Efficient Spherical Designs with Good Geometric Properties

Robert S. Womersley

Dedicated to Ian H. Sloan on the occasion of his 80th birthday in acknowledgement of his many fruitful ideas and generosity.

Abstract Spherical t-designs on $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ provide N nodes for an equal weight numerical integration rule which is exact for all spherical polynomials of degree at most t. This paper considers the generation of efficient, where N is comparable to $(1+t)^d/d$, spherical t-designs with good geometric properties as measured by their mesh ratio, the ratio of the covering radius to the packing radius. Results for $\mathbb{S}^2$ include computed spherical t-designs for $t = 1, ..., 180$ and symmetric (antipodal) t-designs for degrees up to 325, all with low mesh ratios. These point sets provide excellent points for numerical integration on the sphere. The methods can also be used to computationally explore spherical t-designs for $d = 3$ and higher.

1 Introduction

Consider the d-dimensional unit sphere

$$\mathbb{S}^d = \left\{ \boldsymbol{x} \in \mathbb{R}^{d+1} : |\boldsymbol{x}| = 1 \right\}$$

where the standard Euclidean inner product is $\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{i=1}^{d+1} x_i y_i$ and $|\boldsymbol{x}|^2 = \boldsymbol{x} \cdot \boldsymbol{x}$.

A numerical integration (quadrature) rule for $\mathbb{S}^d$ is a set of N points $\boldsymbol{x}_j \in \mathbb{S}^d$, $j = 1, \ldots, N$ and associated weights $w_j > 0$, $j = 1, \ldots, N$ such that

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$$Q_N(f) := \sum_{j=1}^{N} w_j f(\boldsymbol{x}_j) \approx I(f) := \int_{\mathbb{S}^d} f(\boldsymbol{x}) d\sigma_d(\boldsymbol{x}). \tag{1}$$

Here $\sigma_d(\boldsymbol{x})$ is the normalised Lebesgue measure on $\mathbb{S}^d$ with surface area

$$\omega_d := \frac{2\pi^{(d+1)/2}}{\Gamma((d+1)/2)},$$

where $\Gamma(\cdot)$ is the gamma function.

Let $\mathbb{P}_t(\mathbb{S}^d)$ denote the set of all spherical polynomials on $\mathbb{S}^d$ of degree at most t. A *spherical t-design* is a set of N points $X_N = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$ on $\mathbb{S}^d$ such that equal weight quadrature using these nodes is exact for all spherical polynomials of degree at most t, that is

$$\frac{1}{N} \sum_{j=1}^{N} p(\boldsymbol{x}_j) = \int_{\mathbb{S}^d} p(\boldsymbol{x}) \mathrm{d}\sigma_d(\boldsymbol{x}), \quad \forall p \in \mathbb{P}_t(\mathbb{S}^d). \tag{2}$$

Spherical t-designs were introduced by Delsarte, Goethals and Seidel [24] who provided several characterizations and established lower bounds on the number of points N required for a spherical t-design. Seymour and Zaslavsky[55] showed that spherical t-designs exist on $\mathbb{S}^d$ for all N sufficiently large. Bondarenko, Radchenko and Viazovska [8] established that there exists a C_d such that spherical t-designs on $\mathbb{S}^d$ exist for all $N \geq C_d\, t^d$, which is the optimal order. The papers [21, 5, 20] provide a sample of many on spherical designs and algebraic combinatorics on spheres.

An alternative approach, not investigated in this paper, is to relax the condition $w_j = 1/N$ that the quadrature weights are equal so that $|w_j/(1/N) - 1| \leq \varepsilon$ for $j = 1, \ldots, N$ and $0 \leq \varepsilon < 1$, but keeping the condition that the quadrature rule is exact for polynomials of degree t (see [57, 69] for example).

The aim of this paper is not to find spherical t-designs with the minimal number of points, nor to provide proofs that a particular configuration is a spherical t-design. Rather the aim is to find sequences of point sets which are at least computationally spherical t-designs, have a low number of points and are geometrically well-distributed on the sphere. Such point sets provide excellent nodes for numerical integration on the sphere, as well as hyperinterpolation [56, 40, 59] and fully discrete needlet approximation [65]. These methods have a requirement that the quadrature rules are exact for certain degree polynomials. More generally, [41] provides a summary of numerical integration on $\mathbb{S}^2$ with geomathematical applications in mind.

1.1 Spherical Harmonics and Jacobi Polynomials

A *spherical harmonic* of degree ℓ on $\mathbb{S}^d$ is the restriction to $\mathbb{S}^d$ of a homogeneous and harmonic polynomial of total degree ℓ defined on $\mathbb{R}^{d+1}$. Let $\mathbb{H}_\ell$ denote the set of all spherical harmonics of exact degree ℓ on $\mathbb{S}^d$. The dimension of the linear space $\mathbb{H}_\ell$ is

$$Z(d,\ell) := (2\ell+d-1)\frac{\Gamma(\ell+d-1)}{\Gamma(d)\Gamma(\ell+1)} \asymp (\ell+1)^{d-1}, \tag{3}$$

where $a_\ell \asymp b_\ell$ means $c\,b_\ell \le a_\ell \le c'\,b_\ell$ for some positive constants c, c', and the asymptotic estimate uses [26, Eq. 5.11.12].

Each pair $\mathbb{H}_\ell$, $\mathbb{H}_{\ell'}$ for $\ell \neq \ell' \ge 0$ is $\mathbb{L}_2$-orthogonal, $\mathbb{P}_L(\mathbb{S}^d) = \bigoplus_{\ell=0}^{L} \mathbb{H}_\ell$ and the infinite direct sum $\bigoplus_{\ell=0}^{\infty} \mathbb{H}_\ell$ is dense in $\mathbb{L}_p(\mathbb{S}^d)$, $p \ge 2$, see e.g. [64, Ch.1]. The linear span of $\mathbb{H}_\ell$, $\ell = 0,1,\dots,L$, forms the space $\mathbb{P}_L(\mathbb{S}^d)$ of spherical polynomials of degree at most L. The dimension of $\mathbb{P}_L(\mathbb{S}^d)$ is

$$D(d,L) := \dim \mathbb{P}_L(\mathbb{S}^d) = \sum_{\ell=0}^{L} Z(d,\ell) = Z(d+1,L). \tag{4}$$

Let $P_\ell^{(\alpha,\beta)}(z)$, $-1 \le z \le 1$, be the Jacobi polynomial of degree ℓ for $\alpha,\beta > -1$. The Jacobi polynomials form an orthogonal polynomial system with respect to the Jacobi weight $w_{\alpha,\beta}(z) := (1-z)^\alpha(1+z)^\beta$, $-1 \le z \le 1$. We denote the normalised Legendre (or ultraspherical/Gegenbauer) polynomials by

$$P_\ell^{(d+1)}(z) := \frac{P_\ell^{\left(\frac{d-2}{2},\frac{d-2}{2}\right)}(z)}{P_\ell^{\left(\frac{d-2}{2},\frac{d-2}{2}\right)}(1)},$$

where, from [61, (4.1.1)],

$$P_\ell^{(\alpha,\beta)}(1) = \frac{\Gamma(\ell+\alpha+1)}{\Gamma(\ell+1)\Gamma(\alpha+1)}, \tag{5}$$

and [61, Theorem 7.32.2, p. 168],

$$\left|P_\ell^{(d+1)}(z)\right| \le 1, \qquad -1 \le z \le 1. \tag{6}$$

The derivative of the Jacobi polynomial satisfies [61]

$$\frac{\mathrm{d}\,P_\ell^{(\alpha,\beta)}(z)}{\mathrm{d}\,z} = \frac{\ell+\alpha+\beta+1}{2}\,P_{\ell-1}^{(\alpha+1,\beta+1)}(z), \tag{7}$$

so

$$\frac{\mathrm{d}\,P_\ell^{(d+1)}(z)}{\mathrm{d}\,z} = \frac{(\ell+d-1)(\ell+d/2)}{d} P_{\ell-1}^{(d+3)}(z). \tag{8}$$

Also if ℓ is odd then the polynomials $P_\ell^{(d+1)}$ are odd and if ℓ is even the polynomials $P_\ell^{(d+1)}$ are even.

A *zonal function* $K : \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{R}$ depends only on the inner product of the arguments, i.e. $K(\boldsymbol{x},\boldsymbol{y}) = \mathfrak{K}(\boldsymbol{x}\cdot\boldsymbol{y})$, $\boldsymbol{x},\boldsymbol{y} \in \mathbb{S}^d$, for some function $\mathfrak{K} : [-1,1] \to \mathbb{R}$. Frequent use is made of the zonal function $P_\ell^{(d+1)}(\boldsymbol{x}\cdot\boldsymbol{y})$.

Let $\{Y_{\ell,k} : k = 1,\ldots,Z(d,\ell),\ \ell = 0,\ldots,L\}$ be an *orthonormal basis* for $\mathbb{P}_L(\mathbb{S}^d)$. The normalised Legendre polynomial $P_\ell^{(d+1)}(\boldsymbol{x}\cdot\boldsymbol{y})$ satisfies the *addition theorem* (see [61, 64, 3] for example)

$$\sum_{k=1}^{Z(d,\ell)} Y_{\ell,k}(\boldsymbol{x})Y_{\ell,k}(\boldsymbol{y}) = Z(d,\ell)P_\ell^{(d+1)}(\boldsymbol{x}\cdot\boldsymbol{y}). \tag{9}$$

1.2 Number of Points

Delsarte, Goethals and Seidel [24] showed that an N point t-design on $\mathbb{S}^d$ has $N \geq N^*(d,t)$ where

$$N^*(d,t) := \begin{cases} 2\binom{d+k}{d} & \text{if } t = 2k+1, \\ \binom{d+k}{d} + \binom{d+k-1}{d} & \text{if } t = 2k. \end{cases} \tag{10}$$

On $\mathbb{S}^2$

$$N^*(2,t) := \begin{cases} \frac{(t+1)(t+3)}{4} & \text{if } t \text{ odd}, \\ \frac{(t+2)^2}{4} & \text{if } t \text{ even}. \end{cases} \tag{11}$$

Bannai and Damerell [6, 7] showed that *tight spherical t-designs* which achieve the lower bounds (10) cannot exist except for a few special cases (for example except for $t = 1,2,3,5$ on $\mathbb{S}^2$) .

Yudin [68] improved (except for some small values of d,t, see Table 2), the lower bounds (10), by an exponential factor $(4/e)^{d+1}$ as $t \to \infty$, so $N \geq N^+(d,t)$ where

$$N^+(d,t) := 2\frac{\int_0^1 (1-z^2)^{(d-2)/2}\,\mathrm{d}z}{\int_\gamma^1 (1-z^2)^{(d-2)/2}\,\mathrm{d}z} = \frac{\sqrt{\pi}\Gamma(d/2)/\Gamma((d+1)/2)}{\int_\gamma^1 (1-z^2)^{(d-2)/2}\,\mathrm{d}z}, \tag{12}$$

and γ is the largest zero of the derivative $\frac{dP_t^{(d+1)}(z)}{dz}$ and hence the largest zero of $P_{t-1}^{(\alpha+1,\alpha+1)}(z)$ where $\alpha = (d-2)/2$. Bounds [61, 2] on the largest zero of $P_n^{(\alpha,\alpha)}(z)$ are

$$\cos\left(\frac{j_0(\nu)}{n+\alpha+1/2}\right) \leq \gamma \leq \sqrt{\frac{(n-1)(n+2\alpha-1)}{(n+\alpha-3/2)/(n+\alpha-1/2)}}\,\cos\left(\frac{\pi}{n+1}\right), \tag{13}$$

where $j_0(\nu)$ is the first positive zero of the Bessel function $J_\nu(x)$.

Numerically there is strong evidence that spherical t-designs with $N = D(2,t) = (t+1)^2$ points exist, [18] and [17] used interval methods to *prove* existence of spherical t-designs with $N = (t+1)^2$ for all values of t up to 100, but there is no proof yet that spherical t-designs with $N \leq D(2,t)$ points exist for all degrees t. Hardin and Sloane [35],[36] provide tables of designs with modest numbers of points, exploiting icosahedral symmetry. They conjecture that for $d = 2$ spherical t-designs exist with $N = t^2/2 + o(t^2)$ for all t. The numerical experiments reported here and available from [66] strongly support this conjecture.

McLaren [46] defined efficiency E for a quadrature rule as the ratio of the number of independent functions for which the rule is exact to the number of arbitrary constants in the rule. For a spherical t-design with N points on $\mathbb{S}^d$ (and equal weights)

$$E = \frac{\dim \mathbb{P}_t(\mathbb{S}^d)}{dN} = \frac{D(d,t)}{dN}. \tag{14}$$

In these terms the aim is to find spherical t-designs with $E \geq 1$. McLaren [46] exploits symmetry (in particular octahedral and icosahedral) to seek rules with optimal efficiency. The aim here is not to maximise efficiency by finding the minimal number of points for a t-design on $\mathbb{S}^d$, but rather a sequence of *efficient* t-designs with $N \asymp \frac{D(d,t)}{d} \asymp \frac{(1+t)^d}{d}$. Such efficient t-designs provide a practical tool for numerical integration and approximation.

1.3 Geometric Quality

The Geodesic distance between two points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{S}^d$ is

$$\operatorname{dist}(\boldsymbol{x},\boldsymbol{y}) = \cos^{-1}(\boldsymbol{x}\cdot\boldsymbol{y}),$$

while the Euclidean distance is

$$|\boldsymbol{x}-\boldsymbol{y}| = \sqrt{2(1-\boldsymbol{x}\cdot\boldsymbol{y})} = 2\sin(\operatorname{dist}(\boldsymbol{x},\boldsymbol{y})/2).$$

The *spherical cap* with centre $\boldsymbol{z} \in \mathbb{S}^d$ and radius $\eta \in [0,\pi]$ is

$$\mathscr{C}(\boldsymbol{z};\eta) = \left\{\boldsymbol{x} \in \mathbb{S}^d : \operatorname{dist}(\boldsymbol{x},\boldsymbol{z}) \leq \eta\right\}.$$

The *separation distance*

$$\delta(X_N) = \min_{i \neq j} \operatorname{dist}(\boldsymbol{x}_i, \boldsymbol{x}_j)$$

is twice the packing radius for spherical caps of the same radius and centers in X_N. The best packing problem (or Tammes problem) has a long history [21], starting with [62, 53]. A sequence of point sets $\{X_N\}$ with $N \to \infty$ has the optimal order separation if there exists a constant c_d^{pck} independent of N such that

$$\delta(X_N) \geq c_d^{\text{pck}}\, N^{-1/d}.$$

The separation, and all the zonal functions considered in subsequent sections, are determined by the set of inner products

$$\mathscr{A}(X_N) := \left\{ \boldsymbol{x}_i \cdot \boldsymbol{x}_j, i = 1, \ldots, N, j = i+1, \ldots, N \right\} \tag{15}$$

which has been widely used in the study of spherical codes, see [21] for example. Then

$$\max_{z \in \mathscr{A}(X_N)} z = \cos(\delta(X_N)).$$

Point sets are only considered different if the corresponding sets (15) differ, as they are invariant under an orthogonal transformation (rotation) of the point set and permutation (relabelling) of the points.

The *mesh norm* (or fill radius)

$$h(X_N) = \max_{\boldsymbol{x} \in \mathbb{S}^d} \min_{j=1,\ldots,N} \operatorname{dist}(\boldsymbol{x}, \boldsymbol{x}_j)$$

gives the *covering radius* for covering the sphere with spherical caps of the same radius and centers in X_N. A sequence of point sets $\{X_N\}$ with $N \to \infty$ has the optimal order covering if there exists a constant c_d^{cov} independent of N such that

$$h(X_N) \leq c_d^{\text{cov}} N^{-1/d}.$$

The *mesh ratio* is

$$\rho(X_N) = \frac{2h_{X_N}}{\delta_{X_N}} \geq 1.$$

A common assumption in numerical methods is that the mesh ratio is uniformly bounded, that is the point sets are *quasi-uniform*. Minimal Riesz s-energy and best packing points can also produce quasi-uniform point sets [23, 34, 10].

Yudin [67] showed that a spherical t-design with N points has a covering radius of the optimal order $1/t$. Reimer extended this to quadrature rules exact for polynomials of degree t with positive weights. Thus a spherical t-design with $N = O(t^d)$ points provides an optimal order covering.

The union of two spherical t-designs with N points is a spherical t-design with $2N$ points. A spherical design with arbitrarily small separation can be obtained as one N point set is rotated relative to the other. Thus an assumption on the separation of the points of a spherical design is used to derive results, see [38] for example. This simple argument is not possible if N is less than twice a lower bound (10) or (12) on the number of points in a spherical t-design.

Bondarenko, Radchenko and Viazovska [9] have shown that on $\mathbb{S}^d$ well-separated spherical t-designs exist for $N \geq c_d' \, t^d$. This combined with Yudin's result on the covering radius of spherical designs mean that there exist spherical t-designs with $N = O(t^d)$ points and uniformly bounded mesh ratio.

There are many other "geometric" properties that could be used, for example the spherical cap discrepancy, see [32] for example, (using normalised surface measure so $|\mathbb{S}^d| = 1$)

$$\sup_{\boldsymbol{x} \in \mathbb{S}^d, \eta \in [0,\pi]} \left| |\mathscr{C}(\boldsymbol{x}, \eta)| - \frac{|X_N \cap \mathscr{C}(\boldsymbol{x}, \eta)|}{N} \right|,$$

or a Riesz s-energy, see [12] for example,

$$E_s(X_N) = \sum_{1\le i<j\le N} \frac{1}{|\boldsymbol{x}_i - \boldsymbol{x}_j|^s}.$$

In distinguishing between spherical t-designs with the same number N of points we prefer those with lower mesh ratio. Note that some authors, see [34, 10] for example, define the mesh ratio as $\tilde{\rho}(X_N) = h(X_N)/\delta(X_N) \ge 1/2$.

2 Variational Characterizations

Delsarte, Goethals and Seidel [24] showed that $X_N = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\} \subset \mathbb{S}^d$ is a spherical t-design if and only if the Weyl sums satisfy

$$r_{\ell,k}(X_N) := \sum_{j=1}^{N} Y_{\ell,k}(\boldsymbol{x}_j) = 0 \qquad k = 1,\ldots,Z(d,\ell), \quad \ell = 1,\ldots,t, \tag{16}$$

as the integral of all spherical harmonics of degree $\ell \ge 1$ is zero from orthogonality with the constant ($\ell = 0$) polynomial $Y_{0,1} = 1$ which is not included.

In matrix form

$$\boldsymbol{r}(X_N) := \overline{\boldsymbol{Y}}\boldsymbol{e} = \boldsymbol{0}$$

where $\boldsymbol{e} = (1,\ldots,1)^T \in \mathbb{R}^N$ and $\overline{\boldsymbol{Y}} \in \mathbb{R}^{D(d,t)-1\times N}$ is the spherical harmonic basis matrix excluding the first row.

Let $\psi_t : [-1,1] \to \mathbb{R}$ be a polynomial of degree $t \ge 1$ with

$$\psi_t(z) = \sum_{\ell=1}^{t} a_\ell P_\ell^{(d+1)}(z), \qquad a_\ell > 0 \text{ for } \ell = 1,\ldots,t, \tag{17}$$

so the generalised Legendre coefficients a_ℓ for degrees $\ell = 1,\ldots,t$ are all strictly positive. Clearly any such function ψ_t can be scaled by an arbitrary positive constant without changing these properties.

Consider now an arbitrary set X_N of N points on $\mathbb{S}^d$. Sloan and Womersley [58] considered the variational form

$$V_{t,N,\psi}(X_N) := \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \psi_t(\boldsymbol{x}_i \cdot \boldsymbol{x}_j)$$

which from (6) satisfies

$$0 \le V_{t,N,\psi}(X_N) \le \sum_{\ell=1}^{t} a_\ell = \psi_t(1).$$

Moreover the average value is

$$\overline{V}_{t,N,\psi} := \int_{\mathbb{S}^d} \cdots \int_{\mathbb{S}^d} V_{t,N,\psi}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N) d\sigma_d(\boldsymbol{x}_1)\cdots d\sigma_d(\boldsymbol{x}_N) = \frac{\psi_t(1)}{N}.$$

As the upper bound and average of $V_{t,N,\psi}(X_N)$ depend on $\psi_t(1)$, we concentrate on functions ψ for which $\psi_t(1)$ does not grow rapidly with t.

From the addition theorem (9), $V_{t,N,\psi}(X_N)$ is a weighted sum of squares with strictly positive coefficients

$$V_{t,N,\psi}(X_N) = \frac{1}{N^2} \sum_{\ell=1}^{t} \frac{a_\ell}{Z(d,\ell)} \sum_{k=1}^{Z(d,\ell)} \left(r_{\ell,k}(X_N)\right)^2 = \frac{1}{N^2} \boldsymbol{r}(X_N)^T \, \boldsymbol{D}\, \boldsymbol{r}(X_N), \tag{18}$$

where $\boldsymbol{D}$ is the diagonal matrix with strictly positive diagonal elements $\frac{a_\ell}{Z(d,\ell)}$ for $k = 1,\ldots,Z(d,\ell), \ell = 1,\ldots,t$. Thus, from (16), X_N is a spherical t-design if and only if

$$V_{t,N,\psi}(X_N) = 0.$$

Moreover, if the *global* minimum of $V_{t,N,\psi}(X_N) > 0$ then there are no spherical t-designs on $\mathbb{S}^d$ with N points.

Given a polynomial $\widehat{\psi}_t(z)$ of degree t and strictly positive Legendre coefficients, the zero order term may need to be removed to get $\psi_t(z) = \widehat{\psi}_t(z) - a_0$ where for $\mathbb{S}^d$ and $\alpha = (d-2)/2$,

$$a_0 = \int_{-1}^{1} \widehat{\psi}_t(z) \left(1-z^2\right)^{\alpha} dz.$$

Three examples of polynomials on $[-1,1]$ with strictly positive Legendre coefficients for $\mathbb{S}^d$ and zero constant term, with $\alpha = (d-2)/2$ are:

Example 1

$$\psi_{1,t}(z) = z^{t-1} + z^t - a_0 \tag{19}$$

where

$$a_0 = \frac{\Gamma(\alpha+3/2)}{\sqrt{\pi}} \begin{cases} \frac{\Gamma(t/2)}{\Gamma(\alpha+1+t/2)} & t \text{ odd}, \\ \frac{\Gamma((t+1)/2)}{\Gamma(\alpha+3/2+t/2)} & t \text{ even}. \end{cases} \tag{20}$$

For $d = 2$ this simplifies to $a_0 = 1/t$ if t is odd and $a_0 = 1/(t+1)$ if t is even. This function was used by Grabner and Tichy [32] for symmetric point sets where only even values of t need to be considered, as all odd degree polynomials are integrated exactly.

Example 2

$$\psi_{2,t}(z) = \left(\frac{1+z}{2}\right)^t - a_0 \tag{21}$$

where

$$a_0 = \frac{2}{\sqrt{\pi}} 4^{\alpha} \Gamma(\alpha+3/2) \frac{\Gamma(\alpha+1+t)}{\Gamma(2\alpha+2+t)}. \tag{22}$$

For $d = 2$ this simplifies to $a_0 = 1/(1+t)$. This is a scaled version of the function $(1+z)^t$ used by Cohn and Kumar [20] for which a_0 must be scaled by 2^t producing more cancellation errors for large t.

Example 3

$$\psi_{3,t}(z) = P_t^{(\alpha+1,\alpha)}(z) - a_0 \tag{23}$$

where a_0 is given by (22). The expansion in terms of Jacobi polynomials in Szegő [61, Section 4.5] gives

$$\sum_{\ell=0}^{t} Z(d,\ell) P_\ell^{(d+1)}(z) = \frac{1}{a_0} P_t^{(\alpha+1,\alpha)}(z).$$

For S^2 this is equivalent to

$$\sum_{\ell=1}^{t} (2\ell+1) P_\ell^{(d+1)}(z) = (t+1) P_t^{(1,0)}(z) - 1$$

used in Sloan and Womersley [58].

3 Quadrature Error

The error for numerical integration depends on the smoothness of the integrand. Classical results are based on the error of best approximation of the integrand f by polynomials [51], (see also [41] for more details on $\mathbb{S}^2$). For $f \in C^\kappa(\mathbb{S}^d)$, there exists a constant $c = c(\kappa, f)$ such that the numerical integration error satisfies

$$\left| \int_{\mathbb{S}^d} f(\boldsymbol{x}) \mathrm{d}\sigma_d(\boldsymbol{x}) - \frac{1}{N} \sum_{j=1}^{N} f(\boldsymbol{x}_j) \right| \leq c\, t^{-\kappa}.$$

If $N = O(t^d)$ then the right-hand-side becomes $N^{-\kappa/d}$. Thus for functions with reasonable smoothness it pays to increase the degree of precision t.

Similar results are presented in [14], building on the work of [39, 37], for functions f in a Sobolev space $\mathbb{H}^s(\mathbb{S}^d)$, $s > d/2$. The *worst-case-error* for equal weight (quasi Monte-Carlo) numerical integration using an arbitrary point set X_N is

$$WCE(X_N, s, d) := \sup_{f \in \mathbb{H}^s(\mathbb{S}^d), ||f||_{\mathbb{H}^s(\mathbb{S}^d)} \leq 1} \left| \int_{\mathbb{S}^d} f(\boldsymbol{x}) \mathrm{d}\sigma(\boldsymbol{x}) - \frac{1}{N} \sum_{j=1}^{N} f(\boldsymbol{x}_j) \right|. \tag{24}$$

From this it immediately follows that the error for numerical integration satisfies

$$\left| \int_{\mathbb{S}^d} f(\boldsymbol{x}) \mathrm{d}\sigma_d(\boldsymbol{x}) - \frac{1}{N} \sum_{j=1}^{N} f(\boldsymbol{x}_j) \right| \leq WCE(X_N, s, d)\ \|f\|_{\mathbb{H}^s(\mathbb{S}^d)}.$$

Spherical t-designs X_N with $N = O(t^d)$ points satisfy the optimal order rate of decay of the worst case error, for any $s > d/2$, namely

$$WCE(X_N, s, d) = O\left(N^{-s/d}\right), \qquad N \to \infty.$$

Thus spherical t-designs with $N = O(t^d)$ points are ideally suited to the numerical integration of smooth functions.

4 Computational Issues

The aim is to find a spherical t-design with N points on $\mathbb{S}^d$ by finding a point set X_N achieving the global minimum of zero for the variational function $V_{t,N,\psi}(X_N)$. This section considers several computational issues: the evaluation of $V_{t,N,\psi}(X_N)$ either as a double sum or using its representation (18) as a sum of squares; the parametrisation of the point set X_N; the number of points N as a function of t and d; the choice of optimization algorithm which requires evaluation of derivatives with respect to the chosen parameters; exploiting the sum of squares structure which requires evaluating the spherical harmonics and their derivatives; and imposing structure on the point set, for example symmetric (antipodal) point sets. An underlying issue is that optimization problems with points on the sphere typically have many different local minima with different characteristics. Here we are seeking both a global minimizer with value 0 and one with good geometric properties as measured by the mesh ratio.

The calculations were performed using Matlab, on a Linux computational cluster using nodes with up to 16 cores. In all cases analytic expressions for the derivatives with respect to the chosen parametrisation were used.

4.1 Evaluating Criteria

Although the variational functions are nonnegative, there is significant cancellation between the (constant) diagonal elements $\psi_t(1)$ and all the off-diagonal elements with varying signs as

$$V_{t,N,\psi}(X_N) = \frac{1}{N}\psi_t(1) + \sum_{i=1}^{N}\sum_{\substack{j=1 \\ j\neq i}}^{N} \psi_t(\boldsymbol{x}_i \cdot \boldsymbol{x}_j).$$

Accurate calculation of such sums is difficult, see [42] for example, especially getting reproducible results on multi-core architecture with dynamic scheduling of parallel non-associative floating point operations [25]. Example 1 has $\psi_{1,t}(1) = 2$ and Example 2 has $\psi_{2,t}(1) = 1$, both independent of t, while Example 3 has

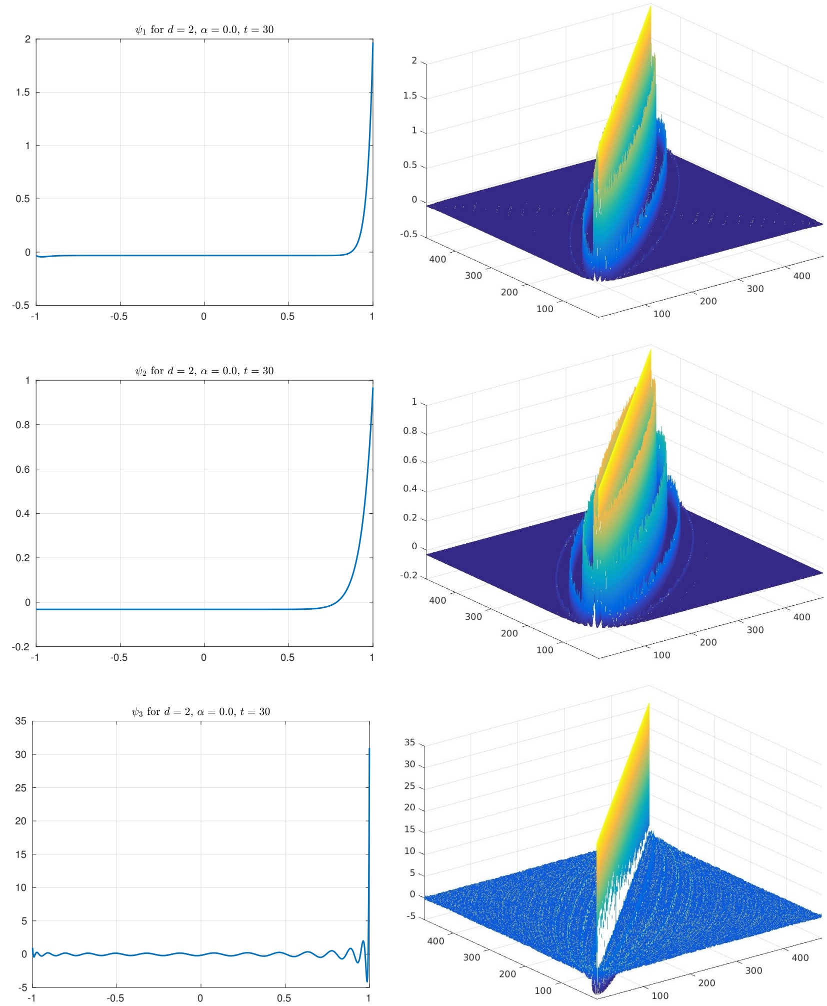

Fig. 1 For $d = 2$, $t = 30$, a spherical t-design with $N = 482$, the functions $\psi_{k,t}$ and arrays $\psi_{k,t}(\boldsymbol{x}_i \cdot \boldsymbol{x}_j)$ for $k = 1, 2, 3$.

$$\psi_{3,t}(1) = \frac{\Gamma(t+\alpha+2)}{\Gamma(t+1)\Gamma(\alpha+2)} - 1,$$

which grows with the degree t (for $d = 2$, $\psi_{3,t}(1) = t$). These functions are illustrated in Fig. 1. As the variational objectives can be scaled by an arbitrary positive constant, you could instead have used $\psi_{3,t}\frac{\Gamma(t+1)\Gamma(\alpha+2)}{\Gamma(t+\alpha+2)}$. Ratios of gamma functions, as in the expressions for a_0, should not be evaluated directly, but rather simplified for small values of d or evaluated using the log-gamma function. The derivatives, essential for large scale non-linear optimization algorithms, are readily calculated using

$$\nabla_{\boldsymbol{x}_k} V_{t,N,\psi}(X_N) = 2 \sum_{\substack{i=1 \\ i \neq k}}^{N} \psi_t'(\boldsymbol{x}_i \cdot \boldsymbol{x}_k) \boldsymbol{x}_i$$

and the Jacobian of the (normalised) spherical parametrisation (see Section 4.2).

Because of the interest in the use of spherical harmonics for the representation of the Earth's gravitational field there has been considerable work, see [43, 44] and [29, Section 7.24.2] for example, on the evaluation of high degree spherical harmonics for $\mathbb{S}^2$. For $(x,y,z)^T \in \mathbb{S}^2$ the real spherical harmonics [54, Chapter 3, Section 18] are usually expressed in terms of the coordinates $z = \cos(\theta)$ and ϕ. In terms of the coordinates $(x, \phi_2) = (\cos(\phi_1), \phi_2)$, see (28) below, they are the $Z(2,\ell) = 2\ell + 1$ functions

$$\begin{aligned} Y_{\ell,\ell+1-k}(x,\phi_2) &:= \hat{c}_{\ell,k}(1-x^2)^{k/2} S_\ell^{(k)}(x) \sin(k\phi_2), \quad k = 1,\ldots,\ell, \\ Y_{\ell,\ell+1}(x,\phi_2) &:= \hat{c}_{\ell,0} S_\ell^{(0)}(x), \\ Y_{\ell,\ell+1+k}(x,\phi_2) &:= \hat{c}_{\ell,k}(1-x^2)^{k/2} S_\ell^{(k)}(x) \cos(k\phi_2), \quad k = 1,\ldots,\ell. \end{aligned} \tag{25}$$

where $S_\ell^{(k)}(x) = \sqrt{\frac{(\ell-k)!}{(\ell+k)!}} P_\ell^k(x)$ are versions of the Schmidt semi-normalised associated Legendre functions for which stable three-term recurrences exist for high (about 2700) degrees and orders. The normalization constants $\hat{c}_{\ell,0}$, $\hat{c}_{\ell,k}$ are, for normalised surface measure,

$$\hat{c}_{\ell,0} = \sqrt{2\ell+1}, \quad \hat{c}_{\ell,k} = \sqrt{2}\sqrt{2\ell+1}, \quad k = 1,\ldots,\ell,$$

For $\mathbb{S}^2$ these expressions can be used to directly evaluate the Weyl sums (16), and hence their sum of squares, and their derivatives.

4.2 Spherical Parametrisations

There are many ways to organise a spherical parametrisation of $\mathbb{S}^d$. For $\phi_i \in [0, \pi]$ for $i = 1,\ldots,d-1$ and $\phi_d \in [0, 2\pi)$ define $\boldsymbol{x} \in \mathbb{S}^d$ by

$$x_1 = \cos(\phi_1) \tag{26}$$

$$x_i = \prod_{k=1}^{i-1} \sin(\phi_k) \cos(\phi_i), \quad i = 2,\ldots,d \tag{27}$$

$$x_{d+1} = \prod_{k=1}^{d} \sin(\phi_k) \tag{28}$$

The inverse transformation used is, for $i = 1,\ldots,d-1$

$$\phi_i = \begin{cases} 0 & \text{if } x_k = 0, \quad k = i, \ldots, d+1, \\ \cos^{-1}\left(x_i \Big/ \sqrt{\sum_{k=i}^{d+1} x_k^2} \right) & \text{otherwise;} \end{cases} \tag{29}$$

$$\phi_d = \tan^{-1}(x_{d+1}/x_d). \tag{30}$$

The last component can be calculated using the four quadrant `atan2` function and periodicity to get $\phi_d \in [0, 2\pi)$. Spherical parametrisations introduce potential singularities when $\phi_i = 0$ or $\phi_i = \pi$ for any $i = 1, \ldots, d-1$.

As all the functions considered are zonal, they are invariant under an orthogonal transformation (rotation). Thus the point sets are normalised so that the $d+1$ by N matrix $\boldsymbol{X} = [\boldsymbol{x}_1 \cdots \boldsymbol{x}_N]$ has

$$\begin{aligned} &\boldsymbol{X}_{i,j} = 0 \quad \text{for } i = j+1, \ldots, d+1, \quad j = 1, \ldots, \min(d, N) \\ &\boldsymbol{X}_{i,i} \geq 0 \quad \text{for } i = 1, \ldots, \min(d, N). \end{aligned}$$

The first normalised point is $\boldsymbol{x}_1 = \boldsymbol{e}_1 = (1, 0, \ldots, 0) \in \mathbb{R}^{d+1}$. Such a rotation can easily be calculated using the QR factorization of $\boldsymbol{X}$ combined with sign changes to the rows Q. The corresponding normalised spherical parametrisation has

$$\Phi_{i,j} = 0 \quad \text{for } i = j, \ldots, d, \quad j = 1, \ldots, \min(d, N),$$

where the jth column of Φ corresponds to the point $\boldsymbol{x}_j$, $j = 1, \ldots, N$. The optimisation variables are then $\Phi_{i,j}, i = 1, \ldots, \min(j-1, d)$, $j = 2, \ldots, N$, stored as the vector $\boldsymbol{\phi} \in \mathbb{R}^n$ where

$$n = \begin{cases} \frac{N(N-1)}{2} & \text{for } N \leq d, \\ Nd - \frac{d(d+1)}{2} & \text{for } N > d, \end{cases} \tag{31}$$

so

$$\begin{aligned} &\boldsymbol{\phi}_p = \Phi_{i,j}, \quad i = 1, \ldots, \min(j-1, d), \quad j = 2, \ldots, N, \\ &p = \begin{cases} \frac{\min(j-1,d)(\min(j-1,d)-1)}{2} + i & \text{for } j = 2, \ldots, \min(d, N) \\ \frac{d(d-1)}{2} + (j-d-1)d + i & \text{for } j = d+1, \ldots, N, \quad N > d. \end{cases} \end{aligned}$$

It is far easier to work with a spherical parametrisation with bound constraints than to impose the quadratic constraints $\boldsymbol{x}_j \cdot \boldsymbol{x}_j = 1, j = 1, \ldots, N$, especially for large N. As the optimization criteria have the effect of moving the points apart, the use of the normalised point sets reduces difficulties with singularities at the boundaries corresponding to $\Phi_{i,j} = 0$ or $\Phi_{i,j} = \pi$, $i = 1, \ldots, d-1$.

For $\mathbb{S}^2$, these normalised point sets may be rotated (the variable components reordered) using

$$Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

to get the commonly [28, 57, 66] used normalization with the first point at the north pole and the second on the prime meridian.

A symmetric (or antipodal) point set ($\boldsymbol{x} \in X_N \iff -\boldsymbol{x} \in X_N$) must have N even, so can be represented as $\boldsymbol{X} = [\overline{\boldsymbol{X}} \ -\overline{\boldsymbol{X}}]$ where the $d+1$ by $N/2$ array of points $\overline{\boldsymbol{X}}$ is normalised as above.

If only zonal function functions depending just on the inner products $\boldsymbol{x}_i \cdot \boldsymbol{x}_j$ are used then you could use the variables $\boldsymbol{Z}_{i,j} = \boldsymbol{x}_i \cdot \boldsymbol{x}_j$, so

$$\boldsymbol{Z} \in \mathbb{R}^{N \times N}, \quad \boldsymbol{Z}^T = \boldsymbol{Z}, \quad \boldsymbol{Z} \succeq 0, \quad \operatorname{diag}(\boldsymbol{Z}) = \boldsymbol{e}, \quad \operatorname{rank}(\boldsymbol{Z}) = d+1.$$

where $\boldsymbol{e} = (1,\ldots,1)^T \in \mathbb{R}^N$ and $\boldsymbol{Z} \succeq 0$ indicates $\boldsymbol{Z}$ is positive semi-definite. The major difficulties with such a parametrisation are the number $N(N-1)/2$ of variables and the rank condition. Semi-definite programming relaxations (without the rank condition) have been used to get bounds on problems involving points on the sphere (see, for example, [4]).

4.3 Degrees of Freedom for $\mathbb{S}^d$

Using a normalised spherical parametrisation of N points on $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ there are $n = Nd - d(d+1)/2$ variables (assuming $N \geq d$). The number of conditions for a t-design is

$$m = \sum_{\ell=1}^{t} Z(d,\ell) = D(d,t) - 1 = Z(d+1,t) - 1.$$

Using the simple criterion that the number of variables n is at least the number of conditions m, gives the number of points as

$$\widehat{N}(d,t) := \left\lceil \frac{1}{d}\left(Z(d+1,t) + \frac{d(d+1)}{2} - 1 \right) \right\rceil. \tag{32}$$

For $\mathbb{S}^2$ there are $n = 2N-3$ variables and $m = (t+1)^2 - 1$ conditions giving

$$\widehat{N}(2,t) := \lceil (t+1)^2)/2 \rceil + 1.$$

Grabner and Sloan [31] obtained separation results for N point spherical t-designs when $N \leq \tau\, 2N^*$ and $\tau < 1$. For $d = 2$, $\widehat{N}$ is less than twice the lower bound N^* as

$$\widehat{N}(2,t) = 2N^*(2,t) - t,$$

but the difference is only a lower order term. The values for $\widehat{N}(2,t)$, $N^*(2,t)$ and the Yudin lower bound $N^+(2,t)$ are available in Tables 2 – 10.

The idea of exploiting symmetry to reduce the number of conditions that a quadrature rule should satisfy at least goes back to Sobolev [60]. For a symmetric point set (both $\boldsymbol{x}_j, -\boldsymbol{x}_j$ in X_N) then all odd degree polynomials $Y_{\ell,k}$ or $P_\ell^{(d+1)}$ are automatically integrated exactly by an equal weight quadrature rule. Thus, for t odd, the number of conditions to be satisfied is

$$m = \sum_{\ell=1}^{(t-1)/2} Z(d,2\ell) = \frac{\Gamma(t+d)}{\Gamma(d+1)\Gamma(t)} - 1. \tag{33}$$

The number of free variables in a normalised symmetric point set $\boldsymbol{X} = [\overline{\boldsymbol{X}} \quad -\overline{\boldsymbol{X}}]$ (assuming $N/2 \geq d$) is

$$n = \left(\frac{Nd}{2} - \frac{d(d+1)}{2}\right). \tag{34}$$

Again the simple requirement that $n \geq m$ gives the number of points as

$$\overline{N}(d,t) := 2\left\lceil \frac{1}{d}\left(\frac{\Gamma(t+d)}{\Gamma(d+1)\Gamma(t)} - 1 + \frac{d(d+1)}{2}\right)\right\rceil. \tag{35}$$

For $d = 2$ this simplifies, again for t odd, to

$$\overline{N}(2,t) := 2\left\lceil \frac{t^2+t+4}{4}\right\rceil.$$

$\overline{N}(2,t)$ is slightly less than $\widehat{N}(2,t)$, comparable to twice the lower bound $N^*(2,t)$ as

$$\overline{N}(2,t) = 2N^*(2,t) - \tfrac{3}{2}t + \begin{cases} \frac{3}{2} & \text{if} \quad \bmod(t,4) = 1, \\ \frac{1}{2} & \text{if} \quad \bmod(t,4) = 3. \end{cases}$$

However $\overline{N}(2,t)$ is not less than $\tau\, 2N^*(2,t)$, $\tau < 1$, as required by Grabner and Sloan [31].

The leading term of both $\widehat{N}(d,t)$ and $\overline{N}(d,t)$ is $D(d,t)/d$, see Table 1, where $D(d,t)$ defined in (4) is the dimension of $\mathbb{P}_t(\mathbb{S}^d)$. From (14), a spherical t-design with $\widehat{N}(d,t)$ or $\overline{N}(d,t)$ points has efficiency $E \approx 1$. Also the leading term of both $\overline{N}(d,t)$ and $\widehat{N}(d,t)$ is $2^d/d$ times the leading term of the lower bound $N^*(d,t)$.

d	$N^*(d,t)$	$\overline{N}(d,t)$	$\widehat{N}(d,t)$	$D(d,t)$
2	$\frac{t^2}{4}+t+O(1)$	$\frac{t^2}{2}+\frac{t}{2}+O(1)$	$\frac{t^2}{2}+t+O(1)$	t^2+2t+1
3	$\frac{t^3}{24}+\frac{3t^2}{8}+O(t)$	$\frac{t^3}{9}+\frac{t^2}{3}+O(t)$	$\frac{t^3}{9}+\frac{t^2}{2}+O(t)$	$\frac{t^3}{3}+\frac{3t^2}{2}+O(t)$
4	$\frac{t^4}{192}+\frac{t^3}{12}+O(t^2)$	$\frac{t^4}{48}+\frac{t^3}{8}+O(t^2)$	$\frac{t^4}{48}+\frac{t^3}{6}+O(t^2)$	$\frac{t^4}{12}+\frac{2t^3}{3}+O(t^2)$
5	$\frac{t^5}{1920}+\frac{5t^4}{384}+O(t^3)$	$\frac{t^5}{300}+\frac{t^4}{30}+O(t^3)$	$\frac{t^5}{300}+\frac{t^4}{24}+O(t^3)$	$\frac{t^5}{60}+\frac{5t^4}{24}+O(t^3)$

Table 1 The lower bound $N^*(d,t)$, the number of points $\overline{N}(d,t)$ (symmetric point set) and $\widehat{N}(d,T)$ to match the number of conditions and the dimension of $\mathbb{P}_t(\mathbb{S}^d)$ for $d = 2,3,4,5$

4.4 Optimization Algorithms

As with many optimization problems on the sphere there are many distinct (not related by an orthogonal transformation or permutation) points sets giving local minima of the optimization objective. For example, Erber and Hockney [27] and Calef et al [16] studied the minimal energy problem for the sphere and the large number of stable configurations.

Gräf and Potts [33] develop optimization methods on general Riemannian manifolds, in particular $\mathbb{S}^2$, and both Newton-like and conjugate gradients methods. Using a fast method for spherical Fourier coefficients at non-equidistant points they obtain approximate spherical designs for high degrees.

While mathematically it is straight forward to conclude that if $V_{t,N,\psi}(X_N) = 0$ then X_N is a spherical t-design, deciding when a quantity is zero with the limits of standard double precision floating point arithmetic with machine precision $\varepsilon = 2.2 \times 10^{16}$ is less clear (should 10^{-14} be regarded as zero?). Extended precision libraries and packages like Maple or Mathematica can help. A point set X_N with $V_{t,N,\psi}(X_N) \approx \varepsilon$ does not give a mathematical proof that is X_N is a spherical t-design, but X_N may still be computationally useful in applications.

On the other hand showing that the global minimum if $V_{t,N,\psi}(X_N)$ is strictly positive, so no spherical t-design with N points exist, is an intrinsically hard problem problem. Semi-definite programming [63] provides an approach [50] to the global optimization of polynomial sum of squares for modest degrees.

For $d = 2$ a variety of gradient based bound constrained optimization methods, for example the limited memory algorithm [15, 47], were tried both to minimise the variational forms $V_{t,N,\psi}(X_N)$. Classically, see [48] for example, methods can exploit the sum of squares structure $\boldsymbol{r}(X_N)^T \boldsymbol{r}(X_N)$. In both cases it is important to provide derivatives of the objective with respect to the parameters. Using the normalised spherical parametrisation $\boldsymbol{\phi}$ of X_N, the Jacobian of the residual $\boldsymbol{r}(\boldsymbol{\phi})$ is $\boldsymbol{A} : \mathbb{R}^n \to \mathbb{R}^{m \times n}$ where $n = dN - d(d+1)/2$ and $m = D(d,t) - 1$

$$\boldsymbol{A}_{i,j}(\boldsymbol{\phi}) = \frac{\partial r_i(\boldsymbol{\phi})}{\partial \phi_j}, \qquad i = 1,\ldots,m, \quad j = 1,\ldots,n,$$

where $i = (\ell-1)Z(d+1,\ell-1)+k$, for $k = 1,\ldots,Z(d,\ell), \ell = 1,\ldots,t$.

For symmetric point sets with $N = \overline{N}(d,t)$ points, the number of variables n is given by (34) and the number of conditions m by (33) corresponding to even degree spherical harmonics.

The well-known structure of a nonlinear least squares problem, see [48] for example, gives, ignoring the $1/N^2$ scaling in (18),

$$f(\boldsymbol{\phi}) = \boldsymbol{r}(\boldsymbol{\phi})^T \boldsymbol{D}\, \boldsymbol{r}(\boldsymbol{\phi}), \tag{36}$$

$$\nabla f(\boldsymbol{\phi}) = 2\boldsymbol{A}(\boldsymbol{\phi})^T \boldsymbol{D}\, \boldsymbol{r}(\boldsymbol{\phi}), \tag{37}$$

$$\nabla^2 f(\boldsymbol{\phi}) = 2\boldsymbol{A}(\boldsymbol{\phi})^T \boldsymbol{D} \boldsymbol{A}(\boldsymbol{\phi}) + 2\sum_{i=1}^{m} r_i(\boldsymbol{\phi}) D_{ii} \nabla^2 r_i(\boldsymbol{\phi}). \tag{38}$$

If $\boldsymbol{\phi}^*$ has $\boldsymbol{r}(\boldsymbol{\phi}^*) = \mathbf{0}$ and $\boldsymbol{A}(\boldsymbol{\phi}^*)$ has rank n, the Hessian $\nabla^2 f(\boldsymbol{\phi}^*) = 2\boldsymbol{A}(\boldsymbol{\phi}^*)^T \boldsymbol{D}\boldsymbol{A}(\boldsymbol{\phi}^*)$ is positive definite and $\boldsymbol{\phi}^*$ is a strict global minimizer. Here this is only possible when $n = m$, for example when $d = 2$ and t is odd, see Tables 2, 3, 4, and in the symmetric case when $t \mod 4 = 3$, see Tables 5, 6, 7. For $d = 2$ the other values of t have $n = m + 1$, so there is generically a one parameter family of solutions even when the Jacobian has full rank. When $d = 3$, the choice $N = \widehat{N}(3,t)$ gives $n = m$, $n = m + 1$ or $n = m + 3$ depending on the value of t, see Table 9. Thus a Levenberg-Marquadt or trust region method, see [48] for example, in which the search direction satisfies

$$\left(\boldsymbol{A}^T \boldsymbol{D}\boldsymbol{A} + \nu \boldsymbol{I}\right) \boldsymbol{d} = \boldsymbol{A}^T \boldsymbol{D}\boldsymbol{r}$$

was used. When $n > m$ the Hessian of the variational form $V_{t,N,\psi}(X_N)$ evaluated using one of the three example functions (19), (21) or (23) will also be singular at the solution. These disadvantages could have been reduced by choosing the number of points N so that $n < m$, but then there may not be solutions with $V_{t,N,\psi}(X_N) = 0$, that is spherical t-designs may not exist for that number of points.

Many local solutions were found as well as (computationally) global solutions which differed depending on the starting point and the algorithm parameters (for example the initial Levenberg-Marquadt parameter ν, initial trust region, line search parameters etc). Even when $n = m$ there are often multiple spherical designs for same t, N, which are strict global minimisers, but have different inner product sets $\mathscr{A}(X_N)$ in (15) and different mesh ratios.

4.5 Structure of Point Sets

There are a number of issues with the spherical designs studied here.

- There is no proof that spherical t-designs on $\mathbb{S}^d$ with $N = t^d/d + O(t^{d-1})$ points exist for all t (that is the constant in the Bondarenko et al result [8] is $C_d = 1/d$ (or lower), as suggested by [35] for $\mathbb{S}^2$).
- The point sets are not nested, that is the points of a spherical t-design are not necessarily a subset of the points of a t'-design for some $t' > t$.
- The point sets do not lie on bands of equal ϕ_1 (latitude on $\mathbb{S}^2$) making them less amenable for FFT based methods.
- The point sets are obtained by extensive calculation, rather than generated by a simple algorithms as for generalized spiral or equal area points on $\mathbb{S}^2$ [52]. Once calculated the point sets are easy to use.

An example of a point set on $\mathbb{S}^2$ that satisfies the last three issues are the HELAPix points[30], which provide a hierarchical, equal area (so exact for constants), iso-latitude set of points widely used in cosmology.

5 Tables of Results

5.1 Spherical t-Designs with no Imposed Symmetry for $\mathbb{S}^2$

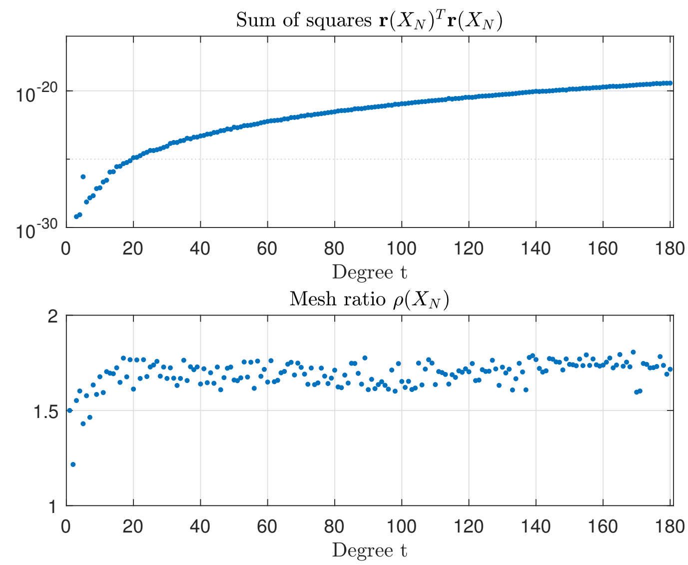

Fig. 2 Sum of squares of Weyl sums and mesh ratios for spherical t-designs on $\mathbb{S}^2$

From Tables 2, 3 and 4 the variational criteria based on the three functions $\psi_{1,t}$, $\psi_{2,t}$ and $\psi_{3,t}$ all have values close to the double precision machine precision of $\varepsilon = 2.2 \times 10^{-16}$ for all degrees $t = 1, \ldots, 180$. Despite being theoretically non-negative, rounding error sometimes gives negative values, but still close to machine precision. The potential values using $\psi_{3,t}$ are slightly larger due to the larger value of $\psi_{3,t}(1)$. The tables also give the unscaled sum of squares $\boldsymbol{r}(X_N)^T \boldsymbol{r}(X_N)$, which is also plotted in Fig. 2. These tables also list both the Delsarte, Goethals and Seidel lower bounds $N^*(2,t)$ and the Yudin lower bound $N^+(2,t)$, plus the actual number of points N. The number of points $N = \widehat{N}(2,t)$, apart from $t = 3,5,7,9,11,13,15$ when $N = \widehat{N}(2,t) - 1$. There may well be spherical t-designs with smaller values of N and special symmetries, see [35] for example. For all these point sets the mesh ratios $\rho(X_N)$ are less than 1.81, see Fig. 2. All these point sets are available from [66].

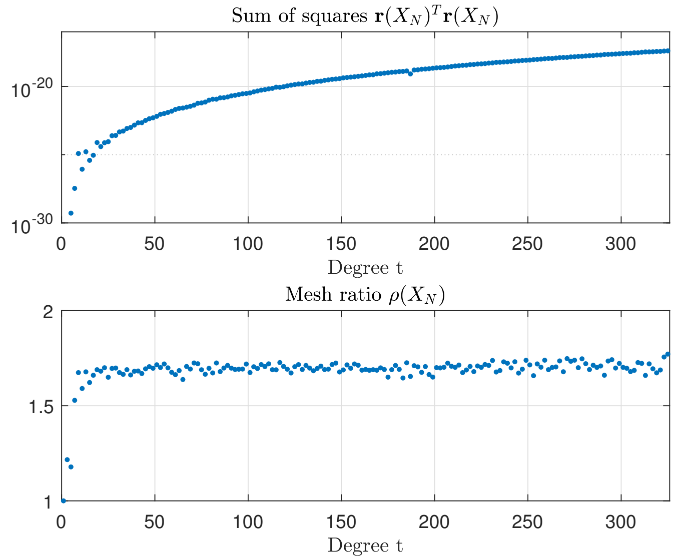

Fig. 3 Sum of squares of Weyl sums and mesh ratios for symmetric spherical t-designs on $\mathbb{S}^2$

5.2 Symmetric Spherical t-Designs for $\mathbb{S}^2$

For $\mathbb{S}^2$ a t-design with a sightly smaller number of points $\overline{N}(2,t)$ can be found by constraining the point sets to be symmetric (antipodal). A major computational advantage of working with symmetric point sets is the reduction (approximately half), for a given degree t, in the number of optimization variables n and the number of terms m in the Weyl sums. Tables 5, 6 and 7 list the characteristics of the calculated t-designs for $t = 1,3,5,\ldots,325$, as a symmetric $2k$-design is automatically a $2k+1$-design. These tables have $t = \overline{N}(2,t)$ except for $t = 1,7,11$. These point sets, again available from [66], provide excellent sets of points for numerical integration on $\mathbb{S}^2$ with mesh ratios all less than 1.78 for degrees up to 325, as illustrated in Fig. 3.

5.3 Designs for $d = 3$

For $d = 3$, $Z(3,\ell) = (\ell+1)^2$, so the dimension of the space of polynomials of degree at most t in $\mathbb{S}^3$ is $D(3,t) = Z(4,t) = (t+1)(t+2)(2t+3)/6$. Comparing the number of variables with the number of conditions, with no symmetry restrictions, gives

$$\widehat{N}(3,t) = \left\lceil \frac{2t^3+9t^2+13t+36}{18} \right\rceil,$$

while for symmetric spherical designs on $\mathbb{S}^3$

$$\overline{N}(3,t) = 2\left\lceil \frac{t^3 + 3t^2 + 2t + 30}{18} \right\rceil.$$

The are six regular convex polytopes with $N = 5, 8, 16, 24, 120$ and 600 vertices on S^3 [22] (the 5-cell, 16-cell, 8-cell, 24-cell, 600-cell and 120-cell respectively) giving spherical t-designs for for $t = 2, 3, 5, 7, 9, 11$ and 11. The energy of regular sets on $\mathbb{S}^3$ with $N = 2, 3, 4, 5, 6, 8, 10, 12, 13, 24, 48$ has been studied by [1]. The $N =$ 24 vertices of the D4 root system [19] provides a one-parameter family of 5-designs on $\mathbb{S}^3$. The Cartesian coordinates of the regular point sets are known, and these can be numerically verified to be spherical designs. The three variational criteria using (19), (21) and (23) are given for these point sets in Table 8. Fig. 4 clearly illustrates the difference between the widely studied [24, 21, 11] inner product set $\mathscr{A}(X_N)$ for a regular point set (the 600-cell with $N = 120$) and a computed spherical 13-design with $N = 340$.

The results of some initial experiments in minimising the three variational criteria are given in Tables 9 and 10. For $d > 2$, it is more difficult to quickly generate a point set with a good mesh ratio to serve as an initial point for the optimization algorithms. One strategy is to randomly generate starting points, but this both makes the optimization problem harder and tends to produce nearby point sets which are local minimisers and have poor mesh ratios as the random initial points may have small separation [13]. Another possibility is the generalisation of equal area points to $d > 2$ by Leopardi [45]. For a given t and N there are still many different point sets with objective values close to 0 and different mesh ratios. To fully explore spherical t-designs for $d > 2$, a stable implementation of the spherical harmonics is needed, so that least squares minimisation can be fully utilised.

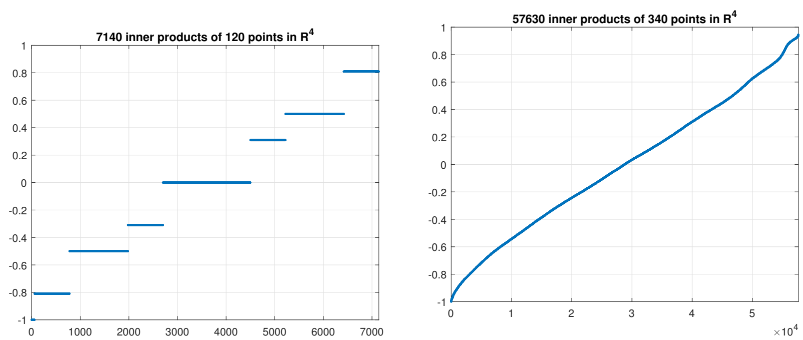

Fig. 4 Inner product sets $\mathscr{A}(X_N)$ for 600-cell with $N = 120$ and 13-design with $N = 340$ on $\mathbb{S}^3$

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t	$N^*(2,t)$	$N^+(2,t)$	N	n	m	$V_{t,N,\psi_1}(X_N)$	$V_{t,N,\psi_2}(X_N)$	$V_{t,N,\psi_3}(X_N)$	$\boldsymbol{r}(X_N)^T\boldsymbol{r}(X_N)$	$\delta(X_N)$	$h(X_N)$	$\rho(X_N)$
1	2	2	3	3	3	3.7e-17	1.9e-17	1.2e-17	3.0e-32	2.0944	1.5708	1.50
2	4	4	6	9	8	3.5e-17	3.5e-17	4.3e-17	1.1e-31	1.5708	0.9553	1.22
3	6	6	8	13	15	3.1e-17	-4.8e-18	-1.0e-17	6.2e-30	1.2310	0.9553	1.55
4	9	9	14	25	24	-2.1e-18	-5.5e-18	4.5e-17	8.6e-30	0.8630	0.6913	1.60
5	12	12	18	33	35	-3.4e-17	6.9e-19	-1.1e-16	5.2e-27	0.8039	0.5749	1.43
6	16	16	26	49	48	2.0e-17	5.0e-18	6.0e-17	7.4e-29	0.6227	0.4911	1.58
7	20	20	32	61	63	1.1e-17	4.5e-18	6.1e-17	1.5e-28	0.5953	0.4357	1.46
8	25	25	42	81	80	-9.6e-18	-2.3e-18	-2.8e-17	2.1e-28	0.4845	0.3958	1.63
9	30	31	50	97	99	1.0e-17	-1.3e-18	-2.4e-17	7.0e-28	0.4555	0.3608	1.58
10	36	37	62	121	120	1.6e-17	2.3e-18	7.7e-17	8.1e-28	0.3945	0.3308	1.68
11	42	43	72	141	143	8.3e-18	8.8e-18	1.2e-16	2.1e-27	0.3750	0.2989	1.59
12	49	50	86	169	168	-9.7e-18	-5.1e-18	-7.6e-17	2.9e-27	0.3241	0.2761	1.70
13	56	58	98	193	195	-7.4e-18	4.3e-18	-6.9e-17	1.1e-26	0.3028	0.2567	1.70
14	64	66	114	225	224	7.0e-18	1.5e-18	-6.4e-18	1.2e-26	0.2838	0.2402	1.69
15	72	75	128	253	255	-4.2e-18	-1.6e-18	-1.2e-16	2.8e-26	0.2644	0.2279	1.72
16	81	84	146	289	288	3.8e-18	6.3e-18	-5.0e-17	3.0e-26	0.2568	0.2115	1.65
17	90	94	163	323	323	2.0e-17	1.1e-17	1.9e-16	4.7e-26	0.2333	0.2070	1.77
18	100	104	182	361	360	-4.1e-18	-5.1e-18	-1.6e-16	5.7e-26	0.2243	0.1880	1.68
19	110	115	201	399	399	1.5e-17	7.7e-18	1.1e-16	7.7e-26	0.2086	0.1843	1.77
20	121	127	222	441	440	1.3e-17	1.3e-17	2.0e-17	1.3e-25	0.2105	0.1697	1.61
21	132	139	243	483	483	1.6e-17	-3.0e-18	-4.3e-17	1.4e-25	0.1900	0.1677	1.77
22	144	151	266	529	528	-7.6e-19	-1.0e-17	8.6e-17	1.8e-25	0.1887	0.1574	1.67
23	156	164	289	575	575	-6.8e-18	1.1e-18	-3.7e-17	2.6e-25	0.1759	0.1554	1.77
24	169	178	314	625	624	2.4e-18	-9.3e-18	-1.2e-16	3.2e-25	0.1730	0.1451	1.68
25	182	192	339	675	675	3.1e-18	1.1e-17	-1.2e-16	4.3e-25	0.1628	0.1407	1.73
26	196	207	366	729	728	6.1e-18	-2.0e-17	-8.6e-17	4.3e-25	0.1533	0.1333	1.74
27	210	222	393	783	783	3.3e-18	1.7e-17	-1.4e-16	4.9e-25	0.1485	0.1305	1.76
28	225	238	422	841	840	-1.8e-17	1.9e-17	-2.5e-16	5.8e-25	0.1490	0.1252	1.68
29	240	254	451	899	899	-2.0e-17	1.4e-17	-3.3e-16	7.2e-25	0.1405	0.1214	1.73
30	256	271	482	961	960	-3.5e-17	4.2e-18	-8.6e-17	8.8e-25	0.1381	0.1152	1.67
31	272	289	513	1023	1023	-3.3e-17	-4.2e-19	-1.7e-16	1.4e-24	0.1313	0.1132	1.72
32	289	307	546	1089	1088	-5.1e-17	-5.6e-18	-2.2e-16	1.7e-24	0.1315	0.1098	1.67
33	306	325	579	1155	1155	-5.0e-17	2.7e-17	-2.1e-16	1.7e-24	0.1292	0.1054	1.63
34	324	344	614	1225	1224	3.1e-17	2.8e-17	-3.0e-16	2.2e-24	0.1235	0.1030	1.67
35	342	364	649	1295	1295	3.0e-17	-1.6e-18	-3.5e-16	2.4e-24	0.1139	0.1005	1.76
36	361	384	686	1369	1368	5.1e-17	-9.6e-19	-3.9e-16	3.4e-24	0.1170	0.0970	1.66
37	380	405	723	1443	1443	5.1e-17	-6.6e-18	-3.1e-16	3.1e-24	0.1113	0.0962	1.73
38	400	426	762	1521	1520	2.8e-17	2.6e-17	-3.6e-16	4.0e-24	0.1079	0.0925	1.71
39	420	448	801	1599	1599	3.0e-17	-6.2e-17	-3.6e-16	4.1e-24	0.1079	0.0933	1.73
40	441	470	842	1681	1680	1.0e-16	5.5e-17	-4.6e-16	5.0e-24	0.1068	0.0875	1.64
41	462	493	883	1763	1763	1.1e-16	2.8e-17	-2.6e-16	5.5e-24	0.0998	0.0858	1.72
42	484	516	926	1849	1848	-1.7e-19	-2.1e-17	-3.8e-16	6.8e-24	0.1007	0.0829	1.65
43	506	540	969	1935	1935	-7.3e-19	-4.7e-17	-4.8e-16	7.0e-24	0.0964	0.0819	1.70
44	529	565	1014	2025	2024	2.9e-17	2.6e-17	-4.0e-16	9.0e-24	0.0980	0.0805	1.64
45	552	590	1059	2115	2115	2.4e-17	-2.5e-17	-3.7e-16	9.4e-24	0.0911	0.0787	1.73
46	576	615	1106	2209	2208	7.6e-17	6.9e-17	-3.5e-16	1.2e-23	0.0949	0.0763	1.61
47	600	642	1153	2303	2303	8.4e-17	-3.7e-17	-2.7e-16	1.3e-23	0.0898	0.0751	1.67
48	625	668	1202	2401	2400	-3.1e-17	-4.2e-17	-3.8e-16	1.6e-23	0.0869	0.0748	1.72
49	650	696	1251	2499	2499	-2.3e-17	1.2e-16	-3.7e-16	1.5e-23	0.0839	0.0725	1.73
50	676	723	1302	2601	2600	3.5e-17	2.9e-17	-1.8e-16	2.2e-23	0.0858	0.0712	1.66
51	702	752	1353	2703	2703	2.6e-17	-1.1e-16	-3.0e-16	2.0e-23	0.0838	0.0694	1.66
52	729	781	1406	2809	2808	1.0e-16	6.5e-17	-3.4e-16	2.3e-23	0.0809	0.0676	1.67
53	756	810	1459	2915	2915	1.1e-16	-5.3e-17	-3.4e-16	2.8e-23	0.0768	0.0674	1.75
54	784	840	1514	3025	3024	-2.6e-17	2.5e-17	-2.5e-16	2.9e-23	0.0783	0.0656	1.68
55	812	870	1569	3135	3135	-3.0e-17	5.1e-17	-3.2e-16	3.1e-23	0.0741	0.0650	1.75
56	841	902	1626	3249	3248	-9.2e-17	-5.7e-17	-2.8e-16	3.5e-23	0.0778	0.0629	1.62
57	870	933	1683	3363	3363	-9.0e-17	-1.2e-16	-2.0e-16	3.8e-23	0.0717	0.0631	1.76
58	900	965	1742	3481	3480	-2.1e-16	-1.7e-16	-2.3e-16	4.6e-23	0.0732	0.0615	1.68
59	930	998	1801	3599	3599	-2.0e-16	-8.5e-17	-2.3e-16	5.2e-23	0.0708	0.0608	1.72
60	961	1031	1862	3721	3720	-9.6e-17	3.7e-18	-1.9e-16	5.8e-23	0.0718	0.0592	1.65

Table 2 Spherical t-designs on $\mathbb{S}^2$ with no symmetry restrictions, $N = \widehat{N}(2,t)$ and degrees $t = 1$ – 60, except $t = 3, 5, 7, 9, 11, 13, 15$ when $N = \widehat{N}(2,t) - 1$

t	$N^*(2,t)$	$N^+(2,t)$	N	n	m	$V_{t,N,\psi_1}(X_N)$	$V_{t,N,\psi_2}(X_N)$	$V_{t,N,\psi_3}(X_N)$	$\boldsymbol{r}(X_N)^T\boldsymbol{r}(X_N)$	$\delta(X_N)$	$h(X_N)$	$\rho(X_N)$
61	992	1065	1923	3843	3843	-9.4e-17	5.3e-17	-4.3e-17	6.4e-23	0.0663	0.0584	1.76
62	1024	1099	1986	3969	3968	2.6e-17	2.4e-17	-1.9e-16	6.7e-23	0.0699	0.0577	1.65
63	1056	1134	2049	4095	4095	2.6e-17	-4.8e-18	-1.5e-16	7.1e-23	0.0680	0.0564	1.66
64	1089	1170	2114	4225	4224	3.3e-17	2.7e-17	6.5e-17	7.4e-23	0.0662	0.0562	1.70
65	1122	1206	2179	4355	4355	2.7e-17	-1.7e-16	9.7e-18	8.9e-23	0.0647	0.0552	1.70
66	1156	1242	2246	4489	4488	-1.0e-16	-1.0e-16	-1.0e-16	8.9e-23	0.0616	0.0537	1.74
67	1190	1279	2313	4623	4623	-1.1e-16	-2.2e-16	-1.7e-16	1.1e-22	0.0609	0.0534	1.75
68	1225	1317	2382	4761	4760	2.5e-16	2.2e-16	-1.5e-17	1.1e-22	0.0620	0.0523	1.69
69	1260	1355	2451	4899	4899	2.5e-16	-1.3e-16	-8.2e-17	1.2e-22	0.0590	0.0516	1.75
70	1296	1394	2522	5041	5040	1.0e-16	4.1e-18	-8.3e-17	1.4e-22	0.0595	0.0513	1.73
71	1332	1433	2593	5183	5183	1.0e-16	-1.7e-16	-3.2e-17	1.5e-22	0.0587	0.0496	1.69
72	1369	1473	2666	5329	5328	1.7e-16	1.7e-16	-6.6e-17	1.7e-22	0.0603	0.0494	1.64
73	1406	1513	2739	5475	5475	1.7e-16	8.5e-17	-2.5e-16	1.6e-22	0.0567	0.0488	1.72
74	1444	1554	2814	5625	5624	1.8e-16	9.0e-17	-2.0e-16	1.9e-22	0.0582	0.0476	1.64
75	1482	1595	2889	5775	5775	1.8e-16	-1.7e-16	-1.6e-16	2.0e-22	0.0577	0.0475	1.64
76	1521	1637	2966	5929	5928	2.8e-16	1.8e-16	-3.8e-16	2.2e-22	0.0547	0.0471	1.72
77	1560	1680	3043	6083	6083	2.8e-16	-2.1e-16	-3.4e-16	2.3e-22	0.0546	0.0459	1.68
78	1600	1723	3122	6241	6240	3.2e-16	2.3e-16	-2.2e-16	2.6e-22	0.0554	0.0455	1.64
79	1640	1766	3201	6399	6399	3.3e-16	2.2e-16	-1.3e-16	2.7e-22	0.0538	0.0449	1.67
80	1681	1810	3282	6561	6560	2.6e-16	2.8e-16	-1.7e-16	3.0e-22	0.0525	0.0450	1.71
81	1722	1855	3363	6723	6723	2.6e-16	2.0e-16	-4.6e-16	3.4e-22	0.0537	0.0436	1.62
82	1764	1900	3446	6889	6888	3.1e-16	2.2e-16	-4.2e-16	3.6e-22	0.0532	0.0431	1.62
83	1806	1946	3529	7055	7055	3.2e-16	-2.9e-16	-3.2e-16	3.6e-22	0.0505	0.0426	1.69
84	1849	1992	3614	7225	7224	-3.9e-17	1.3e-17	-3.9e-16	3.9e-22	0.0516	0.0424	1.64
85	1892	2039	3699	7395	7395	-5.1e-17	1.7e-17	-3.8e-16	4.1e-22	0.0479	0.0418	1.75
86	1936	2087	3786	7569	7568	-1.6e-16	-1.4e-16	-4.4e-16	4.8e-22	0.0488	0.0427	1.75
87	1980	2135	3873	7743	7743	-1.6e-16	-1.4e-16	-4.5e-16	4.9e-22	0.0488	0.0414	1.69
88	2025	2183	3962	7921	7920	2.5e-16	2.3e-16	-4.2e-16	5.0e-22	0.0493	0.0404	1.64
89	2070	2232	4051	8099	8099	2.5e-16	-3.2e-16	-4.9e-16	5.4e-22	0.0454	0.0403	1.78
90	2116	2282	4142	8281	8280	-9.9e-17	-1.6e-17	-4.1e-16	6.0e-22	0.0489	0.0394	1.61
91	2162	2332	4233	8463	8463	-1.1e-16	2.3e-16	-4.0e-16	6.2e-22	0.0473	0.0393	1.66
92	2209	2383	4326	8649	8648	-2.2e-16	-2.9e-16	-5.0e-16	6.7e-22	0.0481	0.0389	1.61
93	2256	2434	4419	8835	8835	-2.1e-16	8.6e-17	-5.1e-16	7.1e-22	0.0467	0.0382	1.64
94	2304	2486	4514	9025	9024	4.3e-17	1.4e-16	-5.6e-16	7.5e-22	0.0463	0.0383	1.65
95	2352	2538	4609	9215	9215	3.5e-17	-1.0e-16	-6.4e-16	7.9e-22	0.0462	0.0377	1.63
96	2401	2591	4706	9409	9408	-1.7e-16	-5.9e-17	-6.8e-16	9.0e-22	0.0458	0.0369	1.61
97	2450	2644	4803	9603	9603	-1.8e-16	-1.2e-16	-6.2e-16	8.8e-22	0.0429	0.0367	1.71
98	2500	2698	4902	9801	9800	-1.6e-16	-1.6e-16	-5.3e-16	1.0e-21	0.0453	0.0362	1.60
99	2550	2753	5001	9999	9999	-1.6e-16	-1.9e-16	-5.5e-16	1.0e-21	0.0424	0.0370	1.75
100	2601	2808	5102	10201	10200	1.4e-16	2.6e-16	-6.4e-16	1.1e-21	0.0432	0.0357	1.65
101	2652	2863	5203	10403	10403	1.5e-16	-2.8e-16	-6.2e-16	1.2e-21	0.0433	0.0351	1.62
102	2704	2919	5306	10609	10608	-1.1e-16	6.7e-18	-5.1e-16	1.3e-21	0.0424	0.0350	1.65
103	2756	2976	5409	10815	10815	-1.2e-16	8.7e-17	-6.7e-16	1.3e-21	0.0428	0.0345	1.61
104	2809	3033	5514	11025	11024	-2.7e-16	-1.3e-16	-7.0e-16	1.5e-21	0.0424	0.0343	1.62
105	2862	3091	5619	11235	11235	-2.8e-16	4.7e-18	-6.4e-16	1.5e-21	0.0395	0.0345	1.75
106	2916	3149	5726	11449	11448	1.9e-16	1.9e-16	-6.1e-16	1.6e-21	0.0410	0.0335	1.63
107	2970	3208	5833	11663	11663	1.9e-16	-1.5e-16	-6.8e-16	1.6e-21	0.0393	0.0337	1.72
108	3025	3267	5942	11881	11880	-4.3e-16	-3.7e-16	-7.0e-16	1.8e-21	0.0385	0.0340	1.77
109	3080	3327	6051	12099	12099	-4.3e-16	2.6e-16	-6.8e-16	1.9e-21	0.0381	0.0333	1.75
110	3136	3388	6162	12321	12320	1.9e-16	3.9e-17	-7.8e-16	2.0e-21	0.0396	0.0324	1.64
111	3192	3449	6273	12543	12543	1.9e-16	-1.5e-16	-8.9e-16	2.1e-21	0.0376	0.0321	1.70
112	3249	3510	6386	12769	12768	6.0e-17	1.6e-16	-8.2e-16	2.2e-21	0.0379	0.0322	1.70
113	3306	3573	6499	12995	12995	6.1e-17	-1.5e-16	-7.3e-16	2.3e-21	0.0373	0.0315	1.69
114	3364	3635	6614	13225	13224	2.2e-16	2.1e-16	-8.1e-16	2.8e-21	0.0381	0.0312	1.64
115	3422	3698	6729	13455	13455	2.2e-16	1.5e-16	-6.7e-16	2.5e-21	0.0367	0.0310	1.69
116	3481	3762	6846	13689	13688	-4.3e-16	-2.6e-16	-7.3e-16	2.8e-21	0.0368	0.0308	1.67
117	3540	3826	6963	13923	13923	-4.3e-16	6.8e-17	-7.9e-16	2.8e-21	0.0355	0.0304	1.71
118	3600	3891	7082	14161	14160	-4.5e-16	-2.8e-16	-7.5e-16	3.0e-21	0.0360	0.0306	1.70
119	3660	3957	7201	14399	14399	-4.5e-16	2.0e-16	-7.5e-16	3.3e-21	0.0358	0.0308	1.72
120	3721	4023	7322	14641	14640	-5.2e-16	-4.7e-16	-8.2e-16	3.4e-21	0.0348	0.0297	1.70

Table 3 Spherical t-designs on $\mathbb{S}^2$ with no symmetry restrictions, $N = \widehat{N}(2,t)$ and degrees $t = 61$ – 120

t	$N^*(2,t)$	$N^+(2,t)$	N	n	m	$V_{t,N,\psi_1}(X_N)$	$V_{t,N,\psi_2}(X_N)$	$V_{t,N,\psi_3}(X_N)$	$\boldsymbol{r}(X_N)^T\boldsymbol{r}(X_N)$	$\delta(X_N)$	$h(X_N)$	$\rho(X_N)$
121	3782	4089	7443	14883	14883	-5.2e-16	-1.5e-16	-9.0e-16	3.3e-21	0.0338	0.0295	1.75
122	3844	4156	7566	15129	15128	1.8e-16	1.7e-16	-8.9e-16	3.6e-21	0.0356	0.0295	1.66
123	3906	4224	7689	15375	15375	1.8e-16	1.3e-16	-6.8e-16	4.0e-21	0.0352	0.0292	1.66
124	3969	4292	7814	15625	15624	-9.1e-17	6.3e-17	-8.4e-16	4.1e-21	0.0348	0.0298	1.71
125	4032	4360	7939	15875	15875	-8.7e-17	-1.9e-16	-1.0e-15	4.2e-21	0.0337	0.0288	1.70
126	4096	4430	8066	16129	16128	-8.7e-17	-1.9e-16	-8.1e-16	4.4e-21	0.0332	0.0283	1.71
127	4160	4499	8193	16383	16383	-7.6e-17	-6.8e-18	-8.1e-16	4.5e-21	0.0321	0.0283	1.76
128	4225	4570	8322	16641	16640	6.9e-16	6.6e-16	-8.6e-16	4.9e-21	0.0330	0.0283	1.72
129	4290	4641	8451	16899	16899	7.1e-16	-6.4e-16	-9.5e-16	5.1e-21	0.0341	0.0279	1.63
130	4356	4712	8582	17161	17160	2.8e-16	2.7e-16	-9.7e-16	5.4e-21	0.0322	0.0278	1.73
131	4422	4784	8713	17423	17423	2.8e-16	-1.6e-16	-8.1e-16	5.6e-21	0.0325	0.0276	1.70
132	4489	4856	8846	17689	17688	2.8e-16	1.3e-16	-1.0e-15	5.8e-21	0.0324	0.0278	1.72
133	4556	4929	8979	17955	17955	2.8e-16	8.4e-17	-9.2e-16	6.1e-21	0.0335	0.0269	1.61
134	4624	5003	9114	18225	18224	4.5e-16	4.1e-16	-1.0e-15	6.4e-21	0.0321	0.0267	1.67
135	4692	5077	9249	18495	18495	4.5e-16	-2.8e-16	-1.0e-15	7.0e-21	0.0309	0.0270	1.75
136	4761	5152	9386	18769	18768	5.4e-16	5.0e-16	-1.0e-15	7.3e-21	0.0306	0.0261	1.70
137	4830	5227	9523	19043	19043	5.5e-16	2.7e-16	-8.8e-16	7.8e-21	0.0323	0.0260	1.61
138	4900	5303	9662	19321	19320	2.5e-16	1.0e-16	-8.8e-16	8.1e-21	0.0301	0.0267	1.78
139	4970	5379	9801	19599	19599	2.5e-16	-3.5e-16	-9.6e-16	8.6e-21	0.0303	0.0271	1.79
140	5041	5456	9942	19881	19880	7.5e-17	6.8e-17	-9.5e-16	9.3e-21	0.0298	0.0263	1.77
141	5112	5533	10083	20163	20163	7.1e-17	-3.8e-16	-9.5e-16	8.9e-21	0.0299	0.0257	1.72
142	5184	5611	10226	20449	20448	2.2e-16	2.2e-16	-9.4e-16	9.4e-21	0.0300	0.0255	1.70
143	5256	5689	10369	20735	20735	2.1e-16	6.1e-16	-1.1e-15	9.4e-21	0.0293	0.0250	1.71
144	5329	5768	10514	21025	21024	-6.5e-16	-6.1e-16	-1.1e-15	9.9e-21	0.0285	0.0253	1.77
145	5402	5848	10659	21315	21315	-6.5e-16	1.3e-16	-1.1e-15	1.0e-20	0.0283	0.0250	1.77
146	5476	5928	10806	21609	21608	6.6e-16	6.2e-16	-1.1e-15	1.1e-20	0.0285	0.0250	1.76
147	5550	6009	10953	21903	21903	6.6e-16	-4.2e-16	-1.1e-15	1.1e-20	0.0287	0.0251	1.75
148	5625	6090	11102	22201	22200	-3.8e-16	-2.3e-16	-1.1e-15	1.2e-20	0.0282	0.0241	1.71
149	5700	6172	11251	22499	22499	-3.8e-16	-3.6e-16	-1.2e-15	1.1e-20	0.0276	0.0245	1.77
150	5776	6254	11402	22801	22800	1.4e-16	1.3e-16	-1.2e-15	1.3e-20	0.0279	0.0243	1.74
151	5852	6337	11553	23103	23103	1.3e-16	5.8e-16	-1.2e-15	1.4e-20	0.0275	0.0239	1.74
152	5929	6420	11706	23409	23408	-8.5e-16	-6.8e-16	-1.4e-15	1.3e-20	0.0274	0.0238	1.73
153	6006	6504	11859	23715	23715	-8.4e-16	4.8e-16	-1.1e-15	1.4e-20	0.0268	0.0237	1.77
154	6084	6589	12014	24025	24024	-4.5e-16	-4.8e-16	-1.2e-15	1.5e-20	0.0271	0.0235	1.74
155	6162	6674	12169	24335	24335	-4.5e-16	2.6e-16	-1.0e-15	1.6e-20	0.0268	0.0240	1.79
156	6241	6759	12326	24649	24648	4.6e-16	3.5e-16	-1.2e-15	1.6e-20	0.0269	0.0234	1.74
157	6320	6845	12483	24963	24963	4.6e-16	-2.7e-16	-1.2e-15	1.7e-20	0.0269	0.0238	1.77
158	6400	6932	12642	25281	25280	3.7e-17	1.0e-16	-1.3e-15	1.7e-20	0.0262	0.0228	1.74
159	6480	7019	12801	25599	25599	2.5e-17	-5.4e-16	-1.3e-15	1.7e-20	0.0262	0.0227	1.73
160	6561	7107	12962	25921	25920	-8.0e-16	-7.6e-16	-1.2e-15	1.9e-20	0.0265	0.0230	1.74
161	6642	7195	13123	26243	26243	-8.0e-16	2.2e-16	-1.2e-15	1.9e-20	0.0256	0.0225	1.75
162	6724	7284	13286	26569	26568	1.2e-16	2.0e-16	-1.3e-15	2.1e-20	0.0256	0.0228	1.78
163	6806	7373	13449	26895	26895	1.1e-16	-6.1e-16	-1.3e-15	2.2e-20	0.0257	0.0221	1.72
164	6889	7463	13614	27225	27224	-5.7e-16	-6.3e-16	-1.5e-15	2.1e-20	0.0257	0.0223	1.74
165	6972	7553	13779	27555	27555	-5.7e-16	-3.7e-16	-1.5e-15	2.2e-20	0.0251	0.0225	1.79
166	7056	7644	13946	27889	27888	-9.6e-16	-8.1e-16	-1.3e-15	2.3e-20	0.0251	0.0217	1.73
167	7140	7736	14113	28223	28223	-9.5e-16	5.6e-16	-1.3e-15	2.3e-20	0.0252	0.0221	1.75
168	7225	7828	14282	28561	28560	-1.8e-16	-1.8e-16	-1.4e-15	2.5e-20	0.0248	0.0214	1.73
169	7310	7921	14451	28899	28899	-1.8e-16	-7.2e-16	-1.3e-15	2.6e-20	0.0249	0.0225	1.81
170	7396	8014	14622	29241	29240	-1.3e-16	-1.3e-16	-1.4e-15	2.7e-20	0.0263	0.0210	1.60
171	7482	8108	14793	29583	29583	-1.3e-16	6.2e-16	-1.4e-15	2.8e-20	0.0261	0.0209	1.60
172	7569	8202	14966	29929	29928	3.5e-16	3.5e-16	-1.4e-15	2.9e-20	0.0242	0.0212	1.75
173	7656	8297	15139	30275	30275	3.4e-16	2.9e-16	-1.3e-15	3.0e-20	0.0243	0.0212	1.74
174	7744	8392	15314	30625	30624	1.4e-16	1.6e-16	-1.4e-15	3.0e-20	0.0244	0.0211	1.72
175	7832	8488	15489	30975	30975	1.3e-16	-1.5e-17	-1.5e-15	3.3e-20	0.0241	0.0208	1.72
176	7921	8584	15666	31329	31328	1.2e-16	-6.0e-18	-1.4e-15	3.5e-20	0.0238	0.0206	1.73
177	8010	8681	15843	31683	31683	1.3e-16	2.2e-16	-1.3e-15	3.4e-20	0.0238	0.0212	1.78
178	8100	8779	16022	32041	32040	-2.3e-16	-1.4e-16	-1.2e-15	3.6e-20	0.0233	0.0202	1.74
179	8190	8877	16201	32399	32399	-2.3e-16	2.2e-16	-1.2e-15	3.6e-20	0.0239	0.0202	1.69
180	8281	8976	16382	32761	32760	-7.8e-16	-6.7e-16	-1.3e-15	3.7e-20	0.0236	0.0202	1.72

Table 4 Spherical t-designs on $\mathbb{S}^2$ with no symmetry restrictions, $N = \widehat{N}(2,t)$ and degrees $t = 121$ – 180

t	$N^*(2,t)$	$N^+(2,t)$	N	n	m	$V_{t,N,\psi_1}(X_N)$	$V_{t,N,\psi_2}(X_N)$	$V_{t,N,\psi_3}(X_N)$	$\boldsymbol{r}(X_N)^T\boldsymbol{r}(X_N)$	$\delta(X_N)$	$h(X_N)$	$\rho(X_N)$
1	2	2	2	0	0	0.0e+00	0.0e+00	0.0e+00	0.0e+00	3.1416	1.5708	1.00
3	6	6	6	3	5	3.5e-17	0.0e+00	0.0e+00	6.1e-31	1.5708	0.9553	1.22
5	12	12	12	9	14	6.9e-18	1.5e-17	-8.5e-17	5.2e-30	1.1071	0.6524	1.18
7	20	20	32	29	27	2.1e-17	4.0e-18	1.9e-16	3.4e-28	0.5863	0.4480	1.53
9	30	31	48	45	44	4.2e-18	-4.1e-18	-1.9e-16	1.2e-25	0.4611	0.3860	1.67
11	42	43	70	67	65	-2.4e-17	-1.9e-18	-1.4e-16	8.6e-27	0.3794	0.3017	1.59
13	56	58	94	91	90	-8.4e-18	4.8e-18	-1.3e-16	1.6e-25	0.3146	0.2639	1.68
15	72	75	120	117	119	7.9e-18	8.0e-18	8.6e-17	3.9e-26	0.2900	0.2352	1.62
17	90	94	156	153	152	8.7e-18	1.0e-17	-6.0e-17	9.1e-26	0.2457	0.2039	1.66
19	110	115	192	189	189	-5.4e-18	-9.2e-19	-1.3e-16	7.8e-25	0.2248	0.1899	1.69
21	132	139	234	231	230	2.2e-17	-7.2e-19	1.2e-16	3.9e-25	0.2009	0.1689	1.68
23	156	164	278	275	275	-3.3e-18	2.1e-18	8.0e-17	7.5e-25	0.1822	0.1548	1.70
25	182	192	328	325	324	-9.5e-18	1.1e-17	-2.4e-16	9.0e-25	0.1722	0.1421	1.65
27	210	222	380	377	377	-4.6e-19	2.0e-17	1.9e-17	2.4e-24	0.1567	0.1328	1.70
29	240	254	438	435	434	5.8e-18	2.2e-17	-9.2e-17	2.6e-24	0.1448	0.1229	1.70
31	272	289	498	495	495	-3.0e-17	-3.2e-18	-3.7e-16	4.6e-24	0.1376	0.1151	1.67
33	306	325	564	561	560	-4.0e-17	2.3e-17	-3.5e-16	5.4e-24	0.1292	0.1075	1.66
35	342	364	632	629	629	2.1e-17	-3.8e-18	-4.2e-16	8.3e-24	0.1197	0.1011	1.69
37	380	405	706	703	702	3.8e-17	-5.0e-18	-2.6e-16	9.9e-24	0.1165	0.0968	1.66
39	420	448	782	779	779	2.6e-17	-6.2e-17	-3.2e-16	1.5e-23	0.1082	0.0910	1.68
41	462	493	864	861	860	9.7e-17	2.7e-17	-2.9e-16	2.2e-23	0.1025	0.0863	1.68
43	506	540	948	945	945	-7.1e-18	-4.5e-17	-2.6e-16	2.2e-23	0.0988	0.0824	1.67
45	552	590	1038	1035	1034	2.5e-17	-2.8e-17	-2.7e-16	3.2e-23	0.0936	0.0793	1.69
47	600	642	1130	1127	1127	7.9e-17	-4.6e-17	-2.3e-16	4.3e-23	0.0889	0.0758	1.71
49	650	696	1228	1225	1224	-3.5e-17	1.2e-16	-2.3e-16	5.0e-23	0.0856	0.0727	1.70
51	702	752	1328	1325	1325	2.8e-17	-1.2e-16	-2.0e-16	6.4e-23	0.0823	0.0706	1.72
53	756	810	1434	1431	1430	8.9e-17	-5.9e-17	-4.3e-16	9.1e-23	0.0799	0.0679	1.70
55	812	870	1542	1539	1539	-1.1e-17	5.5e-17	-3.2e-16	1.0e-22	0.0772	0.0664	1.72
57	870	933	1656	1653	1652	-7.2e-17	-1.2e-16	4.8e-19	1.2e-22	0.0742	0.0630	1.70
59	930	998	1772	1769	1769	-1.9e-16	-1.1e-16	-2.5e-16	1.5e-22	0.0722	0.0605	1.68
61	992	1065	1894	1891	1890	-6.1e-17	5.8e-17	-1.4e-16	2.1e-22	0.0701	0.0583	1.66
63	1056	1134	2018	2015	2015	2.8e-17	-3.4e-18	-9.0e-17	2.5e-22	0.0674	0.0568	1.68
65	1122	1206	2148	2145	2144	3.0e-17	-1.7e-16	-1.1e-16	2.6e-22	0.0664	0.0544	1.64
67	1190	1279	2280	2277	2277	-1.0e-16	-2.2e-16	9.8e-17	3.0e-22	0.0634	0.0541	1.71
69	1260	1355	2418	2415	2414	2.4e-16	-1.1e-16	-2.2e-16	3.5e-22	0.0616	0.0521	1.69
71	1332	1433	2558	2555	2555	7.1e-17	-1.7e-16	-1.8e-16	4.2e-22	0.0596	0.0514	1.72
73	1406	1513	2704	2701	2700	1.7e-16	6.8e-17	-2.9e-16	5.8e-22	0.0575	0.0495	1.72
75	1482	1595	2852	2849	2849	1.5e-16	-1.7e-16	-3.0e-16	6.2e-22	0.0569	0.0480	1.69
77	1560	1680	3006	3003	3002	2.6e-16	-2.0e-16	-1.0e-16	7.1e-22	0.0555	0.0463	1.67
79	1640	1766	3162	3159	3159	2.9e-16	2.2e-16	-2.1e-16	9.8e-22	0.0533	0.0452	1.70
81	1722	1855	3324	3321	3320	2.8e-16	2.1e-16	-2.1e-16	1.2e-21	0.0533	0.0446	1.67
83	1806	1946	3488	3485	3485	2.8e-16	-3.0e-16	-2.1e-16	1.2e-21	0.0506	0.0436	1.72
85	1892	2039	3658	3655	3654	-1.2e-17	9.3e-18	-5.1e-16	1.5e-21	0.0499	0.0419	1.68
87	1980	2135	3830	3827	3827	-1.8e-16	-1.5e-16	-4.6e-16	1.5e-21	0.0490	0.0416	1.70
89	2070	2232	4008	4005	4004	2.5e-16	-3.2e-16	-2.9e-16	1.7e-21	0.0473	0.0405	1.71
91	2162	2332	4188	4185	4185	-6.1e-17	2.3e-16	-3.3e-16	2.0e-21	0.0466	0.0395	1.70
93	2256	2434	4374	4371	4370	-2.6e-16	8.5e-17	-5.5e-16	2.2e-21	0.0460	0.0389	1.69
95	2352	2538	4562	4559	4559	8.8e-17	-1.4e-16	-6.0e-16	2.5e-21	0.0449	0.0380	1.69
97	2450	2644	4756	4753	4752	-1.3e-16	-1.6e-16	-5.7e-16	2.9e-21	0.0439	0.0371	1.69
99	2550	2753	4952	4949	4949	-1.6e-16	-2.3e-16	-5.8e-16	3.1e-21	0.0425	0.0365	1.72
101	2652	2863	5154	5151	5150	2.0e-16	-3.2e-16	-5.6e-16	3.3e-21	0.0422	0.0353	1.67
103	2756	2976	5358	5355	5355	-6.0e-17	1.3e-16	-6.2e-16	4.0e-21	0.0410	0.0349	1.70
105	2862	3091	5568	5565	5564	-2.2e-16	4.5e-17	-7.2e-16	4.6e-21	0.0403	0.0342	1.70
107	2970	3208	5780	5777	5777	1.9e-16	-1.5e-16	-8.4e-16	5.3e-21	0.0397	0.0341	1.72
109	3080	3327	5998	5995	5994	-4.1e-16	2.5e-16	-8.6e-16	5.9e-21	0.0386	0.0329	1.71

Table 5 Symmetric spherical t-designs on $\mathbb{S}^2$ with $N = \overline{N}(2,t)$ and odd degrees $t = 1 - 109$, except for $t = 1$ when $N = \overline{N}(2,t) - 2$ and $t = 7, 11$ when $N = \overline{N}(2,t) + 2$

t	$N^*(2,t)$	$N^+(2,t)$	N	n	m	$V_{t,N,\psi_1}(X_N)$	$V_{t,N,\psi_2}(X_N)$	$V_{t,N,\psi_3}(X_N)$	$\boldsymbol{r}(X_N)^T\boldsymbol{r}(X_N)$	$\delta(X_N)$	$h(X_N)$	$\rho(X_N)$
111	3192	3449	6218	6215	6215	1.3e-16	-1.9e-16	-6.7e-16	6.6e-21	0.0376	0.0324	1.72
113	3306	3573	6444	6441	6440	1.1e-16	-1.5e-16	-7.2e-16	7.2e-21	0.0377	0.0318	1.69
115	3422	3698	6672	6669	6669	2.2e-16	2.0e-16	-6.7e-16	8.8e-21	0.0375	0.0316	1.69
117	3540	3826	6906	6903	6902	-3.7e-16	1.6e-17	-5.6e-16	8.6e-21	0.0358	0.0309	1.73
119	3660	3957	7142	7139	7139	-4.0e-16	2.0e-16	-9.7e-16	9.5e-21	0.0356	0.0303	1.71
121	3782	4089	7384	7381	7380	-5.0e-16	-9.3e-17	-8.7e-16	1.1e-20	0.0351	0.0297	1.69
123	3906	4224	7628	7625	7625	1.7e-16	1.3e-16	-7.7e-16	1.2e-20	0.0349	0.0292	1.67
125	4032	4360	7878	7875	7874	-3.1e-17	-2.5e-16	-6.0e-16	1.4e-20	0.0339	0.0289	1.71
127	4160	4499	8130	8127	8127	-1.1e-16	-7.2e-18	-9.6e-16	1.5e-20	0.0335	0.0288	1.72
129	4290	4641	8388	8385	8384	6.9e-16	-6.4e-16	-1.0e-15	1.5e-20	0.0328	0.0277	1.69
131	4422	4784	8648	8645	8645	2.8e-16	-1.6e-16	-9.6e-16	1.7e-20	0.0320	0.0274	1.71
133	4556	4929	8914	8911	8910	2.3e-16	1.3e-16	-9.0e-16	2.0e-20	0.0317	0.0269	1.70
135	4692	5077	9182	9179	9179	4.3e-16	-2.8e-16	-1.0e-15	2.0e-20	0.0318	0.0267	1.68
137	4830	5227	9456	9453	9452	5.3e-16	2.7e-16	-9.0e-16	2.4e-20	0.0308	0.0261	1.70
139	4970	5379	9732	9729	9729	2.0e-16	-3.0e-16	-8.7e-16	2.4e-20	0.0303	0.0259	1.71
141	5112	5533	10014	10011	10010	6.9e-17	-3.8e-16	-1.1e-15	2.8e-20	0.0300	0.0254	1.69
143	5256	5689	10298	10295	10295	2.2e-16	6.0e-16	-1.0e-15	3.0e-20	0.0302	0.0255	1.69
145	5402	5848	10588	10585	10584	-6.3e-16	1.4e-16	-6.7e-16	3.3e-20	0.0289	0.0248	1.72
147	5550	6009	10880	10877	10877	6.5e-16	-4.2e-16	-1.2e-15	3.5e-20	0.0286	0.0246	1.72
149	5700	6172	11178	11175	11174	-3.3e-16	-3.6e-16	-8.7e-16	3.7e-20	0.0287	0.0241	1.68
151	5852	6337	11478	11475	11475	1.3e-16	5.5e-16	-1.0e-15	4.2e-20	0.0283	0.0239	1.69
153	6006	6504	11784	11781	11780	-7.8e-16	4.8e-16	-1.1e-15	4.5e-20	0.0274	0.0235	1.71
155	6162	6674	12092	12089	12089	-4.8e-16	2.7e-16	-1.0e-15	4.8e-20	0.0275	0.0234	1.70
157	6320	6845	12406	12403	12402	4.3e-16	-2.4e-16	-1.1e-15	5.1e-20	0.0269	0.0232	1.72
159	6480	7019	12722	12719	12719	8.6e-17	-5.6e-16	-1.3e-15	5.5e-20	0.0267	0.0229	1.71
161	6642	7195	13044	13041	13040	-7.8e-16	2.3e-16	-1.2e-15	5.9e-20	0.0267	0.0225	1.69
163	6806	7373	13368	13365	13365	1.8e-16	-6.5e-16	-1.3e-15	6.4e-20	0.0262	0.0222	1.69
165	6972	7553	13698	13695	13694	-6.3e-16	-3.5e-16	-1.3e-15	7.0e-20	0.0257	0.0217	1.69
167	7140	7736	14030	14027	14027	-9.0e-16	5.6e-16	-1.3e-15	7.1e-20	0.0255	0.0216	1.69
169	7310	7921	14368	14365	14364	-1.8e-16	-7.8e-16	-1.2e-15	8.4e-20	0.0253	0.0214	1.69
171	7482	8108	14708	14705	14705	-1.3e-16	6.2e-16	-1.2e-15	8.6e-20	0.0248	0.0210	1.70
173	7656	8297	15054	15051	15050	3.9e-16	3.6e-16	-1.6e-15	9.3e-20	0.0246	0.0208	1.69
175	7832	8488	15402	15399	15399	1.9e-16	-7.7e-17	-1.3e-15	1.0e-19	0.0247	0.0204	1.65
177	8010	8681	15756	15753	15752	3.5e-17	2.2e-16	-1.3e-15	1.1e-19	0.0242	0.0205	1.69
179	8190	8877	16112	16109	16109	-2.0e-16	2.8e-16	-1.3e-15	1.1e-19	0.0236	0.0202	1.71
181	8372	9075	16474	16471	16470	-7.4e-16	-4.6e-17	-1.4e-15	1.2e-19	0.0234	0.0197	1.69
183	8556	9275	16838	16835	16835	1.2e-16	4.5e-17	-1.4e-15	1.3e-19	0.0240	0.0198	1.65
185	8742	9477	17208	17205	17204	-2.8e-17	-2.3e-16	-1.3e-15	1.3e-19	0.0224	0.0193	1.73
187	8930	9681	17580	17577	17577	-1.1e-16	-7.5e-16	-1.8e-15	8.2e-20	0.0232	0.0192	1.65
189	9120	9888	17958	17955	17954	-1.8e-16	-5.4e-16	-1.2e-15	1.6e-19	0.0222	0.0190	1.71
191	9312	10096	18338	18335	18335	2.6e-16	-2.5e-16	-1.3e-15	1.6e-19	0.0221	0.0188	1.70
193	9506	10307	18724	18721	18720	1.2e-15	-9.4e-16	-1.4e-15	1.8e-19	0.0223	0.0187	1.68
195	9702	10520	19112	19109	19109	-7.0e-16	5.6e-16	-1.4e-15	1.9e-19	0.0219	0.0187	1.71
197	9900	10736	19506	19503	19502	6.4e-16	-1.2e-16	-1.2e-15	2.0e-19	0.0218	0.0182	1.66
199	10100	10953	19902	19899	19899	-6.3e-16	-3.4e-16	-1.3e-15	2.1e-19	0.0217	0.0179	1.65
201	10302	11173	20304	20301	20300	1.2e-15	-2.6e-16	-1.2e-15	2.3e-19	0.0210	0.0178	1.70
203	10506	11394	20708	20705	20705	5.0e-16	-3.9e-16	-1.2e-15	2.4e-19	0.0209	0.0178	1.70
205	10712	11618	21118	21115	21114	-6.3e-17	-5.8e-17	-1.4e-15	2.5e-19	0.0206	0.0176	1.70
207	10920	11844	21530	21527	21527	2.5e-16	-5.6e-16	-1.5e-15	2.6e-19	0.0202	0.0174	1.72
209	11130	12073	21948	21945	21944	-1.1e-16	-1.0e-15	-1.2e-15	2.9e-19	0.0201	0.0172	1.71
211	11342	12303	22368	22365	22365	9.1e-16	-8.9e-16	-1.4e-15	3.1e-19	0.0201	0.0171	1.70
213	11556	12536	22794	22791	22790	-5.1e-16	-5.9e-16	-1.3e-15	3.3e-19	0.0195	0.0168	1.71
215	11772	12771	23222	23219	23219	-3.4e-16	6.3e-16	-1.4e-15	3.5e-19	0.0201	0.0168	1.67
217	11990	13008	23656	23653	23652	9.8e-17	2.0e-16	-1.5e-15	3.6e-19	0.0196	0.0166	1.69
219	12210	13247	24092	24089	24089	2.3e-17	-3.1e-16	-1.4e-15	3.9e-19	0.0194	0.0166	1.71

Table 6 Symmetric spherical t-designs on $\mathbb{S}^2$ with $N = \overline{N}(2,t)$ and odd degrees $t = 111 - 219$

t	$N^*(2,t)$	$N^+(2,t)$	N	n	m	$V_{t,N,\psi_1}(X_N)$	$V_{t,N,\psi_2}(X_N)$	$V_{t,N,\psi_3}(X_N)$	$\boldsymbol{r}(X_N)^T\boldsymbol{r}(X_N)$	$\delta(X_N)$	$h(X_N)$	$\rho(X_N)$
221	12432	13488	24534	24531	24530	-1.4e-15	2.6e-16	-1.4e-15	4.0e-19	0.0194	0.0163	1.68
223	12656	13732	24978	24975	24975	-8.1e-17	-1.6e-16	-1.3e-15	4.4e-19	0.0192	0.0164	1.71
225	12882	13978	25428	25425	25424	-4.9e-16	-5.3e-16	-1.3e-15	4.5e-19	0.0188	0.0160	1.70
227	13110	14226	25880	25877	25877	1.5e-16	6.4e-16	-1.4e-15	4.7e-19	0.0185	0.0158	1.72
229	13340	14476	26338	26335	26334	-1.0e-15	1.8e-16	-1.5e-15	5.1e-19	0.0183	0.0156	1.71
231	13572	14728	26798	26795	26795	3.6e-17	-5.2e-16	-1.4e-15	5.3e-19	0.0181	0.0157	1.74
233	13806	14982	27264	27261	27260	-6.2e-16	-2.4e-16	-1.5e-15	5.7e-19	0.0185	0.0155	1.68
235	14042	15239	27732	27729	27729	-9.6e-16	7.0e-17	-1.4e-15	5.9e-19	0.0182	0.0153	1.69
237	14280	15498	28206	28203	28202	1.1e-15	3.9e-16	-1.5e-15	6.1e-19	0.0178	0.0154	1.73
239	14520	15759	28682	28679	28679	-1.5e-15	-3.9e-16	-1.5e-15	6.4e-19	0.0176	0.0152	1.72
241	14762	16022	29164	29161	29160	-1.5e-15	3.8e-16	-1.4e-15	6.7e-19	0.0180	0.0153	1.70
243	15006	16287	29648	29645	29645	8.7e-16	1.9e-16	-1.4e-15	7.2e-19	0.0171	0.0148	1.73
245	15252	16555	30138	30135	30134	8.6e-16	-4.1e-16	-1.5e-15	7.4e-19	0.0174	0.0146	1.67
247	15500	16825	30630	30627	30627	6.9e-16	-4.2e-16	-1.5e-15	7.9e-19	0.0173	0.0146	1.69
249	15750	17097	31128	31125	31124	-5.1e-16	-2.6e-16	-1.6e-15	8.2e-19	0.0168	0.0146	1.74
251	16002	17371	31628	31625	31625	-1.1e-15	4.7e-16	-1.6e-15	8.5e-19	0.0167	0.0143	1.71
253	16256	17647	32134	32131	32130	9.8e-16	4.7e-16	-1.4e-15	9.2e-19	0.0170	0.0141	1.66
255	16512	17925	32642	32639	32639	-2.5e-16	-8.6e-18	-1.8e-15	9.5e-19	0.0165	0.0142	1.72
257	16770	18206	33156	33153	33152	-1.6e-15	6.6e-16	-1.7e-15	1.0e-18	0.0166	0.0141	1.70
259	17030	18489	33672	33669	33669	-5.0e-16	-6.6e-16	-1.5e-15	1.1e-18	0.0163	0.0142	1.74
261	17292	18774	34194	34191	34190	-9.0e-17	-9.6e-16	-1.6e-15	1.1e-18	0.0165	0.0139	1.69
263	17556	19061	34718	34715	34715	7.3e-16	1.5e-15	-1.6e-15	1.1e-18	0.0160	0.0136	1.70
265	17822	19350	35248	35245	35244	9.8e-16	-7.8e-16	-1.5e-15	1.2e-18	0.0162	0.0138	1.70
267	18090	19642	35780	35777	35777	1.7e-16	7.5e-16	-1.7e-15	1.3e-18	0.0154	0.0134	1.74
269	18360	19935	36318	36315	36314	-9.0e-16	-1.2e-15	-1.7e-15	1.3e-18	0.0160	0.0134	1.68
271	18632	20231	36858	36855	36855	1.3e-15	1.5e-15	-1.9e-15	1.3e-18	0.0157	0.0137	1.75
273	18906	20529	37404	37401	37400	-1.7e-15	5.4e-16	-1.7e-15	1.4e-18	0.0152	0.0132	1.73
275	19182	20830	37952	37949	37949	1.6e-15	1.7e-16	-1.6e-15	1.5e-18	0.0152	0.0132	1.74
277	19460	21132	38506	38503	38502	-4.1e-17	5.6e-16	-1.8e-15	1.6e-18	0.0153	0.0130	1.70
279	19740	21437	39062	39059	39059	-9.5e-16	-5.9e-16	-1.9e-15	1.6e-18	0.0152	0.0132	1.75
281	20022	21743	39624	39621	39620	-2.9e-16	-1.4e-15	-1.9e-15	1.7e-18	0.0150	0.0129	1.72
283	20306	22052	40188	40185	40185	1.7e-15	1.3e-15	-1.9e-15	1.8e-18	0.0150	0.0126	1.69
285	20592	22363	40758	40755	40754	-8.5e-16	-1.6e-15	-2.1e-15	1.8e-18	0.0148	0.0127	1.71
287	20880	22677	41330	41327	41327	6.9e-16	-9.4e-16	-1.5e-15	2.0e-18	0.0149	0.0126	1.70
289	21170	22992	41908	41905	41904	-1.1e-15	-6.1e-16	-1.8e-15	2.0e-18	0.0148	0.0126	1.71
291	21462	23310	42488	42485	42485	-4.0e-17	-1.5e-15	-1.9e-15	2.2e-18	0.0149	0.0124	1.66
293	21756	23630	43074	43071	43070	9.5e-16	5.9e-16	-1.9e-15	2.2e-18	0.0142	0.0123	1.74
295	22052	23952	43662	43659	43659	1.7e-15	1.2e-15	-1.5e-15	2.3e-18	0.0143	0.0124	1.74
297	22350	24276	44256	44253	44252	-9.0e-17	7.9e-16	-1.6e-15	2.4e-18	0.0143	0.0121	1.70
299	22650	24602	44852	44849	44849	-8.3e-16	1.4e-15	-1.7e-15	2.5e-18	0.0141	0.0121	1.72
301	22952	24931	45454	45451	45450	5.8e-16	3.6e-17	-1.8e-15	2.6e-18	0.0140	0.0120	1.70
303	23256	25262	46058	46055	46055	3.0e-16	1.0e-15	-1.8e-15	2.7e-18	0.0140	0.0118	1.70
305	23562	25595	46668	46665	46664	5.8e-16	-1.2e-15	-1.9e-15	2.7e-18	0.0140	0.0118	1.68
307	23870	25930	47280	47277	47277	-1.0e-15	-1.4e-15	-1.7e-15	2.9e-18	0.0139	0.0117	1.69
309	24180	26267	47898	47895	47894	-1.8e-15	1.4e-15	-1.6e-15	2.9e-18	0.0136	0.0118	1.73
311	24492	26607	48518	48515	48515	5.9e-16	2.3e-16	-2.0e-15	3.2e-18	0.0135	0.0116	1.72
313	24806	26948	49144	49141	49140	1.6e-15	-9.6e-16	-1.8e-15	3.3e-18	0.0139	0.0115	1.66
315	25122	27292	49772	49769	49769	-3.8e-17	1.0e-15	-2.0e-15	3.4e-18	0.0133	0.0114	1.72
317	25440	27638	50406	50403	50402	-1.2e-15	9.1e-18	-1.8e-15	3.5e-18	0.0134	0.0114	1.69
319	25760	27986	51042	51039	51039	7.8e-16	1.2e-15	-1.7e-15	3.6e-18	0.0134	0.0112	1.67
321	26082	28337	51684	51681	51680	-2.0e-15	1.2e-15	-1.7e-15	3.7e-18	0.0133	0.0112	1.69
323	26406	28689	52328	52325	52325	-3.2e-16	-1.2e-15	-1.7e-15	3.9e-18	0.0131	0.0115	1.76
325	26732	29044	52978	52975	52974	1.2e-15	-1.7e-15	-1.7e-15	4.0e-18	0.0124	0.0110	1.77

Table 7 Symmetric spherical t-designs on $\mathbb{S}^2$ with $N = \overline{N}(2,t)$ and odd degrees $t = 221 - 325$

t	$N^*(3,t)$	$N^+(3,t)$	$\widehat{N}(3,t)$	N	sym	$V_{t,N,\psi_1}(X_N)$	$V_{t,N,\psi_2}(X_N)$	$V_{t,N,\psi_3}(X_N)$	$\delta(X_N)$	$h(X_N)$	$\rho(X_N)$
1	2	2	4	2	1	5.6e-17	0.0e+00	0.0e+00	3.1416	1.5708	1.00
2	5	5	7	5	0	8.4e-17	4.4e-17	1.7e-16	1.8235	1.3181	1.45
3	8	8	12	8	1	1.5e-17	-1.1e-16	-2.6e-17	1.5708	1.0472	1.33
3	8	8	12	16	1	-1.4e-17	-9.6e-17	-1.5e-16	1.0472	1.0472	2.00
5	20	19	32	24	1	-4.7e-17	1.3e-16	-6.8e-17	1.0472	0.7854	1.50
7	40	40	70	48	1	2.8e-17	-1.2e-16	1.8e-16	0.7854	0.6086	1.55
11	112	117	219	120	1	-1.7e-17	-5.6e-17	-5.7e-16	0.6283	0.3881	1.24
11	112	117	219	600	1	2.9e-17	-3.7e-17	-8.1e-17	0.2709	0.3881	2.87

Table 8 Regular spherical t-designs on $\mathbb{S}^3$ for degrees t = 1, 2, 3, 5 ,7 ,11

t	$N^*(3,t)$	$N^+(3,t)$	N	n	m	$V_{t,N,\psi_1}(X_N)$	$V_{t,N,\psi_2}(X_N)$	$V_{t,N,\psi_3}(X_N)$	$\delta(X_N)$	$h(X_N)$	$\rho(X_N)$
1	2	2	4	6	4	6.2e-17	0.0e+00	0.0e+00	1.5708	1.5708	2.00
2	5	5	7	15	13	6.3e-17	5.6e-17	1.0e-16	1.3585	1.1683	1.72
3	8	8	12	30	29	8.1e-18	-9.1e-17	-7.9e-17	1.2311	1.0016	1.63
4	14	13	20	54	54	1.4e-19	5.6e-17	-2.4e-17	0.9414	0.8338	1.77
5	20	19	32	90	90	-3.0e-17	1.3e-16	2.6e-17	0.7816	0.7041	1.80
6	30	28	49	141	139	1.1e-17	1.0e-17	-9.2e-18	0.6883	0.6086	1.77
7	40	40	70	204	203	2.1e-17	-1.2e-16	-1.8e-16	0.5765	0.5454	1.89
8	55	54	97	285	284	-3.2e-17	1.6e-16	-1.5e-17	0.5028	0.4837	1.92
9	70	71	130	384	384	-1.4e-17	7.1e-17	-5.9e-17	0.4688	0.4467	1.91
10	91	92	171	507	505	1.2e-17	6.1e-17	-1.8e-16	0.4404	0.4082	1.85
11	112	117	219	651	649	4.1e-18	-5.2e-17	-2.9e-16	0.3809	0.3748	1.97
12	140	145	275	819	818	3.4e-17	-4.2e-17	-2.2e-16	0.3467	0.3409	1.97
13	168	178	340	1014	1014	3.7e-17	5.6e-17	-3.9e-17	0.3328	0.3225	1.94
14	204	216	415	1239	1239	4.9e-18	-2.0e-17	-1.5e-16	0.3111	0.2982	1.92
15	240	258	501	1497	1495	2.3e-18	1.6e-16	-9.1e-18	0.2909	0.2898	1.99
16	285	306	597	1785	1784	-1.5e-17	-7.6e-17	-1.1e-16	0.2673	0.2616	1.96
17	330	360	705	2109	2108	-3.0e-17	7.9e-17	-4.3e-17	0.2535	0.2507	1.98
18	385	419	825	2469	2469	1.2e-16	1.2e-16	1.1e-15	0.2386	0.2383	2.00
19	440	485	959	2871	2869	3.7e-17	5.5e-18	-6.0e-17	0.2283	0.2267	1.99
20	506	557	1106	3312	3310	1.9e-17	1.0e-16	1.2e-16	0.2163	0.2162	2.00

Table 9 Computed spherical t-designs on $\mathbb{S}^3$ for degrees t = 1,...,20, with $N = \widehat{N}(3,t)$

t	$N^*(3,t)$	$N^+(3,t)$	N	n	m	$V_{t,N,\psi_1}(X_N)$	$V_{t,N,\psi_2}(X_N)$	$V_{t,N,\psi_3}(X_N)$	$\delta(X_N)$	$h(X_N)$	$\rho(X_N)$
1	2	2	4	6	4	6.2e-17	0.0e+00	0.0e+00	1.5708	1.5708	2.00
3	8	8	10	30	29	-2.4e-17	-1.1e-16	-2.6e-16	1.3181	0.9776	1.48
5	20	19	28	90	90	-2.0e-17	1.3e-16	9.2e-17	0.8334	0.7303	1.75
7	40	40	60	204	203	5.7e-18	-1.3e-16	-1.1e-16	0.6324	0.5656	1.79
9	70	71	114	384	384	-2.7e-17	6.4e-17	-1.4e-16	0.4863	0.4548	1.87
11	112	117	194	651	649	1.7e-17	-4.7e-17	-2.2e-16	0.4126	0.3860	1.87
13	168	178	308	1014	1014	3.3e-17	5.5e-17	-1.1e-16	0.3454	0.3220	1.87
15	240	258	458	1497	1495	1.8e-18	1.6e-16	-2.7e-17	0.2914	0.2877	1.98
17	330	360	650	2109	2108	-2.9e-17	8.0e-17	-7.4e-17	0.2649	0.2584	1.95
19	440	485	890	2871	2869	3.5e-17	5.4e-18	-5.5e-17	0.2388	0.2380	1.99
21	572	636	1184	3795	3794	1.0e-17	1.4e-16	4.6e-17	0.2139	0.2113	1.98
23	728	816	1538	4899	4899	1.0e-17	-1.9e-16	-2.5e-16	0.1999	0.1951	1.95
25	910	1027	1954	6201	6200	-1.1e-17	5.3e-17	-2.0e-17	0.1795	0.1778	1.98
27	1120	1272	2440	7713	7713	-7.2e-18	8.9e-17	6.9e-17	0.1586	0.1678	2.12
29	1360	1553	3000	9456	9454	1.2e-15	3.4e-16	3.5e-14	0.1528	0.1565	2.05
31	1632	1872	3642	11439	11439	3.0e-16	-2.0e-17	9.5e-15	0.1438	0.1474	2.05

Table 10 Computed symmetric spherical t-designs on $\mathbb{S}^3$ for degrees t = 1,3,...,31, with $N = \overline{N}(3,t)$