

Numerical integration over bounded intervals

Alvise Sommariva

Padua, autumn, 2024

Doctoral Program in Mathematical Sciences, Padua (I), Autumn 2024

"Numerical cubature and its applications"

Purpose

The purpose is to provide some basics on numerical integration over bounded intervals, that is

- description of symbolic software for computation of integrals;
- introduce the basic ideas on univariate polynomial interpolation;
- show some rules of interpolatory type, in the *Newton-Cotes* family, as midpoint, trapezoidal and Cavalieri-Simpson rules;
- disadvantages of Newton-Cotes rules and pros of composite rules;
- practical examples in Matlab/Octave.

The purpose is to compute

$$\int_a^b f(x) dx$$

where f is typically continuous in the interval $[a, b]$.

Remark

There are of course *more general instances*, e.g. f could have a finite number of jump discontinuities, but we will not take this case into account.

Remark (Pathological continuous functions)

Between the continuous functions there are many odd examples, as the *Weierstrass function*

$$f(x) = \sum_{k=1}^{+\infty} \frac{\sin(\pi k^2 x)}{\pi k^2}$$

that is continuous everywhere but differentiable only on a set of null measure (proof by Hardy).

Poincaré defined it as “an outrage against common sense” and Hermite as a “lamentable scourge”.

- Analytically, a typical approach is to **primitive**) of the integrand.
- In view of the **fundamental theorem of calculus** (known as Torricelli-Barrow theorem), once a primitive is available then one can easily compute the definite integral.
- Unfortunately in many cases there are no known *primitives in terms of elementary functions* and one cannot adopt the technique described above.

Some examples of integrands with no primitive in terms of elementary functions, we have:

- the function $g : \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$g(x) = \exp(x^2)$$

- the function $\text{sinc} : \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$\text{sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0 \end{cases}$$

The proof concerning the absence of primitives in terms of elementary functions depend on the [Liouville theorem](#).

There are several environments for **symbolic calculus**, that allow the computation of definite and indefinite integrals, e.g.

- **Maple**, offering also **online services**;
- **Mathematica**, offering also **online services**;
- **Maxima**, free, with interesting **online demo**.

We will propose some examples to understand their usage.

- 1 As **first example** we consider a rational function and its integration that can be done by **decomposition in partial fractions**.

$$\int \frac{1+x-x^2}{1+x^2} dx = \frac{\log(x^2+1)}{2} - x + 2 \arctan(x) + C$$

This technique has been discovered independently by Johann Bernoulli and Gottfried Leibniz (1702) and usually requires some boring computations.

- 2 As **second example** we consider $\exp(-x^2)$.

For each of them we will compute definite integrals in $[0, 1]$.

We first use Maple via Matlab shell for the computation of indefinite integrals.

```
>> syms x;  
>> int((1+x-x^2)/(1+x^2),x)  
  
ans =  
  
log(x^2 + 1)/2 - x + 2*atan(x)  
  
>> int(exp(-x^2))  
  
ans =  
  
(pi^(1/2)*erf(x))/2  
  
>> help erf  
erf Error function.  
Y = erf(X) is the error function for each element of X. X must be  
real. The error function is defined as:  
  
    erf(x) = 2/sqrt(pi) * integral from 0 to x of exp(-t^2) dt.  
  
See also erfc, erfcx, erfinv, erfcinv.  
  
Documentation for erf  
Other uses of erf  
  
>>
```

Figure: Indefinite integrals in Maple (via Matlab shell).

Now we adopt Maple via Matlab shell for the computation of definite integrals.

```
>> int((1+x-x^2)/(1+x^2),0,1)

ans =

pi/2 + log(2)/2 - 1

>> format long
>> pi/2 + log(2)/2 - 1
|
ans =

0.917369917074869

>> int(exp(-x^2),0,1)

ans =

(pi^(1/2)*erf(1))/2

>> (pi^(1/2)*erf(1))/2

ans =

0.746824132812427

>>
```

Figure: Definite integrals in Maple (via Matlab shell).

Symbolic calculus of integrals

We now make the computations of indefinite integrals via Mathematica, e.g. visiting the website [Online Integral Calculator](#). To complete the computations, use the “equal” symbol on the right.

FROM THE MAKERS OF WOLFRAM LANGUAGE AND MATHEMATICA



$$\int \frac{1+x-x^2}{1+x^2} dx$$

NATURAL LANGUAGE $\int_0^\pi$ MATH INPUT

$\sqrt{}$ ∂f $(::)$ $\sqrt[n]{}$ a_ω ...

$\frac{\square}{\square}$ $\square^\circ$ $\sqrt{\square}$ $\sqrt[n]{\square}$ $\frac{d}{d\square}$ $\frac{d^2}{d^2\square}$ $\int \square$ $\int \square^\circ$ $\sum \square$ $\lim_{\square \rightarrow \square}$ $[a,b]$ $\begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}$

Indefinite integral ☒ Step-by-step solution

$$\int \frac{1+x-x^2}{1+x^2} dx = \frac{1}{2} \log(x^2+1) - x + 2 \tan^{-1}(x) + \text{constant}$$

$\tan^{-1}(x)$ is the inverse tangent function
 $\log(x)$ is the natural logarithm

Plots of the integral  Enlarge  Data  Customize  Plain Text

Figure: Indefinite integrals in Mathematica (via Online Integral Calculator).

Symbolic calculus of integrals

About the second integral, we insert the new integrand and digit
= on the right side of the box.

FROM THE MAKERS OF WOLFRAM LANGUAGE AND MATHEMATICA



$\int \exp(-x^2) dx$

NATURAL LANGUAGE $\int_0^x$ MATH INPUT

★ $\sqrt{\quad}$ ∂f $(::)$ $\sqrt[n]{\quad}$ a_w ...

$\frac{\square}{\square}$ $\square^a$ $\sqrt{\square}$ $\sqrt[n]{\square}$ $\sqrt[n]{\square}$ $\frac{d}{d\square}$ $\frac{d^2}{d^2\square}$ $\int \square$ $\int \square$ $\sum \square$ $\lim \square \cdot \square$ $[\square, \square, \square]$ $\left(\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} \right)$

Indefinite integral

$$\int \exp(-x^2) dx = \frac{1}{2} \sqrt{\pi} \operatorname{erf}(x) + \text{constant}$$

$\operatorname{erf}(x)$ is the error function

Figure: Indefinite integrals in Mathematica (via Online Integral Calculator).

Concerning definite integrals, writing the data in Math Input, clicking on *more digits*:

FROM THE MAKERS OF WOLFRAM LANGUAGE AND MATHEMATICA

WolframAlpha

$$\int_0^1 \frac{1+x-x^2}{1+x^2} dx$$

NATURAL LANGUAGE
MATH INPUT

★
√
∂f
(::)
↯
αω
...

Definite integral

Fewer digits
More digits
☑ Step-by-step solution

$$\int_0^1 \frac{1+x-x^2}{1+x^2} dx = \frac{1}{2} (-2 + \pi + \log(2)) \approx 0.917369917074869$$

log(x) is the natural logarithm

Figure: Definite integrals in Mathematica (via Online Integral Calculator).

We repeat the procedure for the second function.

FROM THE MAKERS OF WOLFRAM LANGUAGE AND MATHEMATICA



$\int_0^1 \exp(-x^2) dx$

NATURAL LANGUAGE  MATH INPUT

★ √ ∂f (:=) √ √ ∞ ∞ ...

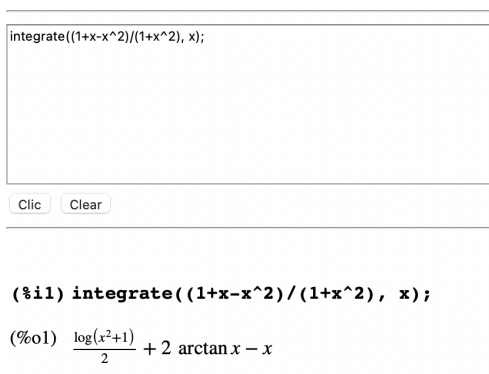
Definite integral Fewer digits More digits

$\int_0^1 \exp(-x^2) dx = \frac{1}{2} \sqrt{\pi} \operatorname{erf}(1) \approx$
0.74682413281242702539946743613185300535449968681260632902765449895`
8605327561772831497848429822901920

$\operatorname{erf}(x)$ is the error function

Figure: Definite integrals in Mathematica (via Online Integral Calculator).

We repeat the computation by means of [Maxima demo homepage](#)



The screenshot shows a web-based interface for the Maxima computer algebra system. At the top, there is a text input field containing the command `integrate((1+x-x^2)/(1+x^2), x);`. Below the input field are two buttons: "Click" and "Clear". The output of the command is displayed below the input field, showing the prompt `(%i1) integrate((1+x-x^2)/(1+x^2), x);` followed by the result `(%o1)  $\frac{\log(x^2+1)}{2} + 2 \arctan x - x$` .

```
integrate((1+x-x^2)/(1+x^2), x);
```

Click Clear

(%i1) integrate((1+x-x^2)/(1+x^2), x);

(%o1) $\frac{\log(x^2+1)}{2} + 2 \arctan x - x$

Figure: Indefinite integrals in Maxima (via online site).

Symbolic calculus of integrals

```
integrate(exp(-x^2), x);
```

```
(%i1) integrate(exp(-x^2), x);
```

```
(%o1) 
$$\frac{\sqrt{\pi} \operatorname{erf}(x)}{2}$$

```

[Yamwi](#)

Figure: Indefinite integrals in Maxima (via online site).

Symbolic calculus of integrals

integrate((1+x-x^2)/(1+x^2),x,0,1);

Clic

Clear

(%i1) integrate((1+x-x^2)/(1+x^2),x,0,1);

(%o1) $\frac{\log 2+\pi-2}{2}$

Figure: Definite integrals in Maxima (via online site).

```
integrate(exp(-x^2),x,0,1);
```

Click Clear

```
(%i1) integrate(exp(-x^2),x,0,1);
```

```
(%o1)  $\frac{\sqrt{\pi} \operatorname{erf}(1)}{2}$ 
```

Figure: Definite integrals in Maxima (via online site).

Remark

The installation of Maxima may be not trivial (e.g. is not immediate on MacOS).

The previous definite integrals

$$\int_0^1 \frac{1+x-x^2}{1+x^2} dx \approx 0.917369917074869$$

$$\int_0^1 \exp(-x^2) dx \approx 0.746824132812427$$

may also be approximate by Matlab/Octave.

- We make the experiments in [Octave](#), taking into account that the Matlab version is equal.
- To this purpose one can use [OctaveOnline](#) as well as [Octave/Matlab](#).

We show how we can compute definite integrals in Matlab/Octave by adaptive algorithm implemented in `integral`, controlling the absolute and relative error, with tolerances decided by the users. This is guaranteed numerically but not mathematically (the code may fail in special and rare examples).

Consider that the function definition in Matlab/Octave requires the knowledge of pointwise operations, using “.”.

```
>> f=@(x) (1+x-x.^2)./(1+x.^2);  
>> Q = integral(f,0,1,'AbsTol',10^(-12),'RelTol',10^(-12));  
>> Q  
Q = 0.917369917074869  
>> g=@(x) exp(-x.^2);  
>> Q = integral(g,0,1,'AbsTol',10^(-12),'RelTol',10^(-12));  
>> Q  
Q = 0.746824132812427  
>> |
```

Figure: Definite integrals in Octave.

Univariate polynomial interpolation (summary)

Problema. (Univariate polynomial interpolation)

Given

- $n + 1$ distinct points $x_0, \dots, x_n$,
- the values $y_0, \dots, y_n$

the problem of *polynomial interpolation* consists in computing

$$p_n(x) = a_0 + \dots + a_n x^n$$

such that

$$p_n(x_i) = y_i, \quad i = 0, \dots, n.$$

Example (Straight line for two given points)

Given 2 distinct points x_0, x_1 and the values y_0, y_1 , determine $p_1 \in \mathbb{P}_1$ such that

$$p_1(x_0) = y_0, \quad p_1(x_1) = y_1 \tag{1}$$

that is the straight line that passes for the couples $(x_0, y_0), (x_1, y_1)$.

Univariate polynomial interpolation (summary)

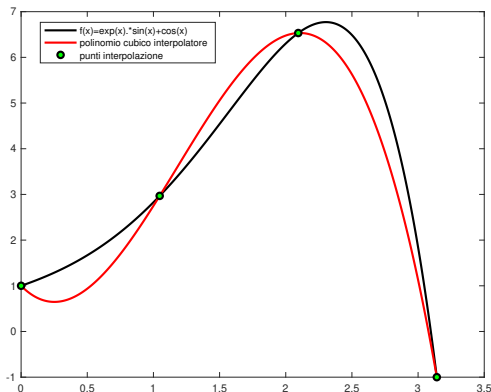


Figure: Example consisting in interpolating the function $f(x) = \exp(x)\sin(x) + \cos(x)$ (in black) by an algebraic polynomial of degree 3 (in red) in 4 equispaced points in $[0, \pi]$ (green circles). We plot the pertinent plots and the points $\{(x_k, y_k)\}_{k=0, \dots, 3}$, where $x_k = k\pi/3$ e $y_k = f(x_k)$, $k = 0, \dots, 3$.

Theorem (Existence and uniqueness of polynomial interpolant)

Given $n + 1$ distinct points $x_0, x_1, \dots, x_n$ and the values $y_0, y_1, \dots, y_n$, the polynomial $p_n \in \mathbb{P}_n$ such that

$$p_n(x_i) = y_i, \quad i = 0, \dots, n.$$

exists and is unique.

Next

$$p_n(x) = \sum_{k=0}^n y_k L_k(x) = y_0 L_0(x) + y_1 L_1(x) + \dots + y_n L_n(x)$$

where

$$L_k(x) := \prod_{j=0, j \neq k}^n \frac{x - x_j}{x_k - x_j} = \frac{(x - x_0) \dots (x - x_{k-1})(x - x_{k+1}) \dots (x - x_n)}{(x_k - x_0) \dots (x_k - x_{k-1})(x_k - x_{k+1}) \dots (x_k - x_n)}$$

is the k -th Lagrange polynomial w.r.t. the nodes $\{x_k\}_{k=0, \dots, n}$.

The [Lagrange polynomials](#) were discovered by

- [Waring](#) in [Problems concerning interpolations](#) in 1779,
- rediscovered by [Euler](#) in 1783,
- published by [Lagrange](#) in [Leçon Cinquième. Sur l'usage des courbes dans la solution des problèmes](#) in 1795 (volume 7, p.286).

For an interesting note see [A Chronology of Interpolation: From Ancient Astronomy to Modern Signal and Image Processing](#).

[59]

VII. *Problems concerning Interpolations.* By Edward Waring, M. D. F. R. S. and of the *Institute of Bononia*, *Lucasian Professor of Mathematics in the University of Cambridge.*

Read Jan. 9, 1779. MR. BRIGGS was the first person, I believe, that invented a method of differences for interpolating logarithms at small intervals from each other: his principles were followed by REGINALD and MOVTON in France. Sir ISAAC NEWTON, from the same principles, discovered a general and elegant solution of the abovementioned problem: perhaps a still more elegant one on some accounts has been since discovered by MEFF. NICHOLE and STIRLING. In the following theorems the same problem is resolved and rendered somewhat more general, without having any recourse to finding the successive differences.

THEOREM I.

Assume an equation $a + bx + cx^2 + dx^3 + \dots + x^{n-1} = y$, in which the co-efficients a, b, c, d, e , &c. are invariable; let

66

Dr. WARING on Interpolations.

$$B = \frac{\frac{x-a}{\beta-a} \times \frac{x-\gamma}{\gamma-\beta} \times \frac{x-\delta}{\delta-\gamma} \times \frac{x-\epsilon}{\epsilon-\delta} \times \frac{x-\zeta}{\zeta-\delta}}{\frac{x-a}{\beta-a} \times \frac{x-\gamma}{\gamma-\beta} \times \frac{x-\delta}{\delta-\gamma} \times \frac{x-\epsilon}{\epsilon-\delta} \times \frac{x-\zeta}{\zeta-\delta}}, \quad C = \frac{\frac{x-a}{\beta-a} \times \frac{x-\gamma}{\gamma-\beta} \times \frac{x-\delta}{\delta-\gamma} \times \frac{x-\epsilon}{\epsilon-\delta} \times \frac{x-\zeta}{\zeta-\delta}}{\frac{x-a}{\beta-a} \times \frac{x-\gamma}{\gamma-\beta} \times \frac{x-\delta}{\delta-\gamma} \times \frac{x-\epsilon}{\epsilon-\delta} \times \frac{x-\zeta}{\zeta-\delta}},$$

&c.: substitute these values for A, B, C, D, E , &c. respectively in the preceding equations ($A + B + C + D + E + \&c. = I$, $A\alpha + B\beta + C\gamma + D\delta + E\epsilon + \&c. = x$, $A\alpha^2 + B\beta^2 + C\gamma^2 + D\delta^2 + E\epsilon^2 + \&c. = x^2$, $A\alpha^3 + B\beta^3 + C\gamma^3 + D\delta^3 + E\epsilon^3 + \&c. = x^3$, &c.)

and there reful the equations (1) $\frac{\frac{x-a}{\beta-a} \times \frac{x-\gamma}{\gamma-\beta} \times \frac{x-\delta}{\delta-\gamma} \times \frac{x-\epsilon}{\epsilon-\delta} \times \frac{x-\zeta}{\zeta-\delta}}{\frac{x-a}{\beta-a} \times \frac{x-\gamma}{\gamma-\beta} \times \frac{x-\delta}{\delta-\gamma} \times \frac{x-\epsilon}{\epsilon-\delta} \times \frac{x-\zeta}{\zeta-\delta}},$

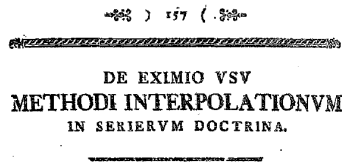
$$+ \frac{\frac{x-a}{\beta-a} \times \frac{x-\gamma}{\gamma-\beta} \times \frac{x-\delta}{\delta-\gamma} \times \frac{x-\epsilon}{\epsilon-\delta} \times \frac{x-\zeta}{\zeta-\delta}}{\frac{x-a}{\beta-a} \times \frac{x-\gamma}{\gamma-\beta} \times \frac{x-\delta}{\delta-\gamma} \times \frac{x-\epsilon}{\epsilon-\delta} \times \frac{x-\zeta}{\zeta-\delta}}, \quad \&c. = I;$$

$$(2) \alpha \times \frac{\frac{x-a}{\beta-a} \times \frac{x-\gamma}{\gamma-\beta} \times \frac{x-\delta}{\delta-\gamma} \times \frac{x-\epsilon}{\epsilon-\delta} \times \frac{x-\zeta}{\zeta-\delta}}{\frac{x-a}{\beta-a} \times \frac{x-\gamma}{\gamma-\beta} \times \frac{x-\delta}{\delta-\gamma} \times \frac{x-\epsilon}{\epsilon-\delta} \times \frac{x-\zeta}{\zeta-\delta}} + \beta \times \frac{\frac{x-a}{\beta-a} \times \frac{x-\gamma}{\gamma-\beta} \times \frac{x-\delta}{\delta-\gamma} \times \frac{x-\epsilon}{\epsilon-\delta} \times \frac{x-\zeta}{\zeta-\delta}}{\frac{x-a}{\beta-a} \times \frac{x-\gamma}{\gamma-\beta} \times \frac{x-\delta}{\delta-\gamma} \times \frac{x-\epsilon}{\epsilon-\delta} \times \frac{x-\zeta}{\zeta-\delta}} + \gamma \times \frac{\frac{x-a}{\beta-a} \times \frac{x-\gamma}{\gamma-\beta} \times \frac{x-\delta}{\delta-\gamma} \times \frac{x-\epsilon}{\epsilon-\delta} \times \frac{x-\zeta}{\zeta-\delta}}{\frac{x-a}{\beta-a} \times \frac{x-\gamma}{\gamma-\beta} \times \frac{x-\delta}{\delta-\gamma} \times \frac{x-\epsilon}{\epsilon-\delta} \times \frac{x-\zeta}{\zeta-\delta}} + \&c. = x;$$

$$(3) \alpha^2 \times \frac{\frac{x-a}{\beta-a} \times \frac{x-\gamma}{\gamma-\beta} \times \frac{x-\delta}{\delta-\gamma} \times \frac{x-\epsilon}{\epsilon-\delta} \times \frac{x-\zeta}{\zeta-\delta}}{\frac{x-a}{\beta-a} \times \frac{x-\gamma}{\gamma-\beta} \times \frac{x-\delta}{\delta-\gamma} \times \frac{x-\epsilon}{\epsilon-\delta} \times \frac{x-\zeta}{\zeta-\delta}} + \beta^2 \times \frac{\frac{x-a}{\beta-a} \times \frac{x-\gamma}{\gamma-\beta} \times \frac{x-\delta}{\delta-\gamma} \times \frac{x-\epsilon}{\epsilon-\delta} \times \frac{x-\zeta}{\zeta-\delta}}{\frac{x-a}{\beta-a} \times \frac{x-\gamma}{\gamma-\beta} \times \frac{x-\delta}{\delta-\gamma} \times \frac{x-\epsilon}{\epsilon-\delta} \times \frac{x-\zeta}{\zeta-\delta}} + \gamma^2 \times \frac{\frac{x-a}{\beta-a} \times \frac{x-\gamma}{\gamma-\beta} \times \frac{x-\delta}{\delta-\gamma} \times \frac{x-\epsilon}{\epsilon-\delta} \times \frac{x-\zeta}{\zeta-\delta}}{\frac{x-a}{\beta-a} \times \frac{x-\gamma}{\gamma-\beta} \times \frac{x-\delta}{\delta-\gamma} \times \frac{x-\epsilon}{\epsilon-\delta} \times \frac{x-\zeta}{\zeta-\delta}} + \&c. = x^2; \text{ and in general,}$$

$\alpha^m \times \frac{\frac{x-a}{\beta-a} \times \frac{x-\gamma}{\gamma-\beta} \times \frac{x-\delta}{\delta-\gamma} \times \frac{x-\epsilon}{\epsilon-\delta} \times \frac{x-\zeta}{\zeta-\delta}}{\frac{x-a}{\beta-a} \times \frac{x-\gamma}{\gamma-\beta} \times \frac{x-\delta}{\delta-\gamma} \times \frac{x-\epsilon}{\epsilon-\delta} \times \frac{x-\zeta}{\zeta-\delta}} + \beta^m \times \frac{\frac{x-a}{\beta-a} \times \frac{x-\gamma}{\gamma-\beta} \times \frac{x-\delta}{\delta-\gamma} \times \frac{x-\epsilon}{\epsilon-\delta} \times \frac{x-\zeta}{\zeta-\delta}}{\frac{x-a}{\beta-a} \times \frac{x-\gamma}{\gamma-\beta} \times \frac{x-\delta}{\delta-\gamma} \times \frac{x-\epsilon}{\epsilon-\delta} \times \frac{x-\zeta}{\zeta-\delta}} + \gamma^m \times \frac{\frac{x-a}{\beta-a} \times \frac{x-\gamma}{\gamma-\beta} \times \frac{x-\delta}{\delta-\gamma} \times \frac{x-\epsilon}{\epsilon-\delta} \times \frac{x-\zeta}{\zeta-\delta}}{\frac{x-a}{\beta-a} \times \frac{x-\gamma}{\gamma-\beta} \times \frac{x-\delta}{\delta-\gamma} \times \frac{x-\epsilon}{\epsilon-\delta} \times \frac{x-\zeta}{\zeta-\delta}} + \&c. = x^m$, whatever may be the values of the quantities $\alpha, \beta, \gamma, \delta, \epsilon$, &c.: reduce all these fractions into terms, proceeding according to the dimensions of the quantity x , and it is evident, that the sum of all the fractions multiplied

Figure: The paper by Waring in which the Lagrange polynomials were discovered.



In methodo interpolationum eiusmodi relatio inter binas variables x et y quaeritur, ut si alteri x successiue dati valores a, b, c, d , etc. tribuantur, altera y inde quoque datos valores p, q, r, s , etc. fortiaur; seu quod eodem redit, aequatio pro eiusmodi linea curua quaeritur, quae per quoscunque puncta data transeat. Quo maior ergo fuerit horum punctorum numerus, eo magis linea curua limitatur: interim tamen iam alia occasione obseruari, etiamsi punctorum numerus in infinitum augeatur, curuam per ea transeuntem non prius determinari, sed semper infinitas adhuc lineas curuas exhiberi posse, quae aequae per cuncta eadem puncta sint transiturae. Quare cum methodus interpolationum pro quouis casu lineam curuam suppeditare determinarem, solutio haec semper pro maxime particulari erit habenda: verum haec ipsa circumstantia singularem quandam indolem solutionis inuenias iouit, quae accuratiorem considerationem meretur. Imprimis autem ista solutionis indoles pendet a ratione, qua interpolatio instituitur, seu a forma, quae aequationi generali tribuitur, in qua aequationem quaesitam contineri

Figure: Paper by **Eulero** in which the **Lagrange** polynomials were rediscovered.

286

LEÇONS ÉLÉMENTAIRES

qu'en faisant $x = p$ on ait

$$A = 1, \quad B = 0, \quad C = 0, \quad \dots;$$

que de même, en faisant $x = q$, on ait

$$A = 0, \quad B = 1, \quad C = 0, \quad D = 0, \quad \dots;$$

qu'en faisant $x = r$, on ait pareillement

$$A = 0, \quad B = 0, \quad C = 1, \quad D = 0, \quad \dots, \quad \text{etc.};$$

d'où il est facile de conclure que les valeurs de $A, B, C, \dots$ doivent être de cette forme

$$A = \frac{(x - q)(x - r)(x - s) \dots}{(p - q)(p - r)(p - s) \dots},$$

$$B = \frac{(x - p)(x - r)(x - s) \dots}{(q - p)(q - r)(q - s) \dots},$$

$$C = \frac{(x - p)(x - q)(x - s) \dots}{(r - p)(r - q)(r - s) \dots},$$

$$\dots\dots\dots,$$

Figure: Work by Lagrange on the relative polynomials.

Choice of interpolation points

One may believe that if we take more and more **equispaced nodes** in $[a, b]$, that is

$$x_k = a + k \frac{(b-a)}{n}, \quad k = 0, \dots, n$$

increasing n , the polynomial p_n will approximate better and better the function f , but it is not so.

Runge discovered a famous counterexample in [Über die Darstellung willkrlicher Functionen und die Interpolation zwischen äquidistanten Ordinaten](#), p.243, (1901), that is $f \in C^\infty([-5, 5])$ defined by

$$f(x) = \frac{1}{1+x^2}, \quad x \in [-5, 5]$$

in which **increasing n , $\max_{x \in [a,b]} |f(x) - p_n(x)|$ does not converge to 0.**

Runge counterexample

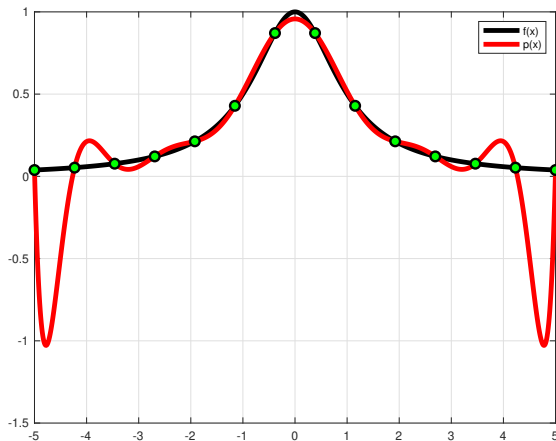


Figure: Plot illustrating the polynomial interpolant (of the Runge function) of degree 12 based on 13 equispaced nodes. The plot in red is the interpolant, while the plot in black is the Runge function. The green dots are the couples to be interpolated. Observe the wild oscillations at extrema. Increasing the degree things get worse!

Über empirische Funktionen und die Interpolation zwischen äquidistanten Ordinaten.

Von C. RUNGE in Hannover.

Die Abhängigkeit zwischen zwei messbaren Grössen kann, streng genommen, durch Beobachtung überhaupt nicht gefunden werden. Denn selbst wenn man von den Beobachtungsfehlern absehen und die Beobachtungen als absolut genau voraussetzen wollte, so bliebe doch immer der Umstand, dass durch Beobachtung immer nur eine diskrete Reihe einander entsprechender Wertepaare der beiden Grössen gefunden werden könnte. Selbst wenn wir die Reihe als unendlich voraussetzen, so würde nicht einmal eine „analytische“¹⁾ Funktion dadurch bestimmt sein. Gesetzt z. B., es seien für eine unendliche Reihe von äquidistanten Werten der einen Grösse die Werte der andern Grösse absolut genau bekannt, so wäre das Abhängigkeitsverhältnis damit noch nicht gegeben, selbst dann nicht, wenn wir nur nach der „analytischen“ Funktion fragen, die das Abhängigkeitsverhältnis darstellen soll. Denn es ist klar, dass man auf mannigfache Weise eine periodische Funktion bilden kann, die für alle jene äquidistanten Werte verschwindet und daher, zu einer Funktion addiert, ihre Werte an jenen Stellen nicht ändert. Dennoch betrachtet man in den beobachtenden Wissenschaften eine Funktion durch eine solche Tabelle ihrer Werte als wohl definiert, sobald die Argumente nur hinreichend nahe aneinander liegen. Wie dicht sie liegen müssen, darüber werden meines Wissens klare Kriterien nicht aufgestellt. Man beschränkt sich darauf zu verlangen, dass die beobachteten Werte graphisch aufgetragen eine „glatte Kurve“ geben. Eine Wellenlinie, die zwischen je zwei aufeinanderfolgenden beobachteten Punkten ein Maximum oder Minimum hätte, würde man stillschweigend ausschliessen.

Dieses übliche Verfahren kann in der That auch mathematisch gerechtfertigt werden.

Man kann nämlich auch durch eine Tabelle eine Funktion wohl definieren, wenn man zugleich ein Interpolationsverfahren vorschreibt.

1) Im Sinne von Weierstrass.

VON C. RUNGE.

243

Es sei z. B. $f(x) = \frac{1}{1+x^2}$ und $a = -5$, $b = +5$. Dann hat $f(x)$ die beiden singulären Stellen $+i$ und $-i$. Statt eines Kreises haben wir dann zwei Kreise auszuschliessen und erhalten, wenn wir die U -Kurve ins Unendliche rücken lassen

$$\frac{1}{1+x^2} = G_n(x) + \frac{i}{2} \frac{g_n(x)}{g_n(i)} \frac{1}{i-x} + \frac{i}{2} \frac{g_n(x)}{g_n(-i)} \frac{1}{-i-x}.$$

Nun ist

$$\begin{aligned} (i-x_1)(i-x_n) &= -(x_1^2+1) \\ (i-x_p)(i-x_{n-1}) &= -(x_p^2+1), \\ &\text{u. s. w.} \end{aligned}$$

Wird daher n als ungrade vorausgesetzt, so muss $g_n(i)$ rein imaginär sein:

$$\begin{aligned} g_n(i) &= \pm i |g_n(i)| \\ g_n(-i) &= \mp i |g_n(i)| \end{aligned}$$

und wir erhalten:

$$\frac{1}{1+x^2} = G_n(x) \pm \frac{g_n(x)}{|g_n(i)|} \frac{x}{1+x^2}.$$

Figure: Original paper on Runge counterexample.

The following theorem holds.

Theorem (Stability of integration)

If $\tilde{f}, f \in C([a, b])$, and $[a, b]$ is a bounded interval then

$$\left| \int_a^b f(x) dx - \int_a^b \tilde{f}(x) dx \right| \leq (b-a) \max_{x \in [a,b]} |f(x) - \tilde{f}(x)|.$$

Thus, if we assume that

- the points $\{x_k\}_{k=0,\dots,n} \subset [a, b]$ are distinct,
- the polynomial $\tilde{f} = p_n$ that interpolates $(x_0, f(x_0)), \dots, (x_n, f(x_n))$ is a good approximation of $f \in C([a, b])$.

then

$$I(p_n) = \int_a^b p_n(x) dx \approx I(f) = \int_a^b f(x) dx$$

with the advantage that the computation of $I(p_n)$ is usually easier than that of $I(f)$.

Interpolatory quadrature rules

In virtue of what we have seen, let L_k be the Lagrange polynomials relatively to the set $\{x_k\}_{k=0,\dots,n}$, and the **weights**

$$w_k = \int_a^b L_k(x) dx$$

and get

$$\begin{aligned} \int_a^b f(x) dx &\approx \int_a^b p_n(x) dx = \int_a^b \sum_{k=0}^n f(x_k) L_k(x) dx \\ &= \sum_{k=0}^n \int_a^b f(x_k) L_k(x) dx = \sum_{k=0}^n f(x_k) \int_a^b L_k(x) dx \\ &= \sum_{k=0}^n w_k f(x_k). \end{aligned} \tag{2}$$

that is

$$\int_a^b f(x) dx \approx \sum_{k=0}^n w_k f(x_k).$$

Interpolatory quadrature rules

Observe that if $f \in \mathbb{P}_n$ then it **it is exactly** the polynomial p_n that **interpolates** the couples $(x_k, f(x_k))$, $k = 0, \dots, n$ (by the uniqueness of the polynomial interpolant) and then

$$\int_a^b f(x) dx = \int_a^b p_n(x) dx = \int_a^b \sum_{k=0}^n f(x_k) L_k(x) dx = \dots = \sum_{k=0}^n w_k f(x_k).$$

Consequently, if $f \in \mathbb{P}_n$ then $\sum_{k=0}^n w_k f(x_k)$ is equal to $\int_a^b f(x) dx$.

We will say that **the degree of exactness of an interpolatory rule in $n+1$ nodes is at least n .**

Thus, an interpolatory rule on

- 1 node, surely integrates exactly the constants;
- 2 nodes, surely integrates exactly the polynomials of degree 0, 1;
- 3 nodes, surely integrates exactly the polynomials of degree 0, 1, 2.

We will see that sometimes **things can be even better.**

The fact that

$$\int_a^b f(x) dx \approx \sum_{k=0}^n w_k f(x_k), \quad w_k = \int_a^b L_k(x) dx$$

says that to approximate the required integral it is not necessary to

- 1 compute the polynomial interpolant,
- 2 determine its primitive,
- 3 apply the fundamental theorem of integral calculus,

but

- 1 compute the weights $\{w_k\}_{k=0,\dots,n}$ relatively to the nodes $\{x_k\}_{k=0,\dots,n}$,
- 2 use the function evaluations $\{f(x_k)\}_{k=0,\dots,n}$.

Remark

This fact is important, since if we change the integrand we minimize the computation, since the weights w_k , $k = 0, \dots, n$ do not change, depending only on the nodes.

Definition (Rectangular rule)

Let

- $f \in C([a, b]), -\infty < a < b < +\infty,$
- $x_0 \in [a, b].$

The **rectangular rule** is defined by

$$\int_a^b f(x) dx \approx w_0 f(x_0) = (b - a) f(x_0) := S_0^*(f). \quad (3)$$

If $x_0 = \frac{a+b}{2}$ we get the so called **midpoint rule**.

For the midpoint rule, the following error estimate holds

Theorem (Midpoint rule error estimate)

If $f \in C^{(2)}([a, b])$ then the midpoint rule error estimate is

$$E_0(f) := I(f) - S_0^*(f) = \frac{(b-a)^3}{24} f^{(2)}(\xi), \quad \xi \in (a, b).$$

Observe that if $f \in \mathbb{P}_1$ allora

$$E_0(f) := I(f) - S_0^*(f) = 0$$

thus the rule is exact.

This is a little surprising since being based on just one function evaluation one may believe it integrates exactly just the constants and not polynomials of degree 1.

Rectangular rule

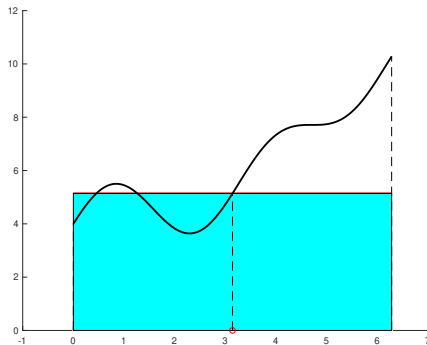


Figure: Rectangular rule with node $x_0 = (a + b)/2$, $a = 0$, $b = 2\pi$, for approximating $\int_0^{2\pi} (3 + \sin(2x) + \cos(x) + x) dx$ (midpoint rule computes the area in cyan).

Trapezoidal rule.

Definition (Trapezoidal rule)

The trapezoidal rule is defined by

$$\int_a^b f(x) dx \approx S_1(f) := \frac{b-a}{2} \cdot (f(a) + f(b)).$$

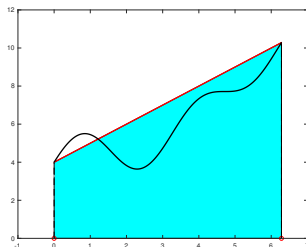


Figure: Trapezoidal rule for the approximation of $\int_0^{2\pi} 3 + \sin(2x) + \cos(x) + x \, dx$ (the rule computes the area in cyan).

Theorem (Error of the Trapezoidal rule)

If $f \in C^2([a, b])$ then the error of the trapezoidal rule is

$$E_1(f) := I(f) - S_1(f) = \frac{-(b-a)^3}{12} f^{(2)}(\xi), \quad \xi \in (a, b).$$

Theorem (Degree of exactness of the trapezoidal rule)

The degree of exactness of the trapezoidal rule is exactly 1.

Definition (Cavalieri-Simpson rule)

The Cavalieri-Simpson rule is defined as $\int_a^b f(x)dx \approx S_2(f)$ with

$$S_2(f) := \frac{b-a}{6} \cdot f(a) + \frac{2(b-a)}{3} \cdot f\left(\frac{a+b}{2}\right) + \frac{b-a}{6} \cdot f(b).$$

This interpolatory rule is equivalent to determine the definite integral for the polynomial with nodes the interval extrema a , b and their midpoint $(a+b)/2$.

Consequently, its degree of exactness is at least 2.

Cavalieri-Simpson rule

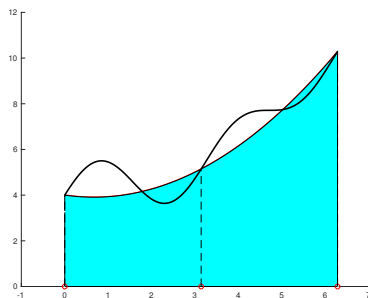


Figure: Cavalieri-Simpson rule for the computation of $\int_0^{2\pi} 3 + \sin(2x) + \cos(x) + x \, dx$, it determines the area in cyan).

Theorem (Cavalieri-Simpson rule error)

If $f \in C^4([a, b])$ the error made by Cavalieri-Simpson rule is

$$E_2(f) := I(f) - S_2(f) = \frac{-h^5}{90} f^{(4)}(\xi), \quad h = \frac{b-a}{2}$$

with $\xi \in (a, b)$.

Theorem (Degree of exactness of Cavalieri-Simpson rule)

The degree of exactness of Cavalieri-Simpson rule is exactly 3.

Remark

As in the case of midpoint rule, this result is remarkable, since it is expected to be 2, since the formula of interpolatory type is based on 3 nodes.

Observe that integration had a complicated history.

Cavalieri did not have the modern concept of integration since he was researcher between 1629 and 1647, while

- the fundamental of integral calculus by (Torricelli-Barrow) appears in a primitive form in *Lectiones geometricae* by Barrow (1670) and in *Geometriae pars universalis* by Gregory (1668), see [?];
- *Riemann integral* was introduced in december 1853 (in his habilitation thesis);
- *Lebesgue integral* was discovered in 1904.

Cavalieri, by means of the so called *method of indivisibles* computed $\int_a^b x^n dx$ for $n = 1, \dots, 9$.

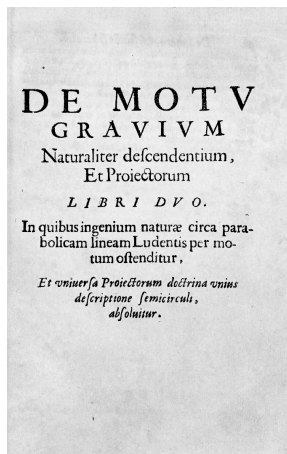
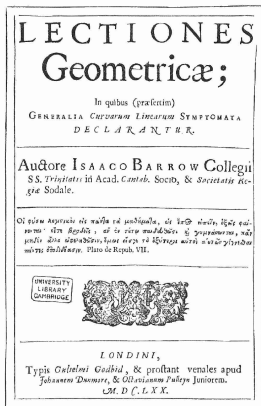


Figure: Work by **Torricelli** in which the fundamental theorem of integral calculus is introduced geometrically (link between displacement on a straight line and velocity).



LECTURE XI

Change of the independent variable in integration. Integration the inverse of differentiation. Differentiation of a quotient. Area and centre of gravity of a paraboliform. Limits for the arc of a circle and a hyperbola. Estimation of π .

NOTE

In the following theorems, Barrow uses his variation of the usual method of summation for the determination of an area. If $ABKJ$ is the area under the curve Ad , he divides BK into an infinite number of equal parts and erects ordinates. In his figures he generally makes four parts do duty for the infinite number.

He then uses the notation already mentioned, namely, that the area $ABKJ$ is equal to the sum of the ordinates AB , CD , EF , GH , JK .

The same idea is involved when he speaks of the sum of the rectangles $CD DB$, $EF FD$, $GH FH$, $JK KH$; for this sum, where commas are used between the quantities instead of a plus sign, does not stand for the area $ABKJ$, but for $R \cdot A'BKJ'$, where an ordinate HG' is such that $R \cdot HG' = HG \cdot FH$, and R is some given length; in other words, ordinates proportional to each of the rectangles are applied to points of the line BK , and their aggregate or sum is found; hence this sum is of three dimensions. On the contrary, he uses the same phrase, with plus signs instead of commas, to stand for a simple summation.



Figure: Work by Barrow in which a geometrical form of fundamental theorem of calculus. On this subject, see also the paper Barrow and Leibniz on the fundamental theorem of the calculus.

GEOMETRIÆ
PARS VNIVERSALIS.
Interviens
Quantitatum Curvarum transmutationi & mensuræ.
AVTHORE
IACOBO GREGORIO
ABREDONENSI
SCOTO.



PATAVII, MDCLXVIII.

Typis Heredum Pauli Frambotti, *Superiorum Perm.*
CVM PRIVILEGIO.



Figure: *Geometriae pars universalis*, published in Padua in 1668.

At the webpage [Riemann integral](#) we find this interesting remark.

The Riemann integral was introduced in Bernhard Riemann's paper "Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe" (On the representability of a function by a trigonometric series; i.e., when can a function be represented by a trigonometric series).

This paper was submitted to the University of Göttingen in 1854 as Riemann's Habilitationsschrift (qualification to become an instructor).

It was published in 1868 in Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen (Proceedings of the Royal Philosophical Society at Göttingen), vol. 13, pages 87-132.

For Riemann's definition of his integral, see section 4, "Über den Begriff eines bestimmten Integrals und den Umfang seiner Gültigkeit" (On the concept of a definite integral and the extent of its validity), pages 101-103.



Figure: Monograph in which the Riemann integral was introduced.

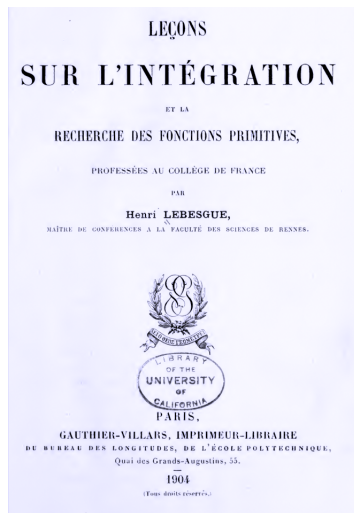


Figure: Monograph in which the Lebesgue integral was introduced.

Cavalieri-Simpson rule

About the ascription to Cavalieri, Peano wrote in [Residuo in Formula de quadratura Cavalieri-Simpson](#) (1916), in a simplified latin language, that this formula was discovered by [Cavalieri](#) (1639), [Gregory](#) (1668), [Cotes](#) (1722) and Simpson (1743).

La projection centrale de la figure 1 sur un ablique à celle-ci f_2 , donne la construction proposée par M. d'Ocagne. Son tracé consiste à mener par A' deux droites quelconques (projection de Δ parallèles par A' dans la fig. 1); par les points d'intersection J et K nous menons deux droites quelconques (projection des parallèles auxiliaires par P et Q dans la fig. 1) sur lesquelles on a deux pointelles homologues déterminées par les trois couples de points JJ' , $Q'P'$ et $Q''P''$. La droite AA' n'étant autre chose que l'axe perspectif des deux pointelles, nous obtenons un point quelconque de l'axe en joignant convenablement les points des deux supports, ici PQ' et $Q''P''$. Le point d'intersection S est un point de la droite cherchée.

Nota de G. Prato (Torino)

Formula de quadratura que nos considera es:

que es vero, si functione f es integro, de grado non superiore

³ Umami vocabulă de latină: *la, de, sau, etc.*; numine habe forma de *Urona*, *id est*: *Pariduo*, *feruine*, *grada*, *intere* (oblivion) *non* *monstrant* *non* *monstrant*.

Ativado presente, troco en « Atli R. As. di Torino, 21, 5, 1913 », en scripto in « lingua sine flexione », uno ex numerosa forma de latetages, intelligibile sine studio al magna publico.

Ce formula occorre sub forma geometrico, in
B. CAVALIERI, *Centuria di varii problemi*, Bologna, a. 1639, p. 446.

Tr. SIMPSON, *Mathematical dissertations*, London, 1743, p. 102.
Es usu de appella « formula di Simpson » formula precedente.
Me adde nomine de primo auctore.

Praxi proba quod formula de Cavallieri-Simpson, es in generale satis approximato. Ergo plure auctore desidera de cognosce una limite de errore in isto approximatione.

In libro: *Applicazioni geometriche del Calcolo infinitesimale*, Torino, Bocca, 1987, pag. 208, me pubblica espressione di residuo, id es de differentia inter primo et secundo membro sub forma:

ubi ξ è un valore medio inter a e b .

Idem resultatu es publicatu ab Markov, in libro edito in S. Petersburg, 1889; et versione cum titulo: MARKOFF, *Differenzenrechnung*, Leipzig, 1896.

Un caso particolare de regola esposito in meo scripto: Resto nelle formule di quadratura, espresso con un integrale definito. « Rendiconti Acc. Lincei », 4 maggio 1913, ibi me da espressione de residui in formula de Cavalieri-Simpson, sub forma de integrale.

In presente scripto, me perveni ad idem resultatu, per via plus
directo et plus elementare.

Versione 3: Pro haube capietate de una vicaria, non multiplicat inno parte de longitudina de uno moggio, dando moggio et uno crocchio ».

Si con nome $\alpha(x)$ de vase pro $\alpha(x)$ de $\mathcal{A}$, se pro origine puncta medio, si longitudina de vase α , et si $1/2$ de area de sectione de vase, in puncto de abscissa x , et normale ad $\alpha(x)$, tunc

velocidade $= \int_{-h/2}^h f(x) dx$; densidade média $= f(h)$; densidade média $= f(h/2) = f(-h/2)$; et re
gula de Cavalieri que que integrate vale

[illegible]

$$\frac{\lambda}{2} [I(0) + I(\lambda)] = \frac{\lambda}{2} [I(0) + I(-\lambda)] + I(\lambda)$$

Figure: Peano paper and historical notes on the ascription of this rule.

It is surprising why Cavalieri for what purpose Cavalieri used this rule.

At that time there were many efforts on solids of rotation. For instance, **Kepler** in *Nova stereometria doliorum vinariorum* that is *New measure of the volume of barrels of wine* justified an empirical method used by austrian coopers, by means of a geometrical approach.

Also **Cavalieri** had some interest on the topic. In *Note - Postille Matematiche*, Gabrio Piola, discusses of the work *Centuria di varii problemi*, by Cavalieri. In particular at page 83, he writes

Problem 80 concerns the measure of elliptical-circular barrels, giving a rule that is exactly the same that nowadays stems from Rossi-Amulis formula, demonstrated in 1806.

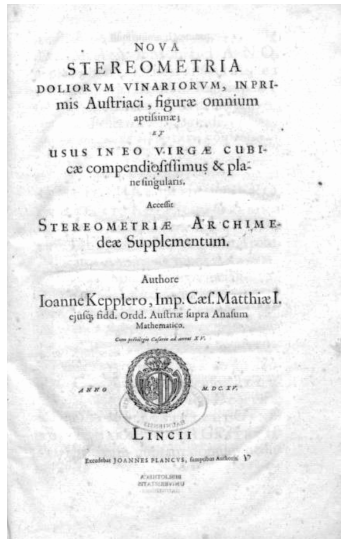


Figure: Cover of Kepler work *Nova stereometria doliorum vinariorum* on the volume of barrels.

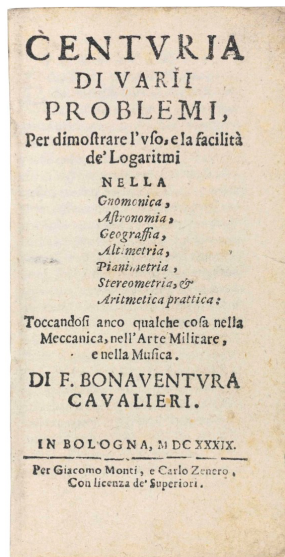


Figure: The cover of the work by Cavalieri, suggested by Peano.

¹ Phrasi de Cavalieri es: « Per havere la capacità della botte moltiplicheremo la terza parte della lunghezza della botte in due cerchi maggiori ed uno dei minori ».

Versione « Pro habere capacitatem de vase vinario, nos multiplicamus tertiam partem de longitudine de vase per duo circulos maiores et uno minorem ».

Si nos sumamus axem de vase pro axi de x , et pro origine punctum medium, si longitudinem de vase es h , et si $f(x)$ es area de sectione de vase, in puncto de abscissa x , et normale ad axem, tunc

volumine $= \int_{-h/2}^{h/2} f(x) dx$; circulus maior $= f(0)$, circulus minor $= f(h/2) = f(-h/2)$; et re-

gula de Cavalieri dic que integrale vale:

$$\frac{h}{3} [2f(0) + f(h/2)] = \frac{h}{6} [4f(0) + f(-h/2) + f(h/2)] .$$

Figure: Remark by Peano on the volume of barrels.

1948. In *Vino Veritas*:—"Si adunque moltiplicheremo la terza parte di IM lunghezza della Botte $BDFH$ in due cerchi maggiore CG e uno de minore BH , DF come in BH , ci verrà la capacità di detta Botte." F. B. Cavalieri, *Una centuria di varii Problemi nella Prattica Astrologica*, (Bologna, 1639), Problema 80. [Per Dr. G. N. Watson. It was probably this formula for the capacity of a cask of wine which suggested to James Gregory (who spent the years 1665–8 in Padua) the more general formula for approximate integration, often written in the form

$$\int_0^{2h} y \, dx = h(y_0 + 4y_1 + y_2)/3,$$

which, in its turn, led to the outstanding generalisations due to Newton and Cotes, and subsequently to the minor generalisation known as 'Simpson's rule'; about this rule (published in 1743) there is not much to be said except that it is more accurate than the 'trapezoidal rule'] [Per Dr. G. N. Watson]

Figure: Remark by G.N. Watson about Cavalieri and the volume of barrels.

Problema 79. 445

quello, che dimostra Archim. ne Lib. de Sphæra, & Cylindro, perciò basterà misurare il cerchio di, BA, semidiametro, cioè giungeremo insieme il log. del cerchio della Tavoletta del prob. 66 che è 0,49715. con il doppio del log. di, BA, cioè con il log. di, EA, 130103. e di, AI, 060206. (perche, EA, AB, AI, sono proporzionali) e ne verrà il logar. 240024. di p. 251.33. e tanto sarà la superficie di detta porzione di sfera. La scio poi l'Essempio per la solidità della porzione dello sferoide, non essendo dissimile l'operazione da quella della sfera.

PROBLEMA 80.

Misurare la capacità delle Botti.

Sia di nuovo posta quā la figura del prob. ant. nella quale siano le due porzioni, ABH, D EF, li cui assi, AI, ME, siano eguali, onde trā le basi di quelle, cioè trā li cerchi, BH, DF, che faranno eguali, resti compreso il corpo,

446 Della Centuria

po, o Botte, BDFH, e per, L, centro pass, CG, diametro del cerchio maggiore, C G, perpendicolare ad, AE.

Per hauere adunque la capacità della Botte, BDFH, si potrà misurare mediante gli assi, AE, CG, tutto lo sferoide, ACEG, per il prob. 77. e poi per il prob. ant. le due porzioni, BAH, DEF, le quali sottratte dallo sferoide, ACEG, ci lasceranno la capacità della Botte, BDFH, ma perche questa è troppo lunga fattura, perciò ci potremo seruire di questi altri modi, come più facili.

Se adunque moltiplicheremo la terza parte di, IM, lunghezza della Botte, BDFH, in due cerchi maggiori, CG, & vno de minori, BH, DF, come m, BH, ci verrà la capacità di detta Botte. E poi quadreremo il cerchio di, CG, moltiplicando, CG, in se stesso,



Problema 80. 447

so, e poi moltiplicando il prodotto di nuovo per 7854. e partendo l'aumento per 10000 (che sono li termini della proporzione del quadrato al cerchio inscritto) poiche il quoziente sarà il cerchio, CG, e così ancora quadreremo il cerchio, BH, e poi moltiplicheremo il terzo di, IM, nelli due cerchi, CG, & vno, BH, e ne verrà la capacità della Botte, BDFH.

E per i log. giungendo insieme il log. della terza parte di, IM, cō il log. del cerchio, CG, cioè con il log. del cerchio della Tavoletta del prob. 66. che è 0,49715. e con il log. duplicato di, CL, e con il log. del Binario, ne verrà il log. di vn primo inuento. Dipoi giungendo parimente insieme il log. della terza parte di, IM, con il log. del cerchio, BH, cioè con il log. del cerchio della Taur. del Prob. 66. e con il log. duplicato di, BL, ne verrà il log. di vn secondo inuento, il quale giunto al primo ci darà la capacità della Botte, BDFH.

ES-

Figure: Problem 80, at p.445 of the *Centuria di varii problemi* by Cavalieri.

The same rule can be found in [Mathematical dissertations On A Variety of Physical and Analytical subjects](#) by Thomas Simpson. It is clear that T. Simpson is thinking to composite rules.

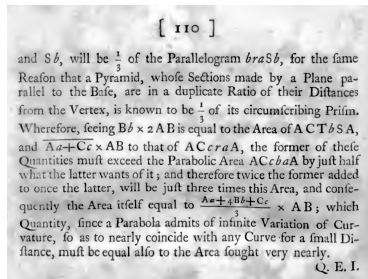
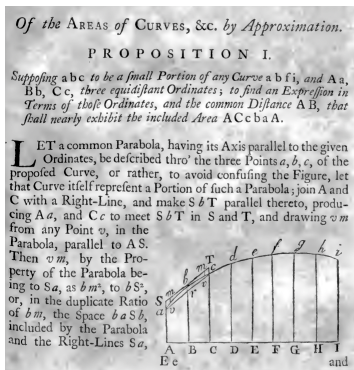


Figure: Excerpt from *Mathematical dissertations On A Variety of Physical and Analytical subjects* by T. Simpson.

Considerazione.

Assume that $f \in C([a, b])$, where $[a, b]$ is bounded.

By Weierstrass theorem, since that function can be approximated arbitrary well and uniformly by polynomials, one may believe that that rules with higher degree may approximate arbitrarily well the definite integral, but we shall see this is not true.

After the previous note on Runge function, one foresees that if the nodes are $\{x_k\}_{k=0,\dots,n}$ some problems may happen.

We will see later the results.

Definition (Newton-Cotes rules)

Let $[a, b]$ be a bounded interval. A rule

$$S_n(f) = \sum_{i=0}^n w_i f(x_i) \approx \int_a^b f(x) dx$$

is of **closed Newton-Cotes** type if

- that the set of nodes is equispaced and contains the extrema, that is

$$x_i = a + \frac{i(b-a)}{n}, \quad i = 0, \dots, n,$$

- the weights are

$$w_i = \int_a^b L_i(x) dx, \quad i = 0, \dots, n, \quad L_i(x) = \prod_{j=0, j \neq i}^n \frac{(x - x_j)}{x_i - x_j}.$$

Remark (Degree of exactness of closed Newton-Cotes formulae)

This rule is interpolatory and has $n + 1$ nodes, thus it has degree of exactness at least n .

Remark (Facoltativa)

- These rules were introduced by Newton in 1676 and described in *Of Quadrature by Ordinates* in 1695. The literature is not clear, it written they had 5 ordinates (so degree of exactness at least 4).
- Cotes, that was editor of the second version of the Principia, generalised the results by Newton, possibly in 1707, though they were published in 1722. In particular Cotes computed rules that had up to 11 nodes.

Considering *Harmonia mensurarum*, it is not easy to found where these rules are. An appendix is sometimes mentioned, but it is not easy to understand where it can be found.

Historians say that Cotes was not cited and paid for his work in the second version of Newton's Principia. In spite of that, when Cotes died at 34 yo, Newton said *If he had lived, we might have known something.*

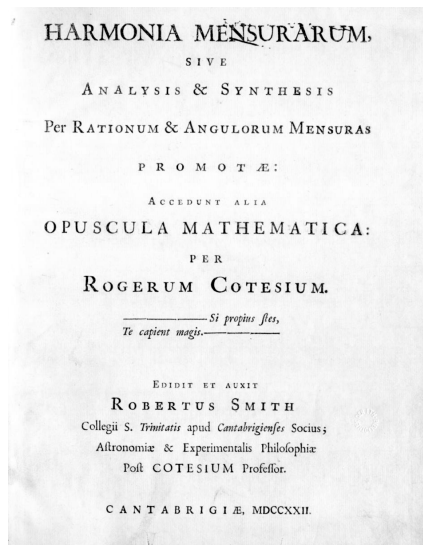


Figure: First page of *Harmonia mensurarum*

Newton-Cotes rules

We show numerically the lack of convergence of the Newton-Cotes closed formula when applied to compute $\int_{-5}^5 1/(1+x^2)dx$ (i.e. we adopt as integrand the Runge function). Since the interpolant is far from approximating the Runge function we believe that also the integrals will be very different.

n	<i>Integral Intp. Pol.</i>	<i>Absolute Error</i>
1	3.846153846153846e - 01	2.362e + 00
2	6.794871794871796e + 00	4.048e + 00
3	2.081447963800905e + 00	6.654e - 01
4	2.374005305039788e + 00	3.728e - 01
5	2.307692307692308e + 00	4.391e - 01
6	3.870448673470800e + 00	1.124e + 00
7	2.898994409748379e + 00	1.522e - 01
8	1.500488907127907e + 00	1.246e + 00
9	2.398617897841837e + 00	3.482e - 01
10	4.673300555653490e + 00	1.926e + 00
15	4.15558992699889e + 00	1.409e + 00
20	-2.684955208653064e + 01	2.960e + 01

Table: Newton-Cotes formulas failing to approximate

$$\int_{-5}^5 1/(1+x^2)dx \approx 2.746801533890032.$$

Midpoint composite rule

composite rules are obtained **integrating a piecewise polynomial interpolant of degree m .**

As example,

- we subdivide $[a, b]$ in L equispaced intervals $[t_0, t_1], \dots, [t_{L-1}, t_L]$;
- define in each $[t_{k-1}, t_k]$ a set of $m + 1$ equispaced points $x_{m(k-1)} < \dots < x_{mk}$ with $t_{k-1} = x_{m(k-1)}$ and $t_k = x_{mk}$;
- let " s_m " the **piecewise polynomial interpolant of degree m** , relatively to the subdivision $[t_k, t_{k+1}]$, with $k = 0, \dots, L$, and couples $(x_j, f(x_j))$, $j = 0, \dots, n = Lm$;
- suppose that $S_m(f, t_{k-1}, t_k)$ is a closed formula of Newton-Cotes close type, in the interval $[t_{k-1}, t_k]$, with $m + 1$ nodes (so having at least degree of exactness m).

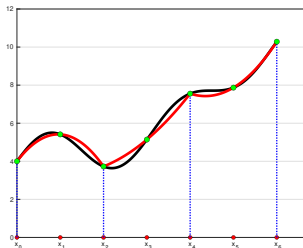


Figure: In black: the function $f(x) = 3 + \sin(2x) + \cos(x) + x$. In red: the piecewise interpolant s_m of degree $m = 2$, relatively to the subdivision $[t_k = x_{2k}, t_{k+1} = x_{2(k+1)}]$, with $k = 0, 1, 2$, and to the couples $(x_j, f(x_j))$, $j = 0, \dots, n = 6$. In green: the couples $(x_j, f(x_j))$, $j = 0, \dots, 6$.

- Being s_m a polynomial of degree m in each $[t_k, t_{k+1}]$,
- since $S_m(f, t_k, t_{(k+1)})$ has degree of exactness m ,

by additivity of integration operator

$$\begin{aligned}
 \int_a^b f(x)dx &\approx \int_a^b s_m(x)dx \\
 &= \int_{t_0=a}^{t_1} s_m(x)dx + \int_{t_1}^{t_2} s_m(x)dx + \dots + \int_{t_{L-1}}^{x_L=b} s_m(x)dx \\
 &= S_m(f, t_0, t_1) + S_m(f, t_1, t_2) + \dots + S_m(f, t_{L-1}, t_L)
 \end{aligned}$$

Thus we obtained the approximation of the integral not by a Newton-Cotes rule with high degree of exactness “ n ”, but **applying the same rule with a low ADE “ m ” in each subinterval, summing up all the contributions.**

Definition (Composite rule)

Let

- 1 $[a, b]$ a bounded interval,
- 2 $t_j = a + jh$ with $h = (b - a)/N$, $j = 0, \dots, N$,
- 3 $S(f, \alpha, \beta)$ a rule in the bounded interval $[\alpha, \beta]$.

The quadrature formula

$$S^{(c)}(f, a, b, N) = \sum_{j=0}^{N-1} S(f, t_j, t_{j+1}) \quad (4)$$

è is a composite rule using S .

Remark

*In this discussion, the points x_j , $j = 0, \dots, N$, will be **equispaced**, but with some effort we can extend the analysis to a different distribution. In other terms:*

- if $t_0 = a < t_1 < \dots < t_N = b$, one partitions $[a, b]$ as union of subintervals $[t_k, t_{k+1}]$, $k = 0, \dots, N - 1$;
- in each subinterval we apply the same rule, summing up the contributions.

Midpoint composite rule

Definition (Midpoint composite rule)

The midpoint composite rule is defined as

$$S_0^{(c)}(f, a, b, N) := \frac{b-a}{N} \sum_{k=0}^{N-1} f(x_k), \quad (5)$$

where x_k is the midpoint of the $k+1$ -th interval, that is

$$x_k = a + \frac{2k+1}{2} \cdot \frac{b-a}{N}, \quad k = 0, \dots, N-1.$$

Subdividing $[a, b]$ in N equispaced interval $[t_k, t_{k+1}]$, $K = 0, \dots, N-1$ with $t_j = a + jh$, $j = 0, \dots, N$, $h = (b-a)/N$, if x_k is the midpoint of $[t_k, t_{k+1}]$ then

$$x_k = \frac{t_k + t_{k+1}}{2} = \frac{a + kh + a + (k+1)h}{2} = a + \frac{(2k+1)h}{2}.$$

and since $w_k = t_{k+1} - t_k = a + (k+1)h - (a + kh) = h = (b-a)/N$ we get

$$S_0^{(c)}(f, a, b, N) := \sum_{k=0}^{N-1} \frac{b-a}{N} f(x_k) = \frac{b-a}{N} \sum_{k=0}^{N-1} f(x_k).$$

Midpoint composite rule

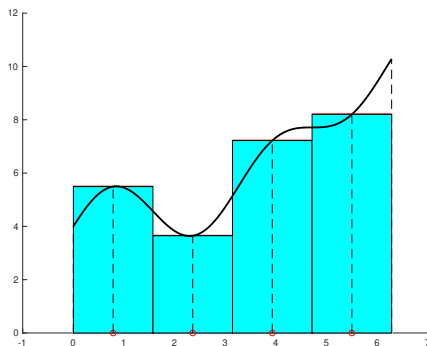


Figure: Midpoint composite rule and approximation of $\int_0^{2\pi} 3 + \sin(2x) + \cos(x) + x \, dx$ (the formula computes the area in cyan).

Theorem (Error of midpoint composite rule)

If $[a, b]$ is subdivided in N equispaced intervals of length $h = \frac{b-a}{N}$, then

$$E_0^{(c)}(f) := I(f) - S_0^{(c)}(f, a, b, N) = \frac{(b-a)}{24} h^2 f^{(2)}(\xi^*), \quad \xi^* \in (a, b)$$

Remark (Degree of exactness)

From the formula above, it is immediate to get that ADE is exactly equal to 2.

Remark (Comparison with the midpoint rule)

The midpoint rule had an error

$$E_0(f) := I(f) - S_0(f) = \frac{(b-a)^3}{24} f^{(2)}(\xi), \quad \xi \in (a, b),$$

while for $N > 1$

$$\frac{(b-a)}{24} h^2 = \frac{(b-a)}{24} \left(\frac{b-a}{N} \right)^2 = \frac{(b-a)^3}{24N^2} < \frac{(b-a)^3}{24}.$$

Consequently, one may think that if f^2 does not vary much, the composite rule tend to provide smaller errors (notice that in general $\xi^ \neq \xi$).*

Trapezoidal composite rule

Purpose. (How it is obtained)

To give some insight, suppose that $[a, b]$ is subdivided $N = 4$ equispaced subintervals $[t_k, t_{k+1}]$, $K = 0, \dots, N - 1 = 3$ con $t_j = a + jh$, $j = 0, \dots, N = 4$,
 $h = (b - a)/N = (b - a)/4$.

Let $S_1(f, \alpha, \beta)$ the application of the trapezoidal rule relatively to f and to the interval $[\alpha, \beta]$,

- $S_1(f, t_0, t_1) = \frac{h}{2}(f(t_0) + f(t_1))$,
- $S_1(f, t_1, t_2) = \frac{h}{2}(f(t_1) + f(t_2))$,
- $S_1(f, t_2, t_3) = \frac{h}{2}(f(t_2) + f(t_3))$,
- $S_1(f, t_3, t_4) = \frac{h}{2}(f(t_3) + f(t_4))$,

thus being $N = 4$

$$\begin{aligned} S_1^{(c)}(f, a, b, 4) &= \frac{h}{2}(f(t_0) + f(t_1) + f(t_1) + f(t_2) + f(t_2) + f(t_3) + f(t_3) + f(t_4)) \\ &= \frac{h}{2}(f(t_0) + 2f(t_1) + 2f(t_2) + 2f(t_3) + f(t_4)) \\ &= \frac{b-a}{N} \left(\frac{1}{2}f(t_0) + f(t_1) + f(t_2) + f(t_3) + \frac{1}{2}f(t_4) \right) \end{aligned}$$

Definition (Trapezoidal composite rule)

Let $x_k = a + kh$, $k = 0, \dots, N$, $h = (b - a)/N$, the **trapezoidal composite rule** is defined as

$$S_1^{(c)}(f, a, b, N) := \frac{b - a}{N} \left[\frac{f(x_0)}{2} + f(x_1) + \dots + f(x_{N-1}) + \frac{f(x_N)}{2} \right],$$

In the previous assumptions,

$$E_1^{(c)}(f) := I(f) - S_1^{(c)}(f, a, b, N) = \frac{-(b - a)}{12} h^2 f^{(2)}(\xi), \quad h = \frac{(b - a)}{N} \quad \xi \in (a, b).$$

Remark (Grado di exactness)

Similarly to the basic rule, the degree of precision is exactly 1 since

$$|E_1^{(c)}(f)| := |I(f) - S_1^{(c)}(f, a, b, N)| = \frac{(b-a)}{12} h^2 |f^{(2)}(\xi)|, \quad h = \frac{(b-a)}{N},$$

Remark (Comparison with the rule)

With regards to the rule, we had

$$|E_1(f)| := |I(f) - S_1(f)| = \frac{(b-a)^3}{12} |f^{(2)}(\xi)|, \quad \xi \in (a, b).$$

but here, being $N > 1$

$$\frac{(b-a)}{12} h^2 = \frac{(b-a)}{12} \left(\frac{(b-a)}{N} \right)^2 = \frac{(b-a)^3}{12N^2} < \frac{(b-a)^3}{12},$$

and as in the case of the midpoint composite rule, we expect an inferior error increasing N if f^2 does not vary much.

Trapezoidal composite rule

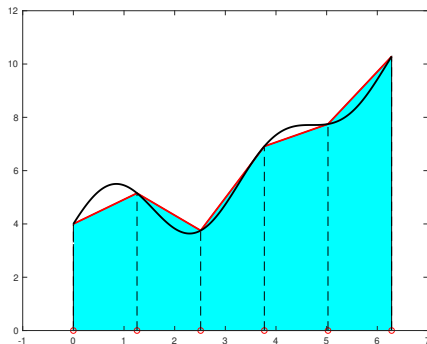


Figure: Trapezoidal composite rule for approximating $\int_0^{2\pi} 3 + \sin(2x) + \cos(x) + x \, dx$ (it computes the area in cyan, that is the area defined by the piecewise interpolant in red).

The trapezoidal composite rule has many interesting features., that are carefully described in [?]. In particular we can consider a note on a work by Poisson.

It appears to have been Poisson, in the 1820s, who first identified this effect [139]. The example Poisson chose has remained a favorite ever since: the perimeter of an ellipse, which he took to have axis lengths $1/\pi$ and $0.6/\pi$, giving the integral

$$(1.1) \quad I = \frac{1}{2\pi} \int_0^{2\pi} \sqrt{1 - 0.36 \sin^2 \theta} d\theta.$$

Poisson used this now-standard notation for definite integrals, but apparently it was not yet standard in the 1820s, for he comments that

pour indiquer [les limites de l'intégrale] en même temps que l'intégrale, nous emploierons la notation très-commode que M. Fourier a proposée.¹

Perhaps in 1826 the spelling of Fourier's name wasn't yet standardized either!

The exact solution of (1.1) is

$$(1.2) \quad I = \frac{2}{\pi} E(0.36) = 0.90277992777219 \dots,$$

where E is the complete elliptic integral of the second kind. As trapezoidal rule approximations we can take

$$I_N = \frac{1}{N} \sum_{k=1}^N \sqrt{1 - 0.36 \sin^2(2\pi k/N)}$$

for any positive integer N , or, equivalently, if N is divisible by 4, exploiting the four-fold symmetry as Poisson did,

$$I_N = \frac{4}{N} \sum_{k=0}^{N/4} \sqrt{1 - 0.36 \sin^2(2\pi k/N)}.$$

Consider the integrand

$$\frac{1}{2\pi} \sqrt{1 - 0.36 \sin^2 \theta}$$

and verify it is periodic with its derivatives in $[0, 2\pi]$. Both the geometry (meaning of the integral) and Maple confirm this.

```
>> syms f(x)
>> f(x)=(1/(2*pi))*sqrt(1-0.36*(sin(x))^2);
>> g=diff(f,x,1); g(2*pi)-g(0)
ans = 0
>> g=diff(f,x,2); g(2*pi)-g(0)
ans = 0
>> g=diff(f,x,3); g(2*pi)-g(0)
ans = 0
>> g=diff(f,x,4); g(2*pi)-g(0)
ans = 0
>> g=diff(f,x,25); g(2*pi)-g(0)
ans = 0
>> g=diff(f,x,30); g(2*pi)-g(0)
ans = 0
```

Euler-Maclaurin formula

Let h_N be the length of the generic subinterval of the equispaced subdivision of $[a, b]$.

Theorem (Euler-Mac Laurin formula, 1735)

If $f \in C^{2M+2}([a, b])$ then

$$\begin{aligned} \int_a^b f(x) dx &= S_1^{(c)}(f, N) + \sum_{k=1}^M \frac{B_{2k}}{(2k)!} h_N^{2k} \left(f^{(2k-1)}(b) - f^{(2k-1)}(a) \right) \\ &\quad - \frac{B_{2M+2}}{(2M+2)!} h_N^{(2M+2)} (b-a) f^{(2M+2)}(\xi), \quad \xi \in (a, b) \end{aligned}$$

where B_k are the *Bernoulli numbers* (Bernoulli, 1713).

If $f \in C^{2M+2}([a, b])$ e $f^{(2k-1)}(b) = f^{(2k-1)}(a)$, for $k = 1, \dots, M$

$$\int_a^b f(x) dx - S_1^{(c)}(f, N) = -\frac{B_{2M+2}}{(2M+2)!} h_N^{(2M+2)} (b-a) f^{(2M+2)}(\xi),$$

where $\xi \in (a, b)$ and from $h_N = (b-a)/N$, we get $\mathcal{E}_1^{(c)}(f) \approx \frac{C}{N^{2M+2}}$.

Euler-Maclaurin formula (Bernoulli numbers)

Bernoulli numbers might have been the first quantities computed numerically by a code on a machine, the so called **Analytical Engine** (1842), that is considered as the first modern calculator.

There is a **controversy** if the programmer was Babbage, the designer of the machine or Ada Lovelace, daughter of Lord Byron.



Figure: A component of the Analytical Engine.

Euler-Maclaurin formula

From Euler-Mac Laurin formula, in the case of $f(\theta) := \frac{1}{2\pi} \sqrt{1 - 0.36 \sin^2 \theta}$ we have that

$$|E_1^{(c)}(f)| := \left| \int_a^b f(x) dx - S_1^{(c)}(f, N) \right|$$

is more rapid than N^{-M} for any M (here N is the number of subintervals).

Actually it is of *geometric* type, that is $|E_1^{(c)}(f)| \approx \alpha \gamma^{-N}$, for suitable α, γ (cf. [?], p.387). he

N	$ E_1^{(c)}(f) $	N	$ E_1^{(c)}(f) $
2	$9.7e-02$	32	0
4	$2.8e-03$	64	0
8	$1.1e-05$	128	0
16	$5.4e-10$	256	0
32	$1.1e-16$	512	0

Table: Error of Trapezoidal composite rule on Poisson example, subdividing $[a, b]$ in N equispaced intervals

Numerical integration of periodic functions is an important subject due to its applications.

It is a fundamental ingredient of **FFT algorithm** that requires the evaluations of these quantities for specific integrands.

The **Fast Fourier Transform** is used in

- elaboration of digital signals (fundamental for the **mp3** compression),
- solution of PDEs;
- algorithms for multiplication of integers of large magnitude.

This algorithm was discovered by Cooley-Tukey in 1965 (but some sources say it was known in some form to **Gauss**!).

Remark

In *IEEE Guest Editors' Introduction: The Top 10 Algorithms*, it is written:

The FFT is perhaps the most ubiquitous algorithm in use today to analyze and manipulate digital or discrete data.

Cavalieri-Simpson composite rule

Purpose. (Composite Cavalieri-Simpson rule)

As illustration, let us subdivide $[a, b]$ in $N = 4$ equispaced intervals $[t_k, t_{k+1}]$, $k = 0, \dots, N-1 = 3$ with $t_j = a + jh$, $j = 0, \dots, N = 4$, $h = (b - a)/N$.

Defining with $S_2(f, t_k, t_{k+1})$ the application of Cavalieri-Simpson rule relatively to f and to the interval $[t_k, t_{k+1}]$, letting $c_k = \frac{t_k + t_{k+1}}{2} = a + \frac{2k+1}{2} \cdot \frac{b-a}{N}$ be the midpoint of $[t_k, t_{k+1}]$,

$$\blacksquare S_2(f, t_0, t_1) = \frac{h}{6}(f(t_0) + 4 \cdot f(c_0) + f(t_1)),$$

$$\blacksquare S_2(f, t_1, t_2) = \frac{h}{6}(f(t_1) + 4 \cdot f(c_1) + f(t_2)),$$

$$\blacksquare S_2(f, t_2, t_3) = \frac{h}{6}(f(t_2) + 4 \cdot f(c_2) + f(t_3)),$$

$$\blacksquare S_2(f, t_3, t_4) = \frac{h}{6}(f(t_3) + 4 \cdot f(c_3) + f(t_4)),$$

we obtain from $N = 4$

$$\begin{aligned} S_2^{(c)}(f, a, b, 4) &= \frac{h}{6}(f(t_0) + 4 \cdot f(c_0) + f(t_1)) + \frac{h}{6}(f(t_1) + 4 \cdot f(c_1) + f(t_2)) \\ &+ \frac{h}{6}(f(t_2) + 4 \cdot f(c_2) + f(t_3)) + \frac{h}{6}(f(t_3) + 4 \cdot f(c_3) + f(t_4)) = \dots \\ &= \frac{h}{6}f(t_0) + \frac{2h}{6}(f(t_1) + f(t_2) + f(t_3)) + \frac{h}{6}f(t_4) \\ &+ \frac{4h}{6}(f(c_0) + f(c_1) + f(c_2) + f(c_3)). \end{aligned} \tag{6}$$

Cavalieri-Simpson composite rule

From

$$\begin{aligned} S_2^{(c)}(f, a, b, 4) &= \frac{h}{6}f(t_0) + \frac{2h}{6}(f(t_1) + f(t_2) + f(t_3)) + \frac{h}{6}f(t_4) \\ &+ \frac{4h}{6}(f(c_0) + f(c_1) + f(c_2) + f(c_3)). \end{aligned} \quad (7)$$

setting $t_0 = x_0$, $c_0 = x_1$, $t_1 = x_2$, $c_1 = x_3$, $t_2 = x_4$, $c_2 = x_5$, $t_3 = x_6$, $c_3 = x_7$, $t_4 = x_8$, we get for $N = 4$ (i.e. the number of subdivisions)

$$S_2^{(c)}(f, a, b, 4) = \frac{h}{6} \left(f(x_0) + 2 \sum_{r=1}^{N-1} f(x_{2r}) + 4 \sum_{s=0}^{N-1} f(x_{2s+1}) + f(x_{2N}) \right) \quad (8)$$

This ideas can be easily generalized to any N .

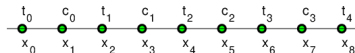


Figure: Relation between t_k , c_k e x_j .

Cavalieri-Simpson composite rule

Definition (Cavalieri-Simpson composite rule)

Set $x_k = a + kh/2$, $k = 0, \dots, 2N$, $h = (b - a)/N$, Cavalieri-Simpson composite rule is defined as

$$S_2^{(c)}(f, a, b, N) = \frac{h}{6} \left[f(x_0) + 2 \sum_{r=1}^{N-1} f(x_{2r}) + 4 \sum_{s=0}^{N-1} f(x_{2s+1}) + f(x_{2N}) \right] \quad (9)$$

Theorem (Error of Cavalieri-Simpson composite rule)

In the assumptions of subdivisions via equispaced intervals of length h , the integration error is

$$E_2^{(c)}(f) := I(f) - S_2^{(c)}(f, a, b, N) = \frac{-(b-a)}{180} \left(\frac{h}{2} \right)^4 f^{(4)}(\xi), \quad \xi \in (a, b)$$

Remark (Degree of exactness of Cavalieri-Simpson composite rule)

The degree of exactness is exactly 3, as Cavalieri-Simpson rule, but if $N > 1$ the length h is inferior and consequently an inferior absolute integration error is expected.

Cavalieri-Simpson composite rule

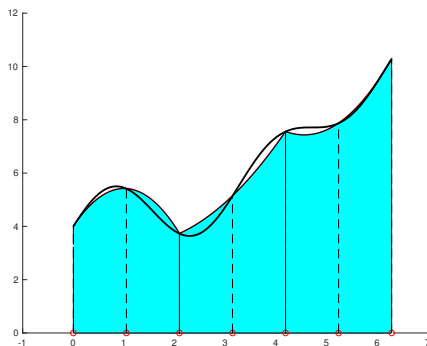


Figure: Cavalieri-Simpson composite rule for the computation of $\int_0^{2\pi} 3 + \sin(2x) + \cos(x) + x \, dx$ (the rule determines the area in cyan).

In this section we analyse some examples in which we apply composite rules to integrate some test integrands in $f \in C([a, b])$.

Example (1)

Approximate the definite integral

$$I = \int_0^{\pi} \exp(x) \cos(x) dx = -(\exp(\pi) + 1)/2.$$

by means of composite rules $S_k^{(c)}(f, 0, \pi, N)$, $N = 1, 2, 4, \dots, 512$, $k = 0, 1, 2$.

Remark

Note that the integrand belongs to $C^\infty([0, 2\pi])$ (actually it is [entire](#)).

Some numerical comparisons: example 1

N	$ E_0^{(c)}(f) $	$ E_1^{(c)}(f) $	$ E_2^{(c)}(f) $	$\#_N^R$	$\#_N^T$	$\#_N^{CS}$
1	$1.2e + 01$	$2.3e + 01$	$4.8e - 01$	1	2	3
2	$2.8e + 00$	$5.3e + 00$	$8.5e - 02$	2	3	5
4	$6.4e - 01$	$1.3e + 00$	$6.1e - 03$	4	5	9
8	$1.6e - 01$	$3.1e - 01$	$3.9e - 04$	8	9	17
16	$3.9e - 02$	$7.8e - 02$	$2.5e - 05$	16	17	33
32	$9.7e - 03$	$1.9e - 02$	$1.6e - 06$	32	33	65
64	$2.4e - 03$	$4.8e - 03$	$9.7e - 08$	64	65	129
128	$6.1e - 04$	$1.2e - 03$	$6.1e - 09$	128	129	257
256	$1.5e - 04$	$3.0e - 04$	$3.8e - 10$	256	257	513
512	$3.8e - 05$	$7.6e - 05$	$2.4e - 11$	512	513	1025

Table: Comparison of midpoint, trapezoidal and Cavalieri-Simpson composite rule, for N equispaced intervals relatively to the computation of $I = \int_0^\pi f(x)dx$ con

$f(x) = \exp(x) \cos(x)dx$, describing the absolute errors

$$|E_0^{(c)}(f)| = |I(f, a, b) - S_0^{(c)}(f, a, b, N)|, \quad |E_1^{(c)}(f)| = |I(f, a, b) - S_1^{(c)}(f, a, b, N)|,$$

$$|E_2^{(c)}(f)| = |I(f, a, b) - S_2^{(c)}(f, a, b, N)|, \text{ for each formula and the respective number of nodes } \#_N^R, \#_N^T, \#_N^{CS}.$$

Remark (optional)

In the second table we show the ratio between two successive errors for each formula. The value $(E_k^{(c)}(f))_N$, $k = 0, 1, 2$, is the absolute integration error by S_k , for computing $\int_a^b f(x)dx$, over N subdivisions, while

$$(r_k^{(c)}(f))_N = \frac{(E_k^{(c)}(f))_N}{(E_k^{(c)}(f))_{2N}}$$

is the ratio for $k = 0, 1, 2$ (in order, midpoint, trapezoidal and Cavalieri-Simpson composite rule).

Some numerical comparisons: example 1 (optional)

N	$(r_0^{(c)}(f))_N$	$(r_1^{(c)}(f))_N$	$(r_2^{(c)}(f))_N$
1	4.33	4.27	5.59
2	4.34	4.20	13.92
4	4.10	4.06	15.54
8	4.03	4.02	15.89
16	4.01	4.00	15.97
32	4.00	4.00	15.99
64	4.00	4.00	16.00
128	4.00	4.00	16.00
256	4.00	4.00	16.00

Table: Ratios relatively to $I = \int_0^\pi f(x)dx$ with $f(x) = \exp(x) \cos(x)$, showing the ratios between two successive absolute errors of the formulas.

Example (2)

Approximate the definite integral

$$I = \int_{-5}^5 \frac{1}{1+x^2} dx \approx 2.7468015338900322319659608$$

by composite rules $S_k^{(c)}(f, -5, 5, N)$, $N = 1, 2, 4, \dots, 1024$,
 $k = 0, 1, 2$.

Remark

The integrand belongs to $C^\infty([-5, 5])$. Thus we can apply the error theorems for all the composite rules.

Some numerical comparisons: example 2

N	$ E_0^{(c)}(f) $	$ E_1^{(c)}(f) $	$ E_2^{(c)}(f) $	$\#_N^R$	$\#_N^T$	$\#_N^{CS}$
1	$7.3e+00$	$2.4e+00$	$4.0e+00$	1	2	3
2	$1.4e+00$	$2.4e+00$	$9.6e-02$	2	3	5
4	$4.6e-01$	$5.4e-01$	$1.3e-01$	4	5	9
8	$3.9e-02$	$3.8e-02$	$1.3e-02$	8	9	17
16	$2.1e-04$	$6.9e-04$	$9.1e-05$	16	17	33
32	$1.2e-04$	$2.4e-04$	$4.5e-08$	32	33	65
64	$3.0e-05$	$6.0e-05$	$2.6e-09$	64	65	129
128	$7.5e-06$	$1.5e-05$	$1.6e-10$	128	129	257
256	$1.9e-06$	$3.8e-06$	$1.0e-11$	256	257	513
512	$4.7e-07$	$9.4e-07$	$6.4e-13$	512	513	1025
1024	$1.2e-07$	$2.4e-07$	$4.0e-14$	1024	1025	2049

Table: Comparison of midpoint, trapezoidal and Cavalieri-Simpson composite rule, for N equispaced intervals relatively to the computation of $I = \int_{-5}^5 f(x)dx$ con $f(x) = 1/(1+x^2)$, describing the absolute errors $|E_0^{(c)}(f)| = |I(f, a, b) - S_0^{(c)}(f, a, b, N)|$, $|E_1^{(c)}(f)| = |I(f, a, b) - S_1^{(c)}(f, a, b, N)|$, $|E_2^{(c)}(f)| = |I(f, a, b) - S_2^{(c)}(f, a, b, N)|$, for each formula and the respective number of nodes $\#_N^R, \#_N^T, \#_N^{CS}$.

Example (3)

Approximate the definite integral

$$I = \int_0^1 x^3 \sqrt{x} dx = 2/9.$$

by composite rules $S_k^{(c)}(f, 0, 1, N)$, $N = 1, 2, 4, \dots, 1024$, $k = 0, 1, 2$.

Remark

The integrand belongs to $C^3([0, 1])$. Thus we can take into account all the error formulas but that of composite Cavalieri-Simpson rule that requires $f \in C^4([0, 1])$

Some numerical comparisons: example 3

N	$ E_0^{(c)}(f) $	$ E_1^{(c)}(f) $	$ E_2^{(c)}(f) $	$\#_N^R$	$\#_N^T$	$\#_N^{CS}$
1	$1.3e-01$	$2.8e-01$	$3.4e-03$	1	2	3
2	$3.6e-02$	$7.2e-02$	$2.3e-04$	2	3	5
4	$9.1e-03$	$1.8e-02$	$1.5e-05$	4	5	9
8	$2.3e-03$	$4.6e-03$	$1.0e-06$	8	9	17
16	$5.7e-04$	$1.1e-03$	$6.5e-08$	16	17	33
32	$1.4e-04$	$2.8e-04$	$4.1e-09$	32	33	65
64	$3.6e-05$	$7.1e-05$	$2.6e-10$	64	65	129
128	$8.9e-06$	$1.8e-05$	$1.7e-11$	128	129	257
256	$2.2e-06$	$4.5e-06$	$1.0e-12$	256	257	513
512	$5.6e-07$	$1.1e-06$	$6.6e-14$	512	513	1025
1024	$1.4e-07$	$2.8e-07$	$4.1e-15$	1024	1025	2049

Table: Comparison of midpoint, trapezoidal and Cavalieri-Simpson composite rule, for N equispaced intervals relatively to the computation of $I = \int_0^1 f(x)dx$ con

$f(x) = x^3\sqrt{x}dx$, describing the absolute errors $|E_0^{(c)}(f)| = |I(f, a, b) - S_0^{(c)}(f, a, b, N)|$, $|E_1^{(c)}(f)| = |I(f, a, b) - S_1^{(c)}(f, a, b, N)|$, $|E_2^{(c)}(f)| = |I(f, a, b) - S_2^{(c)}(f, a, b, N)|$, for each formula and the respective number of nodes $\#_N^R, \#_N^T, \#_N^{CS}$.

Remark (optional)

In the second table we show the ratio between two successive errors for each formula. The value $(E_k^{(c)}(f))_N$, $k = 0, 1, 2$, is the absolute integration error by S_k , for computing $\int_a^b f(x)dx$, over N subdivisions, while

$$(r_k^{(c)}(f))_N = \frac{(E_k^{(c)}(f))_N}{(E_k^{(c)}(f))_{2N}}$$

is the ratio for $k = 0, 1, 2$ (in order, midpoint, trapezoidal and Cavalieri-Simpson composite rule).

From the tables we see that the ratio for

- composite midpoint and trapezoidal rule is approximately 4,
- composite Cavalieri-Simpson rule is approximately 16,

and thus the errors are of the form C^*h^2 and C^*h^4 .