Introduction to numerical cubature over the sphere

Alvise Sommariva

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"Numerical cubature and its applications"

The purpose of this note is to provide a brief introduction to numerical cubature over the sphere.

We

- start introducing some formulas of tensorial type, discussing their pros and cons;
- describe the so called designs and show what it is known about them;
- give a look to some results on rules based on extremal sets;
- show some formula on spherical triangles and spherical rectangles;
- we use them for integration over spherical polygons.

Let us suppose that we have to compute

$$I(f) := \int_{\mathbb{S}^2} f(\boldsymbol{\eta}) dS^2(\boldsymbol{\eta})$$

with $f \in C(\mathbb{S}^2)$.
A first technique consists in reducing $I(f)$ to an integral over a certain rectangle and then apply suitable tensorial rules.
To this purpose, consider the spherical coordinates

$$\boldsymbol{\eta} \to (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta), \quad 0 \le \phi \le 2\pi, \quad 0 \le \theta \le \pi.$$

Thus, taking into account the jacobian determinant of the transformation, we get

$$I(f) = \int_0^{2\pi} \int_0^{\pi} f(\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta) \sin\theta \; d\theta d\phi.$$

Aiming to determine

$$I(f) = \int_0^{2\pi} \int_0^{\pi} f(\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta) \sin\theta \; d\theta d\phi.$$

in view of the periodicity in the variable ϕ, we apply the composite trapezoidal rule with uniform spacing, that is

$$\tilde{I}(g) = \int_0^{2\pi} g(\phi) d\phi \approx \tilde{I}_m(g) = \sum_{j=0}^{m}{}'' g(jh), \quad h = \frac{2\pi}{m}$$

where *prime*′ means the first and last argument of the sum must be halved.
Taking into account the periodicity of the integrand, it is immediate that

$$\tilde{I}_m(g) = \sum_{j=0}^{m}{}'' g(jh) = \sum_{j=1}^{m} g(jh).$$

The integral of

$$I(f) = \int_0^{2\pi} \int_0^{\pi} f(\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta) \sin\theta \; d\theta d\phi.$$

w.r.t. the variable θ is more problematic. Setting $z = \cos(\theta)$, we have

$$I(f) = \int_0^{2\pi} \int_{-1}^{1} f(\cos\phi\sqrt{1-z^2}, \sin\phi\sqrt{1-z^2}, z) \; dz \, d\phi.$$

At his point one can apply

- Gauss-Legendre quadrature over $[-1, 1]$ with n nodes $\{z_k\}_{k=1,\dots,n}$ and weights $\{w_k\}_{k=1,\dots,n}$;
- the trapezoidal rule with nodes $\phi_j = j\frac{\pi}{n}$, $j = 1, \dots, 2n$ and weights $h = \frac{\pi}{n}$.

After these substitutions we get

$$I_n(f) = h \sum_{j=1}^{2n-1} \sum_{k=1}^{n} w_k f(cos\phi_j \sqrt{1-z_k^2}, \sin\phi_j \sqrt{1-z_k^2}, z_k)$$

thus for $\theta_k = \arccos z_k$

$$I_n(f) = h \sum_{j=1}^{2n-1} \sum_{k=1}^{n} w_k f(cos\phi_j \sin\theta_k, \sin\phi_j \sin\theta_k, \cos\theta_k)$$

The following result holds, setting $\mathbb{P}_{2n-1}$ the set of polynomials on the sphere of degree at most $2n-1$.

Theorem

Assume that $f \in \mathbb{P}_{2n-1}$. Then $I(f) = I_n(f)$ for any $f \in \mathbb{P}_{2n-1}$ but for $f(x, y, z) = z^{2n}$ we have that $I(f) \neq I_n(f)$.

In other words the formula has degree of exactness $2n-1$.

Next, introducing the best approximation error at degree n, w.r.t. uniform norm,

$$E_n(f) = \min_{p \in \mathbb{P}_n} \|f - p\|_\infty$$

one can prove that

$$|I(f) - I_n(f)| \leq 8\pi E_{2n-1}(f).$$

The following result holds

Theorem (Atkinson, Han, p.141)

Let $r \geq 1$ be an integer. Assume f is r-times continuously differentiable over $\mathbb{S}^2$. with all such derivatives in $C(\mathbb{S}^2)$. Then

$$E_n(f) \leq \frac{c}{(n+1)^r}.$$

Supposing that we have at hand the Matlab codes

- `rjacobi`,
- `gauss`,

that implement the Gaussian rules w.r.t. a Jacobi weight, we intend to test the product Gauss rule on the computation of the integral

$$I(f) := \int_{\mathbb{S}^2} exp(x) dS^2 \approx 14.76801374576529.$$

To this purpose we define the routines

- `product_gaussian_rule` that computes nodes and weight of the rule for a fixed n,
- `demo_product_gaussian_rule` that tests the results on the approximation of $I(f)$, by means of product Gauss rules for $n = 2, 3, 4, 5, 6$.

Tensorial rules

```
function [nodes,w]=gaussian_product_rule(n)

% Gaussian product rule of degree 2n-1.
% Nodes are in cartesian coordinates.

% Trapezoidal rule
h=(pi/n); phi=(1:2*n)'*h;

% Gauss-Legendre rule
ab=r_jacobi(n,0,0); xw=gauss(n,ab);

% Nodes
z=xw(:,1);
[P,T]=meshgrid(phi,sqrt(1-z.^2));
P=P(:); T=T(:); Z=sqrt(1-T.^2);
nodes=[cos(P).*T sin(P).*T Z];

% Weights
hv=h*ones(2*n,1);
[W1,W2]=meshgrid(hv,xw(:,2));
w=W1(:).*W2(:);
```

Tensorial rules

```
function demo_product_gaussian_rule

% Demo: see
% K.Atkinson, W.Han, Spherical Harmonics and Approximations on the Unit
% Sphere: An introduction, p.172.

f=@(x,y,z) exp(x)+0*y+0*z;

% reference integral via Chebfun
% F=spherefun(f); I=sum2(F)

I=1.476801374576529e+01;
AE=[];

for n=2:6
    [nodes,w]=gaussian_product_rule(n);
    x=nodes(:,1); y=nodes(:,2); z=nodes(:,3);
    fnodes=feval(f,x,y,z);

    In=w'*fnodes;
    AE(end+1)=abs(In-I);

    fprintf('\n \t n: %3.0f nodes: %5.0f error: %1.2e',...
        n,length(x),AE(end))

end

fprintf('\n \n');
```

The routine `demo_product_gaussian_rule` provides the following results

```
>> demo_product_gaussian_rule

   n:   2 nodes:     8 error: 1.17e-02
   n:   3 nodes:    18 error: 4.00e-04
   n:   4 nodes:    32 error: 4.91e-07
   n:   5 nodes:    50 error: 3.84e-09
   n:   6 nodes:    72 error: 2.22e-12

>>
```

Note the fast convergence, in view of the fact that the function $\exp(x)$ is analytic.

In view of the rotational symmetry of the sphere, one prefers points that are not clustered to the poles, but due to the structure of Gauss-Legendre rules, this set does.

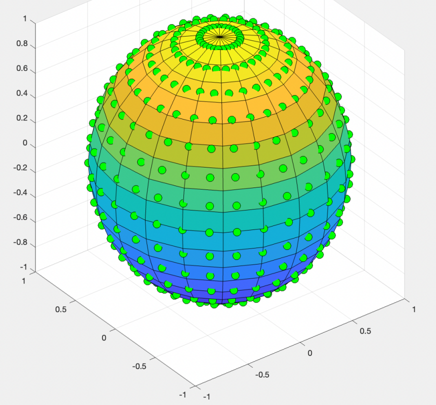

Figure: Gauss Product Rule for $n = 15$.

An interesting set for cubature over $\mathbb{S}^2$ is that of spherical designs.

Definition (Spherical t-design)

A finite subset $X = \{P_i\}_{i=1,\dots,N}$ on $\mathbb{S}^d$ is called a spherical t-design on $\mathbb{S}^d$, if for any $p_t \in \mathbb{P}_t$ we have

$$\int_{\mathbb{S}^d} p_t(\boldsymbol{\eta}) dS^d(\boldsymbol{\eta}) = \frac{\mu(\mathbb{S}^d)}{N} \sum_{i=1}^{N} p_t(P_i),$$

where, $\mu(\mathbb{S}^d)$ is the area of the unit-sphere $\mathbb{S}^2$, e.g., $\mu(\mathbb{S}^2) = 4\pi$.

Remark

Notice that all the weights are equal, i.e. $w_1 = \dots, w_N = \frac{\mu(\mathbb{S}^d)}{N}$.

Purpose.

- *A first question that arises is if such a set X exists, for a fixed degree t.*
- *A second question is if the cardinality N of such a set has some lower-bounds and when they are attained.*
- *Finally one would like to see some examples.*

We will try to respond to all these questions. The theory is rich and connects combinatorics with numerical analysis.

For details see the introduction of Numerical construction of spherical t-designs by Barzilai-Borwein method.

Theorem (Bounds of spherical t-designs)

Let $X = \{P_i\}_{i=1,\dots N}$ be a spherical t-design on $\mathbb{S}^d$ Then

- *if $t = 2e$ then*

$$N \geq N^* = \binom{d+e}{e} + \binom{d+e-1}{e-1}$$

- *if $t = 2e+1$ then*

$$N \geq N^* = 2\binom{d+e}{e}.$$

Definition (Tight Design)

Let X be a spherical t-design on $\mathbb{S}^d$. If the lower bound above is reached as equality, i.e. $N = N^*$, then X is a tight spherical t-design.

Theorem (Bounds of spherical t-designs on $\mathbb{S}^2$)

Let $X = \{P_i\}_{i=1,\dots N}$ be a spherical t-design on $\mathbb{S}^2$ If

- *t is odd then*

$$N \geq N^* = \frac{1}{4}(t+1)(t+3);$$

- *t is even then*

$$N \geq N^* = \frac{1}{4}(t+2)^2,$$

Important things that we know about such is that sets in $\mathbb{S}^d$:

- there is no tight spherical t-design with N^* points except possibly for $t = 1, 2, 3, 4, 5, 7, 11$ (Delsarte, Goethals and Seidel, p.364);
- there are spherical t-designs having $N \geq Ct^2$ points (for a certain C) (Bondarenko, Radchenko and Viazovska, 2013);
- spherical t-designs with $(t+1)^2$ points exist for all degrees t up to 100.

Other interesting features are the following.

1. Extremal systems are sets of $N = (t+1)^2$ points on $\mathbb{S}^2$ which maximize the absolute value of the determinant of the Vandermonde matrix for an arbitrary fixed basis (e.g. spherical harmonics, a well-known basis on the 2-sphere).

For $N = (t+1)^2$, verified that a spherical t-design exists in a neighborhood of an extremal system.

2. For $N \geq (t+1)^2$ verified extremal spherical t-designs exist for all degrees t up to 60 and provided well-conditioned spherical t-designs for interpolation and numerical integration.

At this stage one would like to have examples of spherical t-designs.

1. A good list can be found at Spherical Designs, by Hardy and Sloane.
2. Spherical t-designs for $t = 1, \ldots, 180$ and symmetric (antipodal) t-designs for degrees up to 325, are available at all with low mesh ratios (see Efficient Spherical Designs with Good Geometric Properties). A good list of these points is available at Efficient Spherical Designs with Good Geometric Properties.

   There the author computes
   - Spherical t-designs on the 2-sphere with $N = t^2/2 + t + O(1)$ points;
   - Symmetric (antipodal) spherical t-designs on the 2-sphere with $N = t^2/2 + t/2 + O(1)$ points.

Let us make some numerical examples, looking for tight spherical t-designs, by considering Hardy and Sloane homepage, each one having at best N_t points (being N_t^* the optimal value).

t	N_t^*	N_t
0	1	1
1	2	2
2	4	4
3	6	6
4	9	14
5	12	12
6	16	26
7	20	24
8	25	40
9	30	48
10	36	62
11	42	70
12	49	84

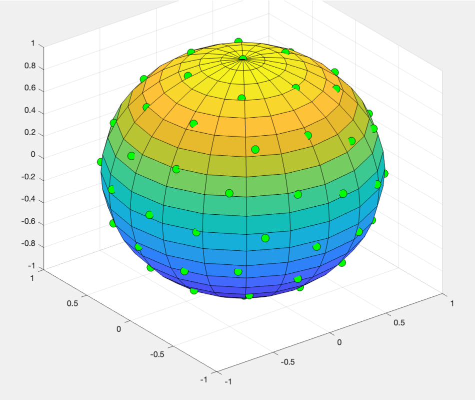

Figure: A spherical $n = 11$-design with 72 points. Notice that the points are well-distributed on the 2-sphere.

Supposing we call the 72 points of the spherical 11-design from the routine `set11`, we can write this demo, to integrate $f(x, y, z) = \exp(x)$ on the 2-sphere.

```
function demo_designs

f=@(x,y,z) exp(x)+0*y+0*z;
I=1.476801374576529e+01;

P=set11;
x=P(:,1); y=P(:,2); z=P(:,3); N=length(x);
w=(4*pi/N)*ones(N,1);
fnodes=feval(f,x,y,z);
In=w'*fnodes;
AE=abs(In-I);
fprintf('\n \t AE: %1.2e',AE)
```

As result we get

```
>> demo_designs

AE =
   6.6258e-13

>>
```

a little better w.r.t. that obtained via tensorial rules with degree of exactness 11.

In the previous part, we have considered two classical methods for integration over the 2-sphere.
Now we focus on two classes of domains on the sphere, i.e.

- spherical triangles,
- spherical rectangles.

These elements are important since many regions can be approximated by spherical polygons, that can be written as union of spherical triangles and hence composite rules can be constructed.

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Let $\mathcal{T} = \widehat{ABC}$ be a spherical triangle, that is, the arcs $\widehat{AB}$, $\widehat{BC}$, $\widehat{AC}$ are geodesic, i.e. arcs of great circles, that is the intersection of a plane containing the origin,with the sphere.

There is no restriction to suppose that the centroid

$$(A+B+C)/\|A+B+C\|_2$$

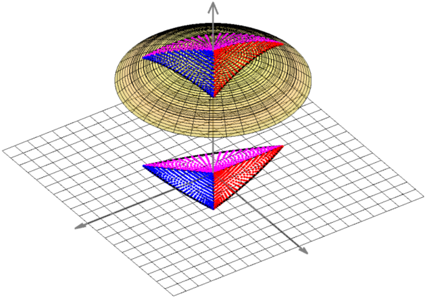
is at the north pole. In fact, if it is not so, one can rotate the original spherical triangle so that it has this property, compute a formula and than rotate it back to the original position.

Then we can write the surface integral of a continuous function $f(x, y, z)$ on such a spherical triangle in cartesian coordinates

$$\int_{\mathcal{T}} f(x, y, z)\, d\sigma = \int_{\Omega} f(x, y, g(x, y))\, \frac{1}{g(x, y)}\, dx\, dy \;, \tag{1}$$

where

- $g(x, y) = \sqrt{1 - x^2 - y^2}$,
- Ω is the projection of $\mathcal{T}$ onto the xy-plane,i.e. the curvilinear triangle with vertices $\hat{A}, \hat{B}, \hat{C}$ being the xy-coordinates of A, B, C.



Notice that the sides of this planar and curvilinear polygon Ω are *arcs of ellipses* centered at the origin, being the projections (and thus transformations by an affine mapping) of great circle arcs (the sides of $\mathcal{T}$).

Then we can split the planar integral into the sum of the integrals on three elliptical sectors S_1, S_2, S_3, obtained by joining the origin with the vertices $\hat{A}, \hat{B}, \hat{C}$, namely

$$\int_{\Omega} f(x, y, g(x, y)) \frac{1}{g(x, y)} \, dx \, dy = \sum_{i=1}^{3} \int_{S_i} f(x, y, g(x, y)) \frac{1}{g(x, y)} \, dx \, dy \; .$$

Now, we seek a quadrature formula which is as close as possible to an algebraic formula (at machine precision), when $f \in \mathbb{P}_n$.

We observe that if the integrand is a monomial as

$$f(x,y,z) = x^{\alpha}y^{\beta}z^{\gamma}, \;, \quad 0 \le \alpha+\beta+\gamma \le n$$

then, being $g(x,y) = \sqrt{1-x^2-y^2}$, we have

$$f(x,y,g(x,y))/g(x,y) = x^{\alpha}y^{\beta}(1-x^2-y^2)^{\gamma/2-1/2}. \qquad (2)$$

We have two distinct situations:

- if γ is *odd*, then $f(x,y,g)/g$ is a polynomial in (x,y) of degree at most $n-1$, namely $f(x,y,g)/g \in \mathbb{P}^2_{\alpha+\beta+\gamma-1} \subseteq \mathbb{P}^2_{n-1}$;
- if γ is *even* (including $\gamma = 0$), since $g \ge 0$, then $g^{\gamma} = (g^2)^{\gamma/2}$ is a polynomial of degree γ and $f(x,y,g)/g \in \frac{1}{g}\,\mathbb{P}^2_{\alpha+\beta+\gamma} \subseteq \frac{1}{g}\,\mathbb{P}^2_n$.

If γ is even, let $p_\varepsilon(x,y)$ be a polynomial of degree $m = m(\varepsilon)$ such that

$$|p_\varepsilon(x,y) - 1/g(x,y)| \leq \varepsilon\,(1/|g(x,y)|)$$

for any $(x,y) \in \Omega$.

Then $fp_\varepsilon \in \mathbb{P}^2_{n+m}$ approximates f/g with a relative error at most ε.

In order to find $m = m(\varepsilon)$, recalling that $g(x,y) = \sqrt{1-(x^2+y^2)}$ and

$$0 \leq x^2 + y^2 \leq \rho = \max\left\{\|\hat{A}\|_2^2, \|\hat{B}\|_2^2, \|\hat{C}\|_2^2\right\} < 1\,,\quad (x,y) \in \Omega\,, \quad (3)$$

it is sufficient to find the degree of a (near) optimal univariate polynomial approximation (up to ε) to the function $1/\sqrt{1-t}$ for $t \in [0,\rho]$.

This can be done efficiently by Chebfun and list the results on a file.

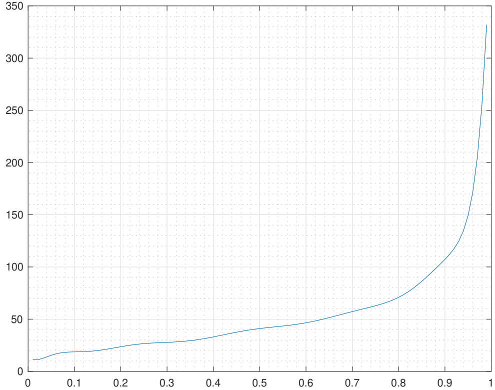


Figure: The degree $m(\varepsilon)$ as a function of ρ.

At this point we are able to solve the problem if we can compute a formula on each curvilinear polygon S_1, S_2, S_3 such that $\Omega = S_1 \cup S_2 \cup S_3$.

This is possible since, for any degree of exactness there are algebraic cubature formulas on circular and elliptical sectors with positive weights and internal nodes, that have been constructed by means of arc blending and *subperiodic* trigonometric gaussian quadrature.

This entails that

1. a quadrature formula with nodes $\{(x_j, y_j)\}$ and positive weights $\{w_j\}$ of exactness degree $n + m$ on Ω will be nearly exact for $f(x, y, g)/g$ if $f \in \mathbb{P}_n(\mathbb{S}_2)$,
2. a formula with nodes $\{(x_j, y_j, g(x_j, y_j))\}$ and weights $\{w_j/g(x_j, y_j)\}$ will be near-algebraic (nearly exact) in $\mathbb{P}_n(\mathcal{T})$, i.e. for spherical polynomials restricted to the spherical triangle $\mathcal{T}$.

We observe that in general, due to the fact that for determining a rule with degree of exatness n on the triangle $\mathcal{T}$, then one must compute one of degree $m+n$ in the curvilinear polygon Ω, the cardinality of the rule, with positive weights and internal nodes, may be pretty high.

There is an algorithm, named Caratheodory-Tchakaloff compression, that allows to compress the rule, so that the resulting one has

- degree of precision $n+m$,
- positive weights,
- a number of nodes at most $(n+m+1)^2$, that is the dimension of the space $P_{n+m}(\mathcal{T})$, taken from those of the initial rule.

As numerical examples, we consider a sphere octant, i.e. the spherical triangle $\mathcal{T}$ with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. For this domain, the parameter ρ ruling the degree $m = m(\epsilon)$, ϵ being the machine precision, is approximatively 0.67.

By a previous figure, this means that $m \approx 60$.

As test functions we take

1. $f_1(x,y,z) = 1 + x + y^2 + x^2 y + x^4 + y^5 + x^2 y^2 z^2$;
2. $f_2(x,y,z) = \cos(10(x+y+z))$;
3. $f_3(x,y,z) = franke(x,y,z)$;
4. $f_4(x,y,z) = (1 + \tanh(9x - 9y + 9z))/9$;
5. $f_5(x,y,z) = (1 + \mathrm{sign}(9x - 9y + 9z))/9$.

We have compute the reference values of these integrals by adaptive codes and reported in the table below the numerical results.

n	$E_{\mathrm{compr}}(f_1)$	$E_{\mathrm{compr}}(f_2)$	$E_{\mathrm{compr}}(f_3)$	$E_{\mathrm{compr}}(f_4)$	$E_{\mathrm{compr}}(f_5)$
5	1e-05	4e-03	6e-02	3e-02	9e-02
10	2e-15	3e-06	9e-04	2e-03	1e-02
15	2e-15	2e-11	2e-04	1e-04	3e-03
20	1e-15	5e-15	2e-05	4e-04	6e-03
25	1e-15	3e-15	2e-07	5e-05	5e-03
30	6e-16	4e-15	4e-08	3e-05	2e-03

Table: Relative errors in the integration of the test functions defined above by the compressed formula, on the sphere octant $\mathcal{T}$ with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.

Finally we report some statistics concerning the *basic* formula and its compressed version, applying *Caratheodory-Tchakaloff algorithm*.

n	# basic	# compr	Cratio	CPU	$E_{\mathrm{basic}}(SPH)$	$E_{\mathrm{compr}}(SPH)$
5	5580	36	155:1	0.02s	3e-15	4e-15
10	6435	121	53:1	0.1s	1e-14	1e-14
15	7560	256	30:1	0.3s	1e-14	3e-14
20	8550	441	19:1	1s	1e-14	5e-14
25	9840	676	15:1	4s	1e-14	6e-14
30	10965	961	11:1	19s	3e-14	1e-13

Table: Cardinalities, compression ratio, CPU time in seconds and average relative errors on Spherical Harmonics (SPH) for the spherical octant $\mathcal{T}$ with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$.

Remark

As for the numerical software, see the Matlab package Software package for polynomial interpolation on spherical triangles.

Let

$$\mathcal{R} = [a_1, b_1] \times [a_2, b_2] \subseteq [0, \pi] \times [0, 2\pi]$$

be a rectangle, and define as spherical rectangle $\Omega_{\mathcal{R}}$ (sometimes also known as geographical rectangle) the subdomain of the sphere $\mathbb{S}^2$ whose points are of the form

$$P = \xi(\theta, \phi) := (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta), \quad (\theta, \phi) \in \mathcal{R}.$$

We observe that depending on $\mathcal{R}$, several well-known subdomains $\Omega_{\mathcal{R}} = \xi(\mathcal{R})$ of the 2-sphere can be defined in this way, as

- caps,
- collars,
- slices,
- more generally spherical rectangles defined by longitudes and latitudes.

Spherical rectangles

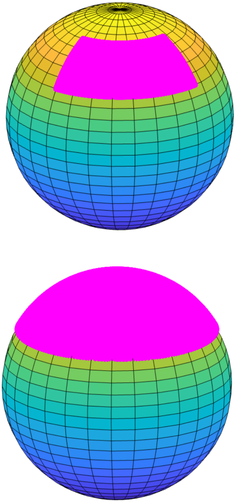


Figure: Above and below: a spherical rectangle $\xi([\pi/6, \pi/3] \times [0, \pi/2])$ and a spherical cap $\xi([0, \pi/3] \times [0, 2\pi])$.

Spherical rectangles

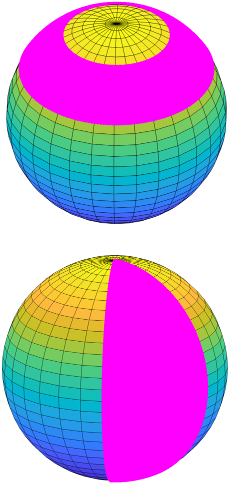


Figure: Above and below: a (spherical) collar $\xi([\pi/6, \pi/3] \times [0, 2\pi])$ and a (spherical) slice $\xi([0, \pi] \times [0, \pi/3])$.

To this purpose, we introduce the following result

Theorem

Let $w \in [0, \pi]$.

- $\mathbb{T}_n([-\omega, \omega]) = span\{1, \cos(k\theta), \sin(k\theta)\}, 1 \leq k \leq n, \theta \in [-\omega, \omega]$,
- $w : [-\omega, \omega] \to \mathbb{R}$ *be a symmetric weight function,*
- $\{\xi_j\}_{j=1,\ldots,n+1}$, $\{\lambda_j\}_{j=1,\ldots,n+1}$ *be respectively the nodes and the weights of an algebraic gaussian rule relatively to the symmetric weight function*

$$\tilde{s}(x) = w(2\arcsin(\sin(\omega/2)x)) \frac{2\sin(\omega/2)}{\sqrt{1 - \sin^2(\omega/2)\, x^2}}, x \in (-1, 1).$$

Then

$$\int_{-\omega}^{\omega} f(\theta) w(\theta)\, d\theta = \sum_{j=1}^{n+1} \lambda_j f(\theta_j), f \in \mathbb{T}_n([-\omega, \omega]) \quad (4)$$

where $\theta_j = 2\arcsin(\sin(\omega/2)\xi_j) \in (-\omega, \omega),\ j = 1, \ldots, n+1$.

This theorem says that if we intend to integrate

$$\int_{\omega}^{\omega} f(\theta)w(\theta)d\theta$$

where f is a trigonometric polynomial of degree n and w a symmetric weight function in $[-\omega, \omega]$ then it is sufficient to

- compute the nodes $\{\xi_j\}$ and weights $\{\lambda_j\}$ of a gaussian rule with n nodes w.r.t. a certain weight function $\tilde{s}$;
- modify the nodes $\{\xi_j\}$ into $\{\theta_j\}$ by a simple transformation.

The hidden difficulty is that the computation of these formula is not trivial since the weight function is a little unusual.

From the previous theorem we have the following one, that determines an algebraic rule over the spherical rectangle $\Omega_{\mathcal{R}} \subseteq \mathbb{S}^2$.

Theorem

Let

- Ω_R *a spherical rectangle, where* $\mathcal{R} = [a_1, b_1] \times [a_2, b_2] \subseteq [0, \pi] \times [0, 2\pi]$;
- $\{\theta_k^{[a_j,b_j]}\}_{k=1,\dots,n+3-j}$ *and* $\{\lambda_k^{[a_j,b_j]}\}_{k=1,\dots,n+3-j}$ *be respectively the nodes and the weights of a gaussian subperiodic trigonometric rule on* $[a_j, b_j]$ *w.r.t.* $w(x) = 1$, *having trigonometric degree of precision* $n + 2 - j$, *for* $j = 1, 2$.

Then the cubature rule

$$S_n(f) = \sum_{j_1=1}^{n+2} \sum_{j_2=1}^{n+1} \lambda_{j_1,j_2} f(\xi_{j_1,j_2})$$

where

$$\xi_{j_1,j_2} = \xi(\theta_{j_1}^{[a_1,b_1]}, \theta_{j_2}^{[a_2,b_2]})$$

$$\lambda_{j_1,j_2} = \prod_{k=1}^{2} \lambda_{j_k} \sin^{2-k}(\theta_{j_k}^{[a_k,b_k]})$$

integrates exactly in $\Omega_{\mathcal{R}}$ *every algebraic polynomial of total degree* n.

Remark (Caps)

The cardinality of these rules is $\approx n^2$*, where* n *is the degree of precision. With some tricks, one can have a formula on the spherical cap with* $\approx n^2/2$ *points.*

Remark (Software)

Though at first sight the result of the theorem is a little complicated, in practice when one provides the nodes and the weights of the subperiodic formula (not easy!), everything become simpler.

As for the numerical software, see the Matlab package Cubature rules on spherical rectangles.

As numerical tests, we consider the cubature of the functions

$$\begin{aligned} f_1(\mathbf{x}) &= \exp(-x^2 - 100\,y^2 - 0.5\,z^2), \\ f_2(\mathbf{x}) &= \sin(-x^2 - 100\,y^2 - 0.5\,z^2), \\ f_3(\mathbf{x}) &= \max(1/4 - ((x - 1/\sqrt{5})^2 + (y - 2/\sqrt{5})^2 + (z - 2/\sqrt{5})^2), 0))^3 \end{aligned}$$

on the spherical rectangle

$$\Omega_{\mathcal{R}} = \xi(\mathcal{R}), \;\; \mathcal{R} = [\pi/6, \pi/3] \times [0, \pi/2].$$

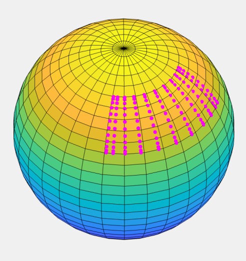


Figure: The spherical rectangle $\xi([\pi/6, \pi/3] \times [0, \pi/2])$ and the nodes of the rule of degree of exactness 10.

Deg.	f_1	f_2	f_3
5	$3.34e-04$	$7.38e-02$	$4.53e-06$
10	$4.89e-06$	$2.69e-02$	$5.44e-07$
15	$9.12e-09$	$5.14e-03$	$4.07e-08$
20	$1.76e-10$	$1.13e-02$	$2.43e-08$
25	$7.73e-14$	$1.13e-02$	$9.53e-09$
30	$3.33e-16$	$1.23e-03$	$2.23e-09$
35	$3.47e-17$	$2.58e-05$	$2.33e-09$
40	$1.14e-16$	$1.96e-07$	$2.82e-10$
45	$3.47e-17$	$6.94e-10$	$8.84e-10$
50	$2.08e-17$	$1.33e-12$	$5.48e-11$

Table: Absolute errors for degrees $5, 10, \ldots, 50$, w.r.t. the integrals on the spherical rectangle $\xi([\pi/6, \pi/3] \times [0, \pi/2])$ on the test functions f_1, f_2, f_3.

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