# Introduction to numerical cubature over the sphere

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Padua, autumn, 2024 Doctoral Program in Mathematical Sciences, Padua (I), Autumn 2024 "Numerical cubature and its applications" The purpose of this note is to provide a brief introduction to numerical cubature over the sphere.

We

- start introducing some formulas of tensorial type, discussing their pros and cons;
- describe the so called designs and show what it is known about them;
- give a look to some results on rules based on extremal sets;
- show some formula on spherical triangles and spherical rectangles;
- we use them for integration over spherical polygons.

Let us suppose that we have to compute

$$l(f) := \int_{\mathbb{S}^2} f(\boldsymbol{\eta}) dS^2(\boldsymbol{\eta})$$

with  $f \in C(\mathbb{S}^2)$ .

A first technique consists in reducing l(f) to an integral over a certain rectangle and then apply suitable tensorial rules. To this purpose, consider the spherical coordinates

$$\eta \to (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta), \ 0 \le \phi \le 2\pi, \ 0 \le \theta \le \pi.$$

Thus, taking into account the jacobian determinant of the transformation, we get

$$I(f) = \int_0^{2\pi} \int_0^{\pi} f(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta) \sin\theta \ d\theta d\phi.$$

Aiming to determine

$$I(f) = \int_0^{2\pi} \int_0^{\pi} f(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta) \sin\theta \ d\theta d\phi.$$

in view of the periodicity in the variable  $\phi,$  we apply the composite trapezoidal rule with uniform spacing, that is

$$\tilde{l}(g) = \int_0^{2\pi} g(\phi) d\phi \approx \tilde{l}_m(g) = \sum_{j=0}^m {}^{\prime\prime}g(jh), \ \ h = \frac{2\pi}{m}$$

where *II* means the first and last argument of the sum must be halved.

Taking into account the periodicity of the integrand, it is immediate that

$$\tilde{l}_m(g) = \sum_{j=0}^m {}''g(jh) = \frac{g(0)}{2} + \sum_{j=1}^{m-1} g(jh) + \frac{g(2\pi)}{2} = \sum_{j=1}^m g(jh).$$

The integral of

$$I(f) = \int_0^{2\pi} \int_0^{\pi} f(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta) \sin\theta \ d\theta d\phi.$$

w.r.t. the variable  $\theta$  is more problematic. Setting  $z=\cos(\theta),$  we have

$$I(f) = \int_0^{2\pi} \int_{-1}^1 f(\cos\phi\sqrt{1-z^2},\sin\phi\sqrt{1-z^2},z) \, dz \, d\phi.$$

At his point one can apply

- Gauss-Legendre quadrature over [-1, 1] with *n* nodes  $\{z_k\}_{k=1,...,n}$  and weights  $\{w_k\}_{k=1,...,n}$ ;
- the trapezoidal rule with nodes  $\phi_j = j\frac{\pi}{n}$ , j = 1, ..., 2n and weights  $h = \frac{\pi}{n}$ .

After these substitutions we get

$$I_n(f) = h \sum_{j=1}^{2n-1} \sum_{k=1}^n w_k f(\cos\phi_j \cdot \sqrt{1-z_k^2}, \sin\phi_j \cdot \sqrt{1-z_k^2}, z_k)$$

thus for  $\theta_k = \arccos z_k$ 

$$I_n(f) = h \sum_{j=1}^{2n-1} \sum_{k=1}^n w_k f(\cos\phi_j \cdot \sin\theta_k, \sin\phi_j \cdot \sin\theta_k, \cos\theta_k)$$

The following result holds, setting  $\mathbb{P}_{2n-1}$  the set of polynomials on the sphere of degree at most 2n - 1.

#### Theorem (Atkinson, Han, p. 169)

If 
$$f \in \mathbb{P}_{2n-1}$$
 then  $l(f) = l_n(f)$ . If  $f(x, y, z) = z^{2n}$  we have that  $l(f) \neq l_n(f)$ .

In other words the formula has degree of exactness 2n - 1.

Next, introducing the best approximation error at degree *n*, w.r.t. uniform norm,

$$\mathsf{E}_n(f) = \min_{p \in \mathbb{P}_n} \|f - p\|_\infty$$

one can prove that

$$|I(f) - I_n(f)| \le 8\pi E_{2n-1}(f).$$

The following result holds

#### Theorem (Atkinson, Han, p.141)

Let  $r \ge 1$  be an integer. Assume f is r-times continuously differentiable over  $\mathbb{S}^2$ . with all such derivatives in  $C(\mathbb{S}^2)$ . Then

$$E_n(f) \leq \frac{c}{(n+1)^r}.$$

Supposing that we have at hand the Matlab codes

- 🔳 rjacobi,
- 🛛 gauss,

that implement the Gaussian rules w.r.t. a Jacobi weight, we intend to test the product Gauss rule on the computation of the integrals

- 1  $h(f) := \int_{\mathbb{S}^2} \exp(x) dS^2 \approx 14.76801374576529;$
- 2  $l_2(f) := \int_{\mathbb{S}^2} \exp(-x^2 0.1 * y^2 2 * z^2); dS^2 \approx 5.028153009823267;$
- **B**  $I_3(f) := \int_{\mathbb{S}^2} (0.1x + y^2 + 0.5z^3)^3 dS^2 \approx 1.969928193450972;$

To this purpose we define the routines

- gaussian\_product\_rule.m that computes nodes and weight of the rule for a fixed n,
- demo\_product\_gaussian\_rule.m that tests the results on the approximation of l(f), by means of product Gauss rules for n = 2, 3, 4, 5, 6.

w = W1(:) . \* W2(:) :

The routine gaussian\_product\_rule.m implements the tensorial rule described above.

```
function [nodes,w]=gaussian_product_rule(n)
% Gaussian product rule of degree 2n-1.
% Nodes are in cartesian coordinates.
% Trapezoidal rule
m = 2*n - 1; h = (2*pi/m); phi = (1:m) '*h;
% Gauss-Legendre rule
ab=r_jacobi(n,0,0); xw=gauss(n,ab);
% Nodes
z = xw(:, 1):
[P,T] = meshgrid (phi, sqrt(1-z.^2));
[PZ,Z]=meshgrid(phi,z);
P=P(:): T=T(:): Z=Z(:):
nodes = [cos(P).*T sin(P).*T Z];
% Weights
hv=h*ones(m,1);
[W1,W2] = meshgrid (hv, xw(:,2));
```

```
function demo_product_gaussian_rule(example)
% Demo: see
% K.Atkinson, W.Han, Spherical Harmonics and Approximations on the Unit
% Sphere: An introduction, p.172.
if nargin < 1, example=3; end
nv = 2.6
T = [1]:
switch example
    case 1
         f=@(x,y,z) exp(x)+0*y+0*z;
         T = 1.476801374576529e + 01:
    case 2
         f=@(x,y,z) exp(-x.^2-0.1*y.^2-2*z.^2);
         T = 5.028153009823267:
      case 3
         f=@(x,y,z) (0.1*x+y.^2+0.5*z.^3).^3;
         T = 1.969928193450972:
end
if isempty(I), F=spherefun(f); I=sum2(F); format long; I, end
AE = []; RE = [];
for n=nv
     [nodes,w]=gaussian_product_rule(n);
    x=nodes(:,1); y=nodes(:,2); z=nodes(:,3);
    fnodes = feval(f, x, y, z);
    In=w'*fnodes:
    AE(end + 1) = abs(In - I); RE(end + 1) = abs(In - I)/(abs(I) + (I = 0));
     fprintf('\n \t n: %3.0f nodes: %5.0f AE: %1.2e RE: %1.2e',...
         n, length(x), AE(end), RE(end))
end
fprintf('n \ n');
```

The routine demo\_product\_gaussian\_rule.m provides the following results, choosing the variable example from 1 to 3.

```
>> demo_product_gaussian_rule(1)
         2 nodes:
                    8 AE: 1.17e-02 RE: 7.94e-04
   n :
   n: 3 nodes: 18 AE: 4.00e-04 RE: 2.71e-05
   n: 4 nodes: 32 AE: 4.91e-07 RE: 3.32e-08
   n: 5 nodes: 50 AE: 3.84e-09 RE: 2.60e-10
n: 6 nodes: 72 AE: 2.21e-12 RE: 1.50e-13
>> demo_product_gaussian_rule(2)
         2 nodes:
                  8 AE: 3.54e-01 RE: 7.04e-02
   n٠
   n:
       3 nodes: 18 AE: 4.77e-02 RE: 9.49e-03
   n: 4 nodes: 32 AE: 5.71e-03 RE: 1.13e-03
   n: 5 nodes: 50 AE: 5.04e-04 RE: 1.00e-04
         6 nodes: 72 AE: 3.71e-05 RE: 7.38e-06
   n:
>> demo_product_gaussian_rule(3)
        2 nodes: 8 AE: 8.11e-03 RE: 4.12e-03
3 nodes: 18 AE: 2.21e-02 RE: 1.12e-02
   n :
   n:
   n: 4 nodes: 32 AE: 2.74e-02 RE: 1.39e-02
   n: 5 nodes: 50 AE: 0.00e+00 BE: 0.00e+00
         6 nodes: 72 AE: 4.44e-16 RE: 2.25e-16
   n:
|>>
```

The third function is a polynomial of degree 9 and the formula has degree of exactness 2n - 1, so per N = 5, n = 6 the relative errors are close to 0.

In view of the rotational symmetry of the sphere, one prefers points that are not clustered to the poles, but due to the structure of Gauss-Legendre rules, this set does.



Figure: Gauss Product Rule for n = 15.

An interesting set for cubature over  $\mathbb{S}^2$  is that of spherical designs.

#### Definition (Spherical t-design)

A finite subset  $X = \{P_i\}_{i=1,...,N}$  on  $\mathbb{S}^d$  is called a spherical t-design on  $\mathbb{S}^d$ , if for any  $p_t \in \mathbb{P}_t$  we have

$$\int_{\mathbb{S}^d} p_t(\boldsymbol{\eta}) dS^d(\boldsymbol{\eta}) = rac{\mu(\mathbb{S}^d)}{N} \sum_{i=1}^N p_t(P_i),$$

where,  $\mu(\mathbb{S}^d)$  is the area of the unit-sphere  $\mathbb{S}^2$ , e.g.,  $\mu(\mathbb{S}^2) = 4\pi$ .

#### Remark

Notice that all the weights are equal, i.e.  $w_1 = \ldots, w_N = \frac{\mu(\mathbb{S}^d)}{N}$ .

#### Purpose.

- A first question that arises is if such a set X exists, for a fixed degree t.
- A second question is if the cardinality N of such a set has some lower-bounds and when they are attained.
- Finally one would like to see some examples.

We will try to respond to all these questions. The theory is rich and connects combinatorics with numerical analysis.

For details see the introduction of Numerical construction of spherical t-designs by Barzilai-Borwein method.

### Theorem (Bounds of spherical t-designs on $\mathbb{S}^2$ )

Let 
$$X = \{P_i\}_{i=1,...N}$$
 be a spherical t-design on  $\mathbb{S}^2$  If  
t is odd then
$$1 (i + 1) (i + 2)$$

$$N \ge N^* = \frac{1}{4}(t+1)(t+3);$$

t is even then

$$N \ge N^* = \frac{1}{4}(t+2)^2,$$

#### Definition (Tight Design)

Let *X* be a spherical *t*-design on  $\mathbb{S}^2$ . If the lower bound above is reached as equality, i.e.  $N = N^*$ , then *X* is a tight spherical *t*-design.

The same ideas can be extended in the more general *d* sphere.

# Theorem (Bounds of spherical t-designs)

Let  $X = \{P_i\}_{i=1,...N}$  be a spherical t-design on  $\mathbb{S}^d$ . Then • if t = 2e then

$$N \ge N^* = \binom{d+e}{e} + \binom{d+e-1}{e-1}$$

• if t = 2e + 1 then

$$N \ge N^* = 2\binom{d+e}{e}.$$

# Spherical designs

Important things that we know about such is that sets in  $\mathbb{S}^d$ :

- there is no tight spherical *t*-design with  $N^*$  points except possibly for t = 1, 2, 3, 4, 5, 7, 11 (Delsarte, Goethals and Seidel, p.364);
- there are spherical *t*-designs having  $N \ge Ct^2$  points (for a certain *C*) (Bondarenko, Radchenko and Viazovska, 2013);
- there are many spherical t-designs that are computed numerically, even for large t (available open-source).

We provide some details about the last item.

1. Extremal systems are sets of  $N = (t + 1)^2$  points on  $S^2$  which maximize the absolute value of the determinant of the Vandermonde matrix for an arbitrary fixed basis (e.g. spherical harmonics, a well-known basis on the 2-sphere).

For  $N = (t + 1)^2$ , verified that a spherical *t*-design exists in a neighborhood of an extremal system.

2. For  $N \ge (t + 1)^2$  verified extremal spherical t-designs exist for all degrees *t* up to 60 and provided well-conditioned spherical *t*-designs for interpolation and numerical integration.

At this stage one would like to have examples of spherical *t*-designs.

- A good list can be found at Spherical Designs, by Hardy and Sloane.
- **2** Spherical t-designs for t = 1, ..., 180 and symmetric (antipodal) t-designs for degrees up to 325, are available at all with low mesh ratios (see the paper Efficient Spherical Designs with Good Geometric Properties). A good list of these points is available at the site Efficient Spherical Designs with Good Geometric Properties.

There the author computes

- Spherical t-designs on the 2-sphere with  $N = t^2/2 + t + O(1)$  points;
- Symmetric (antipodal) spherical t-designs on the 2-sphere with  $N = t^2/2 + t/2 + O(1)$  points.

Let us make some numerical examples, looking for tight spherical t-designs, by considering Hardy and Sloane homepage, each one having at best  $N_t$  points (being  $N_t^*$  the optimal value).

t	$N_t^*$	Nt
0	1	1
1	2	2
2	4	4
3	6	6
4	9	14
5	12	12
6	16	26
7	20	24
8	25	40
9	30	48
10	36	62
11	42	70
12	49	84



Figure: A spherical n = 11-design with 72 points. Notice that the points are well-distributed on the 2-sphere.

# Spherical designs

Supposing we call the 72 points of the spherical 11-design from the routine set11.m, we can write this demo\_designs.m, to make tests on the 2-sphere. In case the integral is not available, the package Spherefun must be installed.

```
function demo_designs(example)
if nargin == 0, example=1; end
I = [];
switch example
    case 1
        f=@(x,y,z) exp(x)+0*y+0*z;
        I = 1.476801374576529e + 01:
    case 2
        f=@(x,y,z) exp(-x.^2-0.1*y.^2-2*z.^2);
        T = 5.028153009823267
end
if isempty(I), F=spherefun(f); I=sum2(F); format long; I, end
P=set11:
x=P(:,1); y=P(:,2); z=P(:,3); N=length(x);
w = (4 * pi / N) * ones (N, 1);
fnodes = feval(f, x, y, z);
In=w'*fnodes:
AE = abs(In - I); RE = AE/(I + (I = 0));
fprintf('\n \t AE: %1.2e RE: %1.2e \n \n', AE, RE);
```

#### As result we get

```
>> demo_designs(1)
    AE: 6.63e.13 RE: 4.49e.14
>> demo_designs(2)
    AE: 2.30e.06 RE: 4.57e.07
>> demo_designs(3)
    AE: 4.44e.16 RE: 2.25e.16
>>
```

similar to those that were obtained via tensorial rules with degree of exactness at most 11 (that were using 66 points, while this design has 72 points). In the previous part, we have considered two classical methods for integration over the 2-sphere.

Now we focus on two classes of domains on the sphere, i.e.

- spherical triangles,
- spherical rectangles.

These elements are important since many regions can be approximated by spherical polygons, that can be written as union of spherical triangles and hence composite rules can be constructed. Let  $\mathcal{T} = \overrightarrow{ABC}$  be a spherical triangle, that is, the arcs  $\overrightarrow{AB}$ ,  $\overrightarrow{BC}$ ,  $\overrightarrow{AC}$  are arcs of great circles, that is the intersection of a plane containing the origin, with the sphere.

Next we suppose it is contained in a cap whose height is inferior to 1 (i.e., it can be rotated so that it can be contained in a hemisphere).

There is no restriction to assume that the centroid of the vertices *A*, *B*, *C*, is at the North-Pole.

If it is not so, one

- can rotate the original spherical triangle so that it has this property,
- compute a formula,
- rotate it back to the original position.

## Spherical triangles

Then we can write the surface integral of a continuous function f(x, y, z) on such a spherical triangle in cartesian coordinates

$$\int_{\mathcal{T}} f(x, y, z) \, d\sigma = \int_{\Omega} f(x, y, g(x, y)) \, \frac{1}{g(x, y)} \, dx \, dy \,, \tag{1}$$

where

• 
$$g(x, y) = \sqrt{1 - x^2 - y^2}$$
,

Ω is the projection of T onto the xy-plane, i.e. the curvilinear triangle with vertices Â, B, C being the xy-coordinates of A, B, C.



- Notice that the sides of this planar and curvilinear polygon Ω are arcs of ellipses centered at the origin, being the projections (and thus transformations by an affine mapping) of great circle arcs (the sides of *T*).
- We can split the planar integral into the sum of the integrals on three elliptical sectors  $S_1$ ,  $S_2$ ,  $S_3$ , obtained by joining the origin with the vertices  $\hat{A}$ ,  $\hat{B}$ ,  $\hat{C}$ , namely

$$\int_{\Omega} f(x, y, g(x, y)) \frac{1}{g(x, y)} \, dx \, dy = \sum_{i=1}^{3} \int_{S_i} f(x, y, g(x, y)) \frac{1}{g(x, y)} \, dx \, dy \; .$$

By determining a suitable formula on each elliptical sector, we can seek a quadrature formula on the spherical which is as close as possible to an algebraic formula (at machine precision), when  $f \in \mathbb{P}_n(\mathbb{S}_2)$ .

## Spherical triangles

Notice that if our cubatura formula is able to integrate exactly all the monomials of total degree *n* then it integrates exactly each polynomial of degree *n*.

If the integrand over  ${\mathcal T}$  is a monomial as

$$f(x, y, z) = x^{\alpha} y^{\beta} z^{\gamma}, \ 0 \le \alpha + \beta + \gamma \le n$$

then, being  $g(x, y) = \sqrt{1 - x^2 - y^2}$ , we have to integrate over the elliptical sectors

$$f(x, y, g(x, y))/g(x, y) = x^{\alpha} y^{\beta} (1 - x^2 - y^2)^{\gamma/2 - 1/2}.$$
 (2)

We have two distinct situations:

- if  $\gamma$  is odd, then f(x, y, g)/g is a polynomial in (x, y) of degree at most n-1, namely  $f(x, y, g)/g \in \mathbb{P}^2_{\alpha+\beta+\gamma-1} \subseteq \mathbb{P}^2_{n-1}$ ;
- if  $\gamma$  is even (including  $\gamma = 0$ ), since  $g \ge 0$ , then  $g^{\gamma} = (g^2)^{\gamma/2}$  is a polynomial of degree  $\gamma$  and  $f(x, y, g)/g \in \frac{1}{g} \mathbb{P}^2_{\alpha+\beta+\gamma} \subseteq \frac{1}{g} \mathbb{P}^2_n$ .

If  $\gamma$  is even, let  $p_{\varepsilon}(x, y)$  be a polynomial of degree  $m = m(\varepsilon)$  such that

$$|p_{\varepsilon}(x,y) - 1/g(x,y)| \leq \varepsilon (1/|g(x,y)|)$$

for any  $(x, y) \in \Omega$ .

Then  $fp_{\varepsilon} \in \mathbb{P}^2_{n+m}$  approximates f/g with a relative error at most  $\varepsilon$ .

For computing  $m = m(\varepsilon)$ , recalling that  $g(x, y) = \sqrt{1 - (x^2 + y^2)}$ and that  $\hat{A}$ ,  $\hat{B}$ ,  $\hat{C}$  are the vertices projected on the equator, since

$$0 \leq x^2 + y^2 \leq 
ho = \max\left\{\|\hat{A}\|_2^2, \|\hat{B}\|_2^2, \|\hat{C}\|_2^2
ight\} < 1\,, \ \ (x,y) \in \Omega\,\,,$$
 (3)

it is sufficient to find the degree of a (near) optimal univariate polynomial approximation (up to  $\varepsilon$ ) to the function  $1/\sqrt{1-t}$  for  $t \in [0, \rho]$ .

# Spherical triangles

This can be done efficiently by Chebfun and list the results on a file.



Figure: The degree  $m(\varepsilon)$  as a function of  $\rho$ .

# Spherical triangles

At this point we are able to solve the problem if we can compute a formula on each curvilinear polygon  $S_1$ ,  $S_2$ ,  $S_3$  such that  $\Omega = S_1 \cup S_2 \cup S_3$ .

This is possible since, for any degree of exactness there are algebraic cubature formulas on circular and elliptical sectors with positive weights and internal nodes, that have been constructed by means of arc blending and *subperiodic* trigonometric gaussian quadrature.

This entails that

- a quadrature formula with nodes  $\{(x_j, y_j)\}$  and positive weights  $\{w_j\}$  of exactness degree n + m on  $\Omega$  will be nearly exact for f(x, y, g)/g if  $f \in \mathbb{P}_n(\mathbb{S}_2)$ ,
- **2** a formula with nodes  $\{(x_j, y_j, g(x_j, y_j))\}$  and weights  $\{w_j/g(x_j, y_j)\}$  will be near-algebraic (nearly exact) in  $\mathbb{P}_n(\mathcal{T})$ , i.e. for spherical polynomials restricted to the spherical triangle  $\mathcal{T}$ .

We observe that in general, due to the fact that for determining a rule with degree of exatness n on the triangle  $\mathcal{T}$ , then one must compute one of degree m + n in the curvilinear polygon  $\Omega$ , the cardinality of the rule, with positive weights and internal nodes, may be pretty high.

There is an algorithm, named Caratheodory-Tchakaloff compression, that allows to compress the rule, so that the resulting one has

- degree of precision n + m,
- positive weights,
- a number of nodes at most  $(n + m + 1)^2$ , that is the dimension of the space  $P_{n+m}(\mathcal{T})$ , taken from those of the initial rule.

Summarizing, given a spherical polygon and a degree of precision n we proceed as follows

- compute the centroid of the vertices A, B, C and rotate it to the North-Pole by the rotation R;
- rotate by *R* all the vertices obtaining  $A_{NP}$ ,  $B_{NP}$ ,  $C_{NP}$ ;
- project A<sub>NP</sub>, B<sub>NP</sub>, C<sub>NP</sub> on the equator obtaining a planar figure that is union of 3 elliptical sectors;
- determine the degree  $m = m(\epsilon)$  after some considerations;
- compute a quadrature rule of degree m + n on each elliptical sector (not easy but can be done!);
- project back the nodes to the spherical triangle at North-Pole and modify weights taking into account the jacobian;
- rotate back nodes to the original spherical polygon, via  $R^{-1}$ .

As numerical examples, we consider a sphere octant, i.e. the spherical triangle  $\mathcal{T}$  with vertices (1, 0, 0), (0, 1, 0), (0, 0, 1). For this domain, the parameter  $\rho$  ruling the degree  $m = m(\epsilon)$ ,  $\epsilon$  being the machine precision, is approximatively 0.67.

By a previous figure, this means that  $m \approx 60$ .

As test functions we take

1 
$$f_1(x, y, z) = 1 + x + y^2 + x^2y + x^4 + y^5 + x^2y^2z^2;$$

2 
$$f_2(x, y, z) = \cos(10(x + y + z));$$

3 
$$f_3(x, y, z) = franke(x, y, z);$$

4 
$$f_4(x, y, z) = (1 + \tanh(9x - 9y + 9z))/9;$$

5 
$$f_5(x, y, z) = (1 + \operatorname{sign}(9x - 9y + 9z))/9.$$

We have compute the reference values of these integrals by adaptive codes and reported in the table below the numerical results.

n	$E_{\rm compr}(f_1)$	$E_{\rm compr}(f_2)$	$E_{\rm compr}(f_3)$	$E_{\rm compr}(f_4)$	$E_{\rm compr}(f_5)$
5	1e-05	4e-03	бе-02	3e-02	9e-02
10	2e-15	3e-06	9e-04	2e-03	1e-02
15	2e-15	2e-11	2e-04	1e-04	3e-03
20	1e-15	5e-15	2e-05	4e-04	6e-03
25	1e-15	3e-15	2e-07	5e-05	5e-03
30	6e-16	4e-15	4e-08	3e-05	2e-03

Table: Relative errors in the integration of the test functions defined above by the compressed formula, on the sphere octant T with vertices (1, 0, 0), (0, 1, 0), (0, 0, 1).

# Spherical triangles

Finally we report some statistics concerning the *basic* formula and its compressed version, applying *Caratheodory-Tchakaloff algorithm*.

n	# basic	# compr	Cratio	CPU	$E_{\text{basic}}(SPH)$	$E_{\rm compr}(SPH)$
5	5580	36	155:1	0.02s	3e-15	4e-15
10	6435	121	53:1	0.1s	1e-14	1e-14
15	7560	256	30:1	0.3s	1e-14	3e-14
20	8550	441	19:1	1s	1e-14	5e-14
25	9840	676	15:1	4s	1e-14	6e-14
30	10965	961	11:1	19s	3e-14	1e-13

Table: Cardinalities, compression ratio, CPU time in seconds and average relative errors on Spherical Harmonics (SPH) for the spherical octant T with vertices (1,0,0), (0,1,0), (0,0,1).

#### Remark

As for the numerical software, see the Matlab package Software package for polynomial interpolation on spherical triangles.

Let

$$\mathcal{R} = [a_1, b_1] \times [a_2, b_2] \subseteq [0, \pi] \times [0, 2\pi]$$

be a rectangle, and define as spherical rectangle  $\Omega_{\mathcal{R}}$  (sometimes also known as geographical rectangle) the subdomain of the sphere  $\mathbb{S}^2$  whose points are of the form

$$P = \xi(\theta, \phi) := (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta), \ (\theta, \phi) \in \mathcal{R}.$$

We observe that depending on  $\mathcal{R}$ , several well-known subdomains  $\Omega_{\mathcal{R}} = \xi(\mathcal{R})$  of the 2-sphere can be defined in this way, as

- caps,
- collars,
- slices,
- more generally spherical rectangles defined by longitudes and latitudes.



Figure: Above and below: a spherical rectangle  $\xi([\pi/6, \pi/3] \times [0, \pi/2])$  and a spherical cap  $\xi([0, \pi/3] \times [0, 2\pi])$ .



Figure: Above and below: a (spherical) collar  $\xi([\pi/6, \pi/3] \times [0, 2\pi])$  and a (spherical) slice  $\xi([0, \pi] \times [0, \pi/3])$ .

### To this purpose, we introduce the following result

#### Theorem

# Let $w \in [0, \pi]$ .

- $\mathbb{T}_n([-\omega, \omega]) = span\{1, \cos(k\theta), \sin(k\theta)\}, 1 \le k \le n, \theta \in [-\omega, \omega],$
- $w: [-\omega, \omega] \to \mathbb{R}$  be a symmetric weight function,
- {ξ<sub>j</sub>}<sub>j=1,...,n+1</sub>, {λ<sub>j</sub>}<sub>j=1,...,n+1</sub> be respectively the nodes and the weights of an algebraic gaussian rule relatively to the symmetric weight function

$$\widetilde{s}(x) = w(2 \arcsin\left(\sin\left(\omega/2\right)x\right)) \frac{2\sin\left(\omega/2\right)}{\sqrt{1-\sin^2(\omega/2) x^2}}, x \in (-1, 1).$$

Then

$$\int_{-\omega}^{\omega} f(\theta) w(\theta) \, d\theta = \sum_{j=1}^{n+1} \lambda_j f(\theta_j), f \in \mathbb{T}_n([-\omega, \omega]) \tag{4}$$

where  $\theta_j = 2 \arcsin (\sin (\omega/2)\xi_j) \in (-\omega, \omega), \ j = 1, \dots, n+1.$ 

This theorem says that if we intend to integrate

 $\int_{\omega}^{\omega} f(\theta) w(\theta) d\theta$ 

where f is a trigonometric polynomial of degree n and w a symmetric weight function in  $[-\omega, \omega]$  then it is sufficient to

compute the nodes {ξ<sub>j</sub>} and weights {λ<sub>j</sub>} of a gaussian rule with *n* nodes w.r.t. a certain weight function s̃;

modify the nodes  $\{\xi_j\}$  into  $\{\theta_j\}$  by a simple transformation. The hidden difficulty is that the computation of these formula is not trivial since the weight function is a little unusual.

We provide an algebraic rule over the spherical rectangle  $\Omega_{\mathcal{R}} \subseteq \mathbb{S}^2$ .

Theorem

#### Let

- $\Omega_R$  a spherical rectangle, where  $\mathcal{R} = [a_1, b_1] \times [a_2, b_2] \subseteq [0, \pi] \times [0, 2\pi];$
- $\{\theta_k^{[a_1,b_1]}\}_{k=1,...,n+2}$  and  $\{\lambda_k^{[a_1,b_1]}\}_{k=1,...,n+2}$  be the nodes and the weights of a gaussian subperiodic trigonometric rule on  $[a_1, b_1]$  w.r.t. w(x) = 1, having trigonometric degree of precision n + 1;
- $\{\theta_k^{[a_2,b_2]}\}_{k=1,...,n+1}$  and  $\{\lambda_k^{[a_2,b_2]}\}_{k=1,...,n+1}$  be the nodes and the weights of a gaussian subperiodic trigonometric rule on  $[a_2, b_2]$  w.r.t. w(x) = 1, having trigonometric degree of precision n.

Then the tensorial cubature rule

$$S_n(f) = \sum_{j_1=1}^{n+2} \sum_{j_2=1}^{n+1} \lambda_{j_1,j_2} f(\xi_{j_1,j_2})$$

$$\xi_{\dot{h},\dot{h}_2} = \xi(\theta_{\dot{h}}^{[a_1,b_1]},\theta_{\dot{h}_2}^{[a_2,b_2]}), \quad \lambda_{\dot{h},\dot{h}_2} = \lambda_{\dot{h}}\lambda_{\dot{h}_2}\sin(\theta_{\dot{h}}^{[a_1,b_1]})$$

integrates exactly in  $\Omega_{\mathcal{R}}$  every algebraic polynomial of total degree n.

The previous theorem seems complicated, but it comes directly from observing that similarly to the tensorial rule on the unit sphere

$$I_{\Omega_{\mathcal{R}}} = \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(\cos\theta_2 \sin\theta_1, \sin\theta_2 \sin\theta_1, \cos\theta_1) \sin\theta_1 \, d\theta_1 d\theta_2.$$

and thus if f is a polynomial of total degree n then the integrand on the r.h.s. is

- **a** trigonometric polynomial of degree n + 1 in the variable  $\theta_1$ ;
- a trigonometric polynomial of degree *n* in the variable  $\theta_2$ ; and thus to provide a tensorial rule of the desired degree we need
  - a formula with trigonometric degree n + 1 in the variable θ<sub>1</sub> (ranging in [a<sub>1</sub>, b<sub>1</sub>]);
  - a formula with trigonometric degree n in the variable θ<sub>2</sub> (ranging in [a<sub>1</sub>, b<sub>1</sub>]);

The rest is the determination of a tensorial rule by means of univariate rules to obtain the desired formula.

# Remark (Caps)

The cardinality of these rules is  $\approx n^2$ , where n is the degree of precision. With some tricks, one can have a formula on the spherical cap with  $\approx n^2/2$  points.

## Remark (Software)

Though at first sight the result of the theorem is a little complicated, in practice when one provides the nodes and the weights of the subperiodic formula (not easy!), everything become simpler.

As for the numerical software, see the Matlab package Cubature rules on spherical rectangles.

As numerical tests, we consider the cubature of the functions

$$f_{1}(\mathbf{x}) = \exp(-x^{2} - 100 y^{2} - 0.5 z^{2}),$$
  

$$f_{2}(\mathbf{x}) = \sin(-x^{2} - 100 y^{2} - 0.5 z^{2}),$$
  

$$f_{3}(\mathbf{x}) = \max(1/4 - ((x - 1/\sqrt{5})^{2} + (y - 2/\sqrt{5})^{2} + (z - 2/\sqrt{5})^{2}), 0))^{3}$$

on the spherical rectangle

$$\Omega_{\mathcal{R}} = \xi(\mathcal{R}), \ \mathcal{R} = [\pi/6, \, \pi/3] \times [0, \, \pi/2].$$



Figure: The spherical rectangle  $\xi([\pi/6, \pi/3] \times [0, \pi/2])$  and the nodes of the rule of degree of exactness 10. 44/46

Deg.	<i>f</i> 1	f2	fз
5	3.34 <i>e</i> - 04	7.38 <i>e</i> – 02	4.53 <i>e</i> – 06
10	4.89 <i>e</i> – 06	2.69 <i>e</i> – 02	5.44 <i>e</i> - 07
15	9.12 <i>e</i> – 09	5.14 <i>e</i> - 03	4.07 <i>e</i> - 08
20	1.76 <i>e -</i> 10	1.13 <i>e</i> — 02	2.43 <i>e</i> - 08
25	7.73 <i>e</i> – 14	1.13 <i>e</i> – 02	9.53 <i>e</i> – 09
30	3.33 <i>e</i> - 16	1.23 <i>e</i> – 03	2.23 <i>e</i> – 09
35	3.47e – 17	2.58 <i>e</i> – 05	2.33 <i>e</i> – 09
40	1.14 <i>e</i> - 16	1.96 <i>e</i> – 07	2.82 <i>e</i> - 10
45	3.47e – 17	6.94 <i>e</i> - 10	8.84 <i>e</i> - 10
50	2.08 <i>e</i> - 17	1.33 <i>e –</i> 12	5.48 <i>e</i> - 11

Table: Absolute errors for degrees 5, 10, ..., 50, w.r.t. the integrals on the spherical rectangle  $\xi([\pi/6, \pi/3] \times [0, \pi/2])$  on the test functions  $f_1$ ,  $f_2$ ,  $f_3$ .

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