridge, 1997.

[4] K.E. Atkinson, A survey of numerical methods for solving nonlinear integral equations., *J. Integral Equations Appl.*, **4** (1992), 15-46.

[5] C.T. Baker, *The Numerical treatment of Integral Equations*, Monographs on Numerical Analysis, Oxford University Press, 1977.

[6] A. Brandt and C.H. Venner, Multilevel evaluation of integral transforms with asymptotically smooth kernels, *SIAM J. Sci. Comput.* **19** (1998), 468-492.

[7] R.H. Chan, F.R. Lin, and C.F. Chan, A fast solver for Fredholm equations of the second kind with weakly singular kernels, *J. Numer. Math.* **10** (2002), 13-36.

[8] W. Dahmen, Wavelet and multiscale methods for operator equations, Acta Numer. **6** (1997), 55-228.

[9] S. De Marchi and M. Vianello, Approximating the approximant: a numerical code for polynomial compression of discrete operators, *Numer. Algorithms* **28** (2001), 101-116.

[10] S. De Marchi and M. Vianello, Fast evaluation of discrete integral transforms by Chebyshev and Leja polynomial approximation, in "*Constructive functions theory*", Varna 2002, B. Bojanov Ed., DARBA, Sofia 2002.

[11] L. Greengard, Fast algorithms for classical physics, Science **265** (1994), 909-914.

[12] L. Greengard and V. Rohklin, A fast algorithm for particle simulations, *J. Comput. Phys.* **73** (1987), 325-348.

[13] A. Griewank, The local convergence of Broyden-like methods on Lipschitzian problems in Hilbert space, *SIAM J. Num. Anal.* **24** (1987), 684-705.

[14] W. Hackbusch *Integral Equations. Theory and Numerical Treatment*, International series of numerical mathematics **120**, Birkhäuser, Basel, 1995.

[15] H. Kaneko and R. Noren, An application of approximation theory to numerical solutions for Fredholm integral equations of the second kind, *Numer. Funct. Anal. and Optimiz.* **12** (1991), 517-523.

[16] C.T. Kelley and E. Sachs, Broyden's method for approximate solution of nonlinear integral equations, *J. Integral Equations* **9** (1985), 25-43.

[17] C.T. Kelley, *Iterative Methods for Linear and Nonlinear Equations*, Frontiers in Applied Mathematics **16**, SIAM, Philadelphia, 1995.

[18] S. Kumar, Superconvergence of a collocation-type method for Hammerstein equations, *IMA J. Numerical Anal.* **7** (1987), 313-326.

[19] S. Kumar and I. Sloan, A new collocation-type method for Hammerstein equations, *Math. Comp.* **48** (1987), 585-593.

[20] J.C. Mason and D.C. Handscomb *Chebyshev Polynomials*, Chapman & Hall/CRC, Boca Raton, FL, 2003.

[21] G. Monegato and L. Scuderi, High order methods for weakly singular integral equations with nonsmooth input functions, *Math. Comp.* **67** (1998), 1493-1515.

[22] L. Reichel, Fast Solution Methods for Fredholm Integral Equations of The Second Kind, *Numer. Math* **57** (1989), 719-736.

[23] T.J. Rivlin, *The Chebyshev Polynomials*, Pure and Applied Mathematics, Wiley-Interscience, New York-London-Sidney, 1974.

[24] I. Sloan, Four Variants of the Galerkin Method for Integral Equations of the Second Kind, *IMA J. Numerical Anal.* **4** (1984), 9-17.

[25] M. Vianello, Chebyshev-like compression of linear and nonlinear discretized integral operators, *Neural, Parallel and Sci. Comput.* **8** (2000), 327-353.