Basics on tensorial rules

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"Numerical cubature and its applications"

In this presentation we consider the basics of the tensorial type rules (sometimes known as *product rules*), for numerical integration over a domain $\Omega \subset \mathbb{R}^d$ via a weighted sum, that is

$$\int_{\Omega} f(\mathbf{x}) d\Omega \approx \sum_{k=1}^{n} w_k f(\mathbf{x}_k).$$

These formulas are usually based on univariate rules of Gaussian type, in virtue of all their favourable properties.

We will consider the basic case of domains Ω as

- the hypercube $[-1, 1]^d$;
- the simplex;
- the disk and more general specific domains obtained by *linear blending*.

For details, see e.g. [**?**, p.361].

In order to show the basic idea behind this approach, we consider first the example of the sometimes called *normal domain*.

To introduce this technique, we consider the case of bivariate normal domains

$$\Omega = \{(x, y) : \quad a \leq x \leq b, \quad \psi(x) \leq y \leq \phi(x)\},$$

being $\psi, \phi : [a, b] \rightarrow \mathbb{R}$ two sufficiently regular functions.

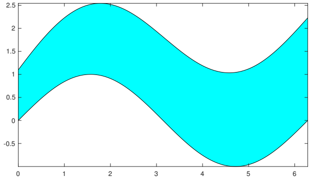

Figure: A normal domain Ω where $a = 0$, $b = 2\pi$, $\psi(x) = \sin(x)$, $\phi(x) = \sin(x) + \log(x + 3)$.

Since

$$\Omega = \{(x, y) : \quad a \leq x \leq b, \quad \psi(x) \leq y \leq \phi(x)\},$$

- setting $g(x) := \int_{\psi(x)}^{\phi(x)} f(x, y)\, dy$,
- using the rule $\int_a^b g(x)\, dx \approx \sum_{i=1}^n w_i g(x_i)$,

we havee from basic calculus,

$$\begin{aligned} I(f) : \; &= \int_{\Omega} f(\mathbf{x}) d\Omega = \int_a^b \left(\int_{\psi(x)}^{\phi(x)} f(x, y)\, dy \right) dx = \int_a^b g(x)\, dx \\ &\approx \sum_{i=1}^n w_i g(x_i) = \sum_{i=1}^n w_i \int_{\psi(x_i)}^{\phi(x_i)} f(x_i, y)\, dy \end{aligned} \tag{1}$$

We observe that we can approximated the n inner integrals of

$$I(f) \approx \sum_{i=1}^{n} w_i \int_{\psi(x_i)}^{\phi(x_i)} f(x_i, y)\, dy$$

with a suitable m-point rule.

Notice that the domain of the integral may vary with the index "i ", but that this is not a problem, since we can scale the rule (e.g. one can use a shifted Gauss-Legendre rule, from $[-1, 1]$ to $[\psi(x_i), \phi(x_i)]$). If

$$\int_{\psi(x_i)}^{\phi(x_i)} f(x_i, y)\, dy \approx \sum_{j=1}^{m} v_{j,i} f(x_i, y_{j,i})$$

we finally get the formula with cardinality mn

$$I(f) \approx \sum_{i=1}^{n} w_i \sum_{j=1}^{m} v_{j,i} f(x_i, y_{j,i}) = \sum_{i=1}^{n} \sum_{j=1}^{m} w_i v_{j,i} f(x_i, y_{j,i})$$

We observe that in the construction of the formula

$$I(f) \approx \sum_{i=1}^{n} w_i \sum_{j=1}^{m} v_{j,i} f(x_i, y_{j,i}) = \sum_{i=1}^{n} \sum_{j=1}^{m} w_i v_{j,i} f(x_i, y_{j,i})$$

we did not make assumptions on the degree of exactness of

$$\int_a^b g(x)\, dx \approx \sum_{i=1}^{n} w_i g(x_i)$$

and of each

$$\int_{\psi(x_i)}^{\phi(x_i)} f(x_i, y)\, dy \approx \sum_{j=1}^{m} v_{j,i} f(x_i, y_{j,i}).$$

Except for specific cases, e.g. ϕ, ψ polynomials, it will not be possible to choose m, n so to have formulas with a fixed degree of exactness.

We define some Matlab codes to illustrate these formulas. We start with a routine `define_normal_rule` that computes the nodes and weights on a normal domain defined by the interval $[a, b]$ and the functions ψ, ϕ.

```
function [nodes,weights]=define_normal_rule(n,m,a,b,psi,phi)

% Rule direction "x".
abn=r_jacobi(n,0,0); xw=gauss(n,abn); % Gauss-Legendre
x=xw(:,1); w=xw(:,2); x=(a+b)/2+(b-a)*x/2; w=(b-a)*w/2;

% Rule direction "y".
abm=r_jacobi(m,0,0); yv=gauss(m,abm); % Gauss-Legendre

% Rule on the normal domain
y=yv(:,1); v=yv(:,2);

nodes=[]; weights=[];

for i=1:n
    psi_i=feval(psi,x(i)); phi_i=feval(phi,x(i));
    y_i=(psi_i+phi_i)/2+((phi_i-psi_i)/2)*y; % scaled nodes
    v_i=((phi_i-psi_i)/2)*v; % scaled weights

    nodes_add=[x(i)*ones(size(y_i)) y_i]; % rule nodes/weights to add
    nodes=[nodes; nodes_add];
    weights_add=w(i)*v_i;
    weights=[weights; weights_add];
end
```

Normal domains

Next we implement a demo, to study the case in which

- $a = 0$, $b = 2\pi$, $\psi(x) = \sin(x)$ and $\phi(x) = \sin(x) + \log(x+3)$;
- the integrand is $f(x,y) = (x + 0.5 * y)^{10}$ and $I(f) = 234913153.2071612\ldots$.

```
function demo_normal_domain

a=0; b=2*pi; % Define "normal domain".
psi=@(x) sin(x);
phi=@(x) sin(x)+log(x+3);
f=@(x,y) (x+0.5*y).^10; % integrand
Iex=2.349131532071612e+08; % integral computed with high order rule
n=10; m=11; % Define "n", "m" (cardinality of the rules).

% External routine that computes nodes and weights.
[nodes,weights]=define_normal_rule(n,m,a,b,psi,phi);

% Compute integral.
fnodes=feval(f,nodes(:,1),nodes(:,2));
Inum=weights'*feval(f,nodes(:,1),nodes(:,2));

fprintf('\n \t * I : %-1.15e',Inum);
fprintf('\n \t * AE: %-1.3e',abs(Inum-Iex));
fprintf('\n \t * RE: %-1.3e \n',abs(Inum-Iex)/abs(Iex));

% Plot normal domain (external subroutine)
plot_normal_domain(a,b,psi,phi);
plot(nodes(:,1),nodes(:,2),'go','MarkerEdgeColor','k',...
    'MarkerFaceColor','g','MarkerSize',4);
axis equal; axis tight;
hold off;
```

Normal domains

Below we mention the routine for plotting the domain.

```
function plot_normal_domain(a,b,psi,phi)

t=linspace(a,b,1000); t=t';
psi_t=feval(psi,t);
phi_t=feval(phi,t);

pts_bound_low=[t psi_t];
pts_bound_up=[flipud(t) flipud(phi_t)];
pts=[pts_bound_low; pts_bound_up; pts_bound_low(1,:)];

fill(pts(:,1),pts(:,2),'c');
hold on;
```

As numerical results we see that the formula is not exact for degree 10, since the integrand belongs to $\mathbb{P}_{10}$.

```
>> demo_normal_domain

   * I : 2.349132020504614e+08
   * AE: 4.884e+01
   * RE: 2.079e-07
>>
```

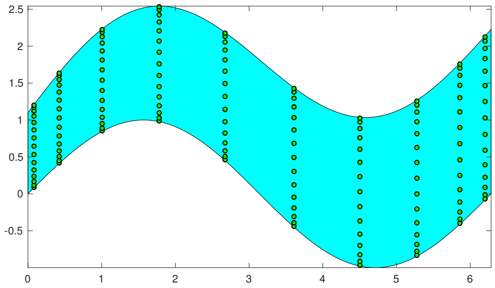

Figure: The normal domain Ω where $a = 0$, $b = 2\pi$, $\psi(x) = \sin(x)$, $\phi(x) = \sin(x) + \log(x+3)$ and the cubature nodes achieved from the usage of Gauss-Legendre rules in which $n = 10$ and $m = 15$.

- This technique can be used for computing integrals over hypercubes $\Omega = [-1, 1]^d$ (thus, by shifting, also on hyperectangles).
- This time we ask the rule must have a fixed degree of exactness $\mathrm{ADE} = \delta$.
- Following the ideas described in the part about normal domains, we adopt a Gauss-Legendre rule

$$\sum_{-1}^{1} g(x)dx \approx \sum_{i=1}^{n} w_i g(x_i)$$

with $n = \lceil \frac{\delta+1}{2} \rceil$ nodes, so having at least $\mathrm{ADE} = \delta$.

Thus we have

$$\begin{aligned} I(f) : &= \int_{\Omega} f(\mathbf{x}) d\Omega = \int_{-1}^{1} \dots \int_{-1}^{1} f(x_1, \dots, x_d)\, dx_1 \dots dx_d \\ &\approx \sum_{i_1=1}^{n} \dots \sum_{i_d=1}^{n} w_{i_1} \dots w_{i_d} f(x_{i_1}, \dots, x_{i_d}). \end{aligned} \quad (2)$$

That is a formula with cardinality

$$\#_{\delta,d} = n^d = \left(\left\lceil \frac{\delta+1}{2} \right\rceil \right)^d \approx \left(\frac{\delta}{2} \right)^d .$$

Since it grows exponentially with the dimension d, this formula maybe not suitable for d high, causing the so called *curse of dimensionality*.

For example, if $ADE = \delta = 20$ and $d = 10$, one needs 10^{10} function evaluations (possibly expensive, in view of the number of variables involved).

In general this kind of rules are very used in low dimension (e.g. 2 or 3), but they are not minimal, in the sense that there are rules with much lower number of nodes, sharing the same cardinality.

In the case of the square $[-1, 1]$, a rule with $ADE = \delta$, in view of a bound by Möller must have at least cardinality

$$n_\delta = \begin{cases} \frac{(k+1)(k+2)}{2}, & d = 2k \\ \frac{(k+1)(k+2)}{2} + \lfloor \frac{(k+1)}{2} \rfloor, & d = 2k+1 \end{cases}$$

and there are rules that go closer to this bound than those of tensorial type.

ADE	*MB*	*AMR*	*TR*
5	7	7	9
10	21	22	36
15	40	46	64
20	66	77	121
25	97	113	169
30	136	166	256
35	180	222	324
40	231	287	441
45	287	361	529
50	351	442	676

Table: Formulas on the unit square. Algebraic degree of exactness `ADE`, the Möeller bound `MB`, the cardinality of almost minimal rules `AMR` and that of tensorial rules `TR`.

We now approximate certain integrals on the unit-square and unit-cube.

```
function demo_hypercube

ADE=10; d=2;

% Define integrand
switch d
    case 2
        f=@(x,y) (0.3*x+0.9*y).^10;  I=5.002201832727280e-01;
    case 3
        f=@(x,y,z) (0.3*x+0.9*y+0.8*z).^10;  I=4.377443514181815e+01;
end

% Gaussian rule with degree ADE.
n=ceil((ADE+1)/2);
abn=r_jacobi(n,0,0); xw=gauss(n,abn); % Gauss-Legendre
x=xw(:,1); w=xw(:,2);
switch d
    case 2
        [x1,x2] = meshgrid(x); [w1,w2] = meshgrid(w);
        fP=feval(f,x1,x2); w=w1.*w2;
        Inum=sum(sum(w.*fP));
    case 3
        [x1,x2,x3] = ndgrid(x); [w1,w2,w3] = ndgrid(w);
        fP=feval(f,x1,x2,x3); w=w1.*w2.*w3;
        Inum=sum(sum(sum(w.*fP)));
end

fprintf('\n \t * I : %1.15e',Inum)
fprintf('\n \t * AE: %1.3e',abs(I-Inum))
fprintf('\n \t * RE: %1.3e \n',abs(I-Inum)/abs(Inum))
```

1. We consider the case of the formula on $[-1,1]^2$. It has degree 10 and we show it integrates exactly (in the numerical sense!) $p(x,y) = (0.3x + 0.9y)^{10}$.

```
>> f=@(x,y) (0.3*x+0.9*y).^10;
>> I=integral2(f,-1,1,-1,1,'AbsTol',10^(-15),'RelTol',10^(-15));
>> format long e
>> I
I =
     5.002201832727280e-01
>> demo_hypercube

    * I : 5.002201832727267e-01
    * AE: 1.332e-15
    * RE: 2.663e-15
>>
```

2. We consider the case of the formula on $[-1,1]^3$. It has degree 10 and we show it integrates exactly (in the numerical sense!) $p(x,y,z) = (0.3x + 0.9y + 0.8z)^{10}$.

```
>> f=@(x,y,z) (0.3*x+0.9*y+0.8*z).^10;
>> I=integral3(f,-1,1,-1,1,-1,1,'AbsTol',10^(-15),'RelTol',10^(-15));
>> format long; I
I =
  43.774435141818145
>> demo_hypercube

    * I : 4.377443514181813e+01
    * AE: 2.132e-14
    * RE: 4.870e-16
>>
```

Similar rules can be established for the d-dimensional unit-ball. For sake of simplicity we restrict our attention to the bivariate unit-disk, i.e. $\Omega \equiv B(0,1)$.
We observe that in this case, after the transformation in polar coordinates, taking into account the determinant of the jacobian matrix,

$$\int_{\Omega} f(\mathbf{x})\, d\Omega = \int_0^1 \int_0^{2\pi} f(r\cos(\theta), r\sin(\theta)) \cdot r \, d\theta \, dr$$

Observe that the r.h.s. consists of an integral over a rectangle $[a,b] \times [0,2\pi]$ where the integrand is

$$g(r,\theta) = f(r\cos(\theta), r\sin(\theta)) \cdot r.$$

As rules, a common choice, to get a formula with $ADE = \delta$, is to adopt

- a Gauss-Legendre rule, shifted in $[0,1]$, with $ADE = \delta + 1$, in the variable "r ",
- a trapezoidal rule, on $\delta + 2$ equispaced points, including the extrema, on the angular interval $[0, 2\pi]$, that can be proved to be exact over *trigonometric* polynomials of degree δ.

In view of the fact that

- the Gauss-Legendre rule does not have nodes at the extrema 0, 1,
- the trapezoidal rule has nodes in 0, 2π and $(r\cos(0), r\sin(0)) = (r\cos(2\pi), r\sin(2\pi))$,

after some computation one can see that such a product rule has cardinality $\lceil(\delta + 1)/2\rceil(\delta + 1)$.

In what follows, we implement these tensorial rules and show some numerical examples.

```
function [nodes,weights]=define_rule_disk(ade)

% Cubature rule on the unit disk with ADE equal to ade.
% Note: As output, the nodes will be in cartesian coordinates.

% shifted gaussian rule: r direction
m=ceil((ade+1)/2);
abm=r_jacobi(m,0,0); yv=gauss(m,abm); % Gauss-Legendre
rnodes=(yv(:,1)+1)/2; rw=yv(:,2)/2;

% trapezoidal rule, consider first and last node repetition
N=ade+2;
t=linspace(0,2*pi,N); t=t(1:end-1); t=t';
tw=(2*pi/(N-1))*ones(N-1,1);

% define tensorial rule
[r_mat,th_mat]=meshgrid(rnodes,t);
[r_matw,th_matw]=meshgrid(rw,tw);

x_mat=r_mat.*cos(th_mat); y_mat=r_mat.*sin(th_mat);
w_mat=r_matw.*th_matw.*r_mat;

nodes=[x_mat(:) y_mat(:)]; weights=w_mat(:);
```

Unit-disk

```
function demo_disk

ade=10; % Define "n", "m" (cardinality of the rules).
example=1; % define example

switch example
    case 1
        f=@(x,y) (x+0.5*y).^10; % integrand
        Iex=3.932323797070195e-01 ; % numerically exact integral
    otherwise
        f=@(x,y) (1+x+0.5*y).^11; % integrand
        Iex=5.546261116442703e+02 ; % numerically exact integral
end

% External routine that computes nodes and weights.
[nodes,weights]=define_rule_disk(ade);

% Compute integral.
fnodes=feval(f,nodes(:,1),nodes(:,2));
Inum=weights'*feval(f,nodes(:,1),nodes(:,2));

% Statistics
fprintf('\n \t * # : %-8.0f',length(weights));
fprintf('\n \t * #T: %-8.0f',ceil((ade+1)/2)*(ade+1));
fprintf('\n \t * I  : %-1.15e',Inum);
fprintf('\n \t * AE: %-1.3e',abs(Inum-Iex));
fprintf('\n \t * RE: %-1.3e \n',abs(Inum-Iex)/abs(Iex));

% Plot disk and pointset
th=linspace(0,2*pi,100); gray_color=[211, 211, 211]/256;
fill(cos(th),sin(th),gray_color); hold on;
plot(nodes(:,1),nodes(:,2),'go','MarkerEdgeColor','k',...
    'MarkerFaceColor','g','MarkerSize',6);
axis equal; axis tight;
hold off;
```

1. As first experiment we integrate a polynomial of degree 10, by a rule with ADE equal to 10. To this purpose we set in `ade=10` and `example=1` in the file `demo_disk`, getting

```
>> demo_disk
   * # : 66
   * #T: 66
   * I : 3.932323797070124e-01
   * AE: 7.161e-15
   * RE: 1.821e-14
>>
```

2. As second experiment we integrate a polynomial of degree 11, by a rule with ADE equal to 11. To this purpose we set in `ade=11` and `example=2` in the file `demo_disk`, getting

```
>> demo_disk
   * # : 72
   * #T: 72
   * I : 3.932323797070125e-01
   * AE: 6.994e-15
   * RE: 1.779e-14
>>
```

We observe that the numerical approximation of the desired integral can be obtained by the following adaptive routine,

```
function exact_integral_disk

example=1;

switch example
    case 1
        fpolar=@(r,t) (1*r.*cos(t)+0.5*r.*sin(t)).^10.*r;
        I=integral2(fpolar,0,1,0,2*pi,'AbsTol',10^(-15),'RelTol',10^(-15));
    case 2
        fpolar=@(r,t) (1+1*r.*cos(t)+0.5*r.*sin(t)).^11.*r;
        I=integral2(fpolar,0,1,0,2*pi,'AbsTol',10^(-15),'RelTol',10^(-15));
end

fprintf('\n \t I : %1.15e \n',I)
```

or alternatively, by means of `chebfun` environment,

```
function exact_integral_disk_chebfun

example=1;

switch example
    case 1
        f=@(x,y) (1*x+0.5*y).^10;
    case 2
        f=@(x,y) (1+1*x+0.5*y).^11;
end

fc=diskfun(f); Ic=sum2(fc);
fprintf('\n \t Ic: %1.15e \n',Ic)
```

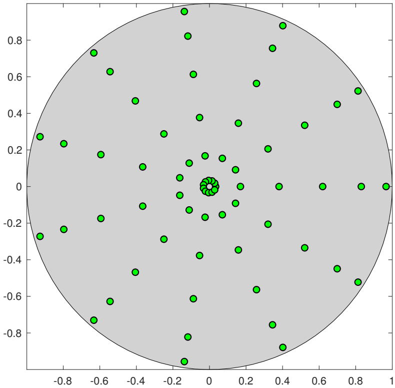

Figure: Nodes of the tensorial rule on the unit-disk, for $ADE = 10$.

A tensorial rule can be also found for the n-simplex. For simplicity we shall take into account the case of a triangle, see e.g. [**?**] for a survey on the topic.

In general, there are many reference triangles, depending on the purpose. We will consider first the triangle $\mathcal{T}$ with vertices $(0,0)$, $(1,0)$, $(0,1)$.

It can be easily seen, setting $y = ux$ and shifting the variables that is $s = 2x - 1$, $t = 2u - 1$

$$\begin{aligned}
I(f) &:= \int_{\mathcal{T}} f(x,y)\,dxdy = \int_0^1 \int_0^x f(x,y)\,dx\,dy \\
&= \int_0^1 x \int_0^1 f(x,xu)\,dx\,du = \ldots = \\
&= \int_{-1}^1 \int_{-1}^1 \frac{1}{8} f\left(\frac{s+1}{2}, \frac{(s+1)(t+1)}{4}\right)(1+s)\,ds\,dt \qquad (3)
\end{aligned}$$

Thus, we have reduced to a certain interval on the square $[-1,1]^2$.

Setting

$$\phi(s,t) = \frac{1}{8} f\left(\frac{s+1}{2}, \frac{(s+1)(t+1)}{4}\right)$$

we get

$$\begin{aligned} I(f) &= \int_{-1}^{1}\int_{-1}^{1} \frac{1}{8} f\left(\frac{s+1}{2}, \frac{(s+1)(t+1)}{4}\right)(1+s)\,ds\,dt \\ &= \int_{-1}^{1}\int_{-1}^{1} \phi(s,t)(1+s)\,ds\,dt \qquad (4) \end{aligned}$$

Defining

- for the direction s, a Gauss-Jacobi rule with degree of exactness $ADE = \delta$, w.r.t. the weight $(1-s)^0(1+s)^1$,
- for the direction t, a Gauss-Legendre rule with degree of exactness $ADE = \delta$, i.e. w.r.t. the weight $(1-s)^0(1+s)^0$,

we get a formula with positive weights, internal nodes, $ADE = \delta$ on the simplex, with cardinality $(\lceil \frac{\delta+1}{2} \rceil)^2 \approx \frac{\delta^2}{4}$.

More precisely, letting

1. $n = \lceil \frac{\delta+1}{2} \rceil$,
2. $\phi(s,t) = \frac{1}{8} f\left(\frac{s+1}{2}, \frac{(s+1)}{2}\frac{(t+1)}{4}\right)$,
3. $\int_{-1}^{1} g(s)(1+s)ds = \sum_{i=1}^{n} w_i^{(GJ)} g(x_i^{(GJ)})$,
4. $\int_{-1}^{1} g(t)dt = \sum_{i=1}^{n} w_i^{(GL)} g(x_i^{(GL)})$,

we have

$$\begin{aligned}
I(f) &= \int_{-1}^{1}\int_{-1}^{1} \phi(s,t)(1+s)\, ds\, dt \\
&\approx \int_{-1}^{1} \sum_{i=1}^{n} w_i^{(GJ)} \phi(x_i^{(GJ)}, t)dt = \sum_{i=1}^{n} w_i^{(GJ)} \int_{-1}^{1} \phi(x_i^{(GJ)}, t)dt \\
&\approx \sum_{i=1}^{n}\sum_{j=1}^{n} w_i^{(GL)} w_j^{(GJ)} \phi(x_i^{(GJ)}, x_j^{(GL)}) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{w_i^{(GL)} w_j^{(GJ)}}{8} f\left(\frac{x_i^{(GJ)}+1}{2}, \frac{x_i^{(GJ)}+1}{2} \cdot \frac{x_j^{(GL)}+1}{2}\right).
\end{aligned}$$

As first thing, we define a routine `define_rule_simplex` that determines such a formula on the reference simplex.

```
function [nodes,weights]=define_rule_simplex(ade)

% Cubature rule on the unit simplex
% * with vertices (0,0), (1,0), (1,1),
% * with ADE equal to ade.

% Gaussian-Jacobi rule.
m=ceil((ade+1)/2);
ab_GJ=r_jacobi(m,0,1); xw_GJ=gauss(m,ab_GJ); % Gauss-Jacobi
x_GJ=(xw_GJ(:,1)+1)/2; w_GJ=xw_GJ(:,2);

% Gaussian-Legendre rule.
ab_GL=r_jacobi(m,0,0); xw_GL=gauss(m,ab_GL); % Gauss-Legendre
x_GL=(xw_GL(:,1)+1)/2; w_GL=xw_GL(:,2);

% Define tensorial rule
[x_mat_GJ,x_mat_GL]=meshgrid(x_GJ,x_GL);
X_mat=x_mat_GJ; Y_mat=x_mat_GJ.*x_mat_GL;

[w_mat_GJ,w_mat_GL]=meshgrid(w_GJ,w_GL);
W_mat=(1/8)*w_mat_GJ.*w_mat_GL;

nodes=[X_mat(:) Y_mat(:)];
weights=W_mat(:);
```

Next we present a Matlab demo `demo_simplex` in which we test the polynomial exactness and plot the nodes of the formula.

```
function demo_simplex

ade=10;

f=@(x,y) (0.3*x+0.9*y).^10;
I=6.254277723408297e-02;


% Gaussian rule with degree ADE.
[nodes,weights]=define_rule_simplex(ade);
fP=feval(f,nodes(:,1),nodes(:,2));

Inum=weights'*fP;

% Stats
fprintf('\n \t * ade: %-8.0f',ade)
fprintf('\n \t * #   : %-8.0f',length(weights))
fprintf('\n \t * I   : %-1.15e',Inum)
fprintf('\n \t * AE  : %-1.3e',abs(I-Inum))
fprintf('\n \t * RE  : %-1.3e \n',abs(I-Inum)/abs(Inum))

gray_color=[211, 211, 211]/256;
fill([0 1 1 0],[0 0 1 0],gray_color);
hold on;
plot(nodes(:,1),nodes(:,2),'go','MarkerEdgeColor','k',...
    'MarkerFaceColor','g','MarkerSize',6);
axis equal; axis tight;
hold off;
```

For the computation of the reference value `I`, we have written the routine `exact_integral_simplex`, based on adaptive procedure `integral2` over a rectangle.

1. As first method for approximating $\int_{\mathcal{T}} f(x,y)\,dxdy$ we took into account an integrand on $[0,1]^2$, equal to $f \cdot \mathcal{X}_{\mathcal{T}}$, where $\mathcal{X}_{\mathcal{T}}$ is the carachteristic function on the simplex $\mathcal{T}$.
2. Alternatively we replaced the desired integral with one on a square, as described in **??**

```
function exact_integral_simplex

f=@(x,y) (0.3*x+0.9*y).^10; % integrand on the simplex

method=2;

switch method
    case 1
        F=@(x,y) f(x,y).*(y <= x);
        I=integral2(F,0,1,0,1,'AbsTol',10^(-15),'RelTol',10^(-15));
    case 2
        F=@(s,t) (1/8)*f( (s+1)/2,(s+1).*(t+1)/4) .*(1+s);
        I=integral2(F,-1,1,-1,1,'AbsTol',10^(-15),'RelTol',10^(-15));
end

fprintf('\n \t I : %1.15e \n ',I)
```

Running the demo we get

```
>> demo_simplex

   * ade: 10
   * #  : 36
   * I  : 6.254277723409099e-02
   * AE : 8.021e-15
   * RE : 1.283e-13
>>
```

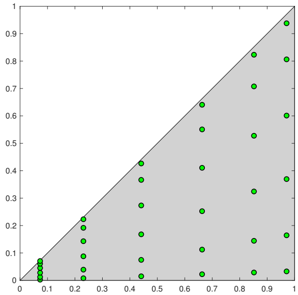

Figure: Nodes of the tensorial rule on the unit-simplex, for $ADE = 10$.

As before, these rules are easily at hand, but they are far from being the best around in terms of cardinality.

For example, at degree 10, the tensorial rule above had 36 nodes, but it is known there is one with these feature having only 24 positive weights and internal nodes (sometime known with the acronym *PI type*, see table below).

δ	N^*_δ	δ	N^*_δ	δ	N^*_δ	δ	N^*_δ	δ	N^*_δ
1	1	11	27	21	85	31	181	41	309
2	3	12	32	22	93	32	193	42	324
3	4	13	36	23	100	33	204	43	339
4	6	14	42	24	109	34	214	44	354
5	7	15	46	25	117	35	228	45	370
6	11	16	52	26	130	36	243	46	385
7	12	17	57	27	141	37	252	47	399
8	16	18	66	28	150	38	267	48	423
9	19	19	70	29	159	39	282	49	435
10	24	20	78	30	171	40	295	50	453

Table: Cardinality N^*_δ of (almost) minimal rules on triangles with $ADE = \delta$.

We investigate the case of circular regions that can be obtained by the so called *linear blending* of elliptical arcs.

Let two elliptical arcs defined respictively by

$$\begin{aligned} P(\theta) &= A_1 \cos(\theta) + B_1 \sin(\theta) + C_1, \\ Q(\theta) &= A_2 \cos(\theta) + B_2 \sin(\theta) + C_2, \end{aligned}$$

where $\theta \in [\alpha, \beta]$, $0 \leq \beta - \alpha \leq 2\pi$ and

$$A_i = (a_{i1}, a_{i2}), \quad B_i = (b_{i1}, b_{i2}), \quad C_i = (c_{i1}, c_{i2}), \quad i = 1, 2.$$

The region

$$\mathcal{S} = \{(x, y) = U(t, \theta) = tP(\theta) + (1 - t)Q(\theta),\ (t, \theta) \in [0, 1] \times [\alpha, \beta]\}$$

is known as linear blending of elliptical arcs.

We provide some examples, for different choices of the parameters.

Example

Set in (**??**)

1. $A_1 = (r, 0)$, $B_1 = (0, r)$, $C_1 = (0, 0)$,
2. $A_2 = (r, 0)$, $B_2 = (0, -r)$, $C_2 = (0, 0)$.

and consider the interval $[0, \beta]$ with $0 < \beta \leq \pi$.
The regions that we obtain are circular segments. In particular for $\beta = \pi$ we get the unit-disk.

Linear blending: circular segments 1

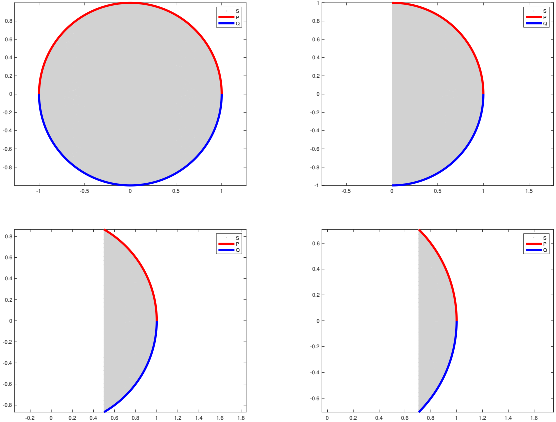

Figure: Example 1, with $\beta = \pi$, $\beta = \pi/2$, $\beta = \pi/3$, $\beta = \pi/4$.

Example

Set in (**??**)

1. $A_1 = (r, 0)$, $B_1 = (0, r)$, $C_1 = (\cos(\beta), 0)$,
2. $A_2 = (r, 0)$, $B_2 = (0, r)$, $C_2 = (0, 0)$.

and consider the interval $[-\beta, \beta]$ with $0 < \beta \leq \pi/2$.
The regions that we obtain are again circular segments.

Linear blending: circular segments 2

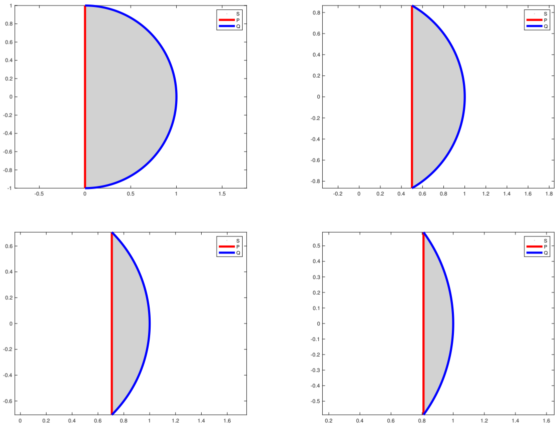

Figure: Example 2, with $\beta = \pi$, $\beta = \pi/2$, $\beta = \pi/3$, $\beta = \pi/4$, $\beta = \pi/5$.

Example

Set in (??)

1. $A_1 = (r\cos(\beta), 0)$, $B_1 = (0, r)$, $C_1 = (\cos(\beta), 0)$,
2. $A_2 = (r, 0)$, $B_2 = (0, r)$, $C_2 = (0, 0)$.

and consider the interval $[-\beta, \beta]$ with $0 < \beta \leq \pi/2$.
The regions that we obtain are again circular segments.

Linear blending: circular segments 3

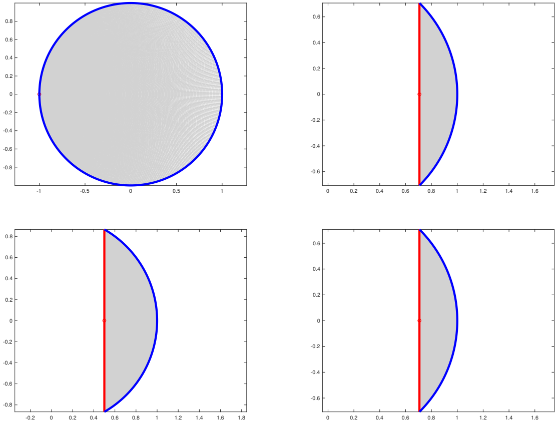

Figure: Example 3, with $\beta = \pi$, $\beta = \pi/2$, $\beta = \pi/3$, $\beta = \pi/4$, $\beta = \pi/5$.

Example

Set in (??)

1. $A_1 = (r_1, 0)$, $B_1 = (0, r_1)$, $C_1 = (0, 0)$,
2. $A_2 = (r_2, 0)$, $B_2 = (0, r_2)$, $C_2 = (0, 0)$.

and consider the interval $[\alpha, \beta]$ with $0 < \beta - \alpha \leq 2\pi$.
The regions that we obtain are symmetric annular sectors.

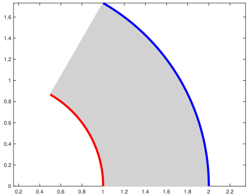

Figure: Symmetric annular sector, with $r_1 = 1$, $r_2 = 2$, $\alpha = 0$, $\beta = \pi/3$.

Example

Set in (??)

1. $A_1 = (0,0)$, $B_1 = (0,0)$, $C_1 = (0,0)$,
2. $A_2 = (r,0)$, $B_2 = (0,r)$, $C_2 = (0,0)$.

and consider the interval $[\alpha, \beta]$ with $0 < \beta - \alpha \leq 2\pi$.
The regions that we obtain are sectors.

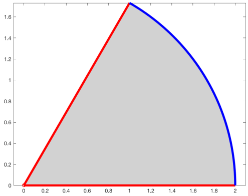

Figure: Symmetric annular sector, with $r = 1$, $\alpha = 0$, $\beta = \pi/3$.

As one may understand, depending on the parameters, many other circular regions can be defined as

- Asymmetric sectors and asymmetric annuli,
- circular zones,
- circular lenses,
- butterfly-shaped and candy-shaped regions.

See [**?**] for more details.

Theorem

Consider the planar domain generated by linear blending of two parametrics arcs

$$\mathcal{S} = \{(x,y) = U(t,\theta) = tP(\theta)+(1-t)Q(theta) \in [0,1]\times[\alpha,\beta], \quad 0 < \beta-\alpha$$

where

$$\begin{array}{rcl} P(\theta) & = & A_1\cos(\theta) + B_1\sin(\theta) + C_1, \\ Q(\theta) & = & A_2\cos(\theta) + B_2\sin(\theta) + C_2, \end{array}$$

in which $\theta \in [\alpha,\beta]$, $0 \leq \beta-\alpha \leq 2\pi$ and

$$A_i = (a_{i1}, a_{i2}), \quad B_i = (b_{i1}, b_{i2}), \quad C_i = (c_{i1}, c_{i2}), \quad i = 1,2.$$

Assume that the transformation U is injective for $(t,\theta) \in (0,1)\times(\alpha,\beta)$, and let

$$u_0 = (a_{11}-a_{21})(b_{12}-b_{22}) + (a_{12}-a_{22})(b_{21}-b_{11})$$

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