

Numerical construction of spherical t -designs by Barzilai-Borwein method [☆]

Congpei An ^{a,*}, Yuchen Xiao ^b

^a School of Economic Mathematics, Southwestern University of Finance and Economics, Liutai Avenue, Chengdu 611130, China

^b Department of Mathematics, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong SAR China



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ABSTRACT

A point set X_N on the unit sphere is a spherical t -design is equivalent to the nonnegative quantity $A_{N,t+1}$ vanished. We show that if X_N is a stationary point set of $A_{N,t+1}$ and the minimal singular value of basis matrix is positive, then X_N is a spherical t -design. Moreover, the numerical construction of spherical t -designs is valid by using Barzilai-Borwein method. We obtain numerical spherical t -designs with $N = (t + 2)^2$ points for $t + 1$ up to 127.

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1. Introduction

Distributing finite points on the unit sphere is a challenging problem in the 21st century [17]. Spherical t -design is to find the “good” finite sets of points on the unit sphere $\mathbb{S}^d := \{\mathbf{x} \in \mathbb{R}^{d+1} \mid \|\mathbf{x}\| = 1\}$ for spherical polynomial approximations, where $\|\cdot\|$ is the Euclidean 2-norm. Spherical t -design is very useful in approximation theory, geometry and combinatorics. Recently, it has been applied in quantum mechanics (for quantum t -design) and statistics (for rotatable design) [20].

Definition 1.1. A finite set $X_N := \{\mathbf{x}_1, \dots, \mathbf{x}_N\} \subset \mathbb{S}^d$ is a spherical t -design if for any polynomial $p : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$ of degree at most t such that the average value of p on the X_N equals the average value of p on \mathbb{S}^d , i.e.,

$$\frac{1}{N} \sum_{i=1}^N p(\mathbf{x}_i) = \frac{1}{|\mathbb{S}^d|} \int_{\mathbb{S}^d} p(\mathbf{x}) d\omega(\mathbf{x}) \quad \forall p \in \Pi_t, \quad (1)$$

where $|\mathbb{S}^d|$ is the surface of the whole unit sphere \mathbb{S}^d , $\Pi_t := \Pi_t(\mathbb{S}^d)$ is the space of spherical polynomials on \mathbb{S}^d with degree at most t and $d\omega(\mathbf{x})$ denotes the surface measure on \mathbb{S}^d .

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* Corresponding author.

E-mail addresses: andbachcp@gmail.com, ancp@swufe.edu.cn (C. An), petyercxiao@gmail.com (Y. Xiao).

The concept of spherical t -design was introduced by Delsarte et al. [11] in 1977. From then on, spherical t -designs have been studied extensively [15,8,4,16,3,9,1,7]. In this paper, we pay attention to 2-dimensional unit sphere \mathbb{S}^2 .

A lower bound on the number of points N to construct a spherical t -design for any $t \geq 1$ on \mathbb{S}^2 was given in [11]:

$$N \geq N^* = \begin{cases} \frac{1}{4}(t+1)(t+3), & t \text{ is odd,} \\ \frac{1}{4}(t+2)^2, & t \text{ is even.} \end{cases}$$

When a spherical t -design achieves the lower bound, we call it a tight spherical t -design. However, there is no tight spherical t -design with N^* points except $t = 1, 2, 3, 4, 5, 7, 11$ [4]. Bondarenko et al. [7] proved spherical t -designs exist for $O(t^2)$ points. From the work of Chen et al. [9], we know that spherical t -designs with $(t+1)^2$ points exist for all degrees t up to 100 on \mathbb{S}^2 . This encourages us to find higher degrees t for spherical t -designs.

Extremal systems are sets of $N = (t+1)^2$ points on \mathbb{S}^2 which maximize the determinant of a basis matrix for an arbitrary basis of Π_t [15]. For $N = (t+1)^2$, Chen and Womersley verified a spherical t -design exist in a neighborhood of an extremal system [8]. For $N \geq (t+1)^2$, An et al. [3] verified extremal spherical t -designs exist for all degrees t up to 60 and provided well conditioned spherical t -designs for interpolation and numerical integration.

By now, numerical methods have been developed for finding spherical t -designs. The problem of finding a spherical t -design is expressed as solving nonlinear equations or optimization problems [16,3]. However, the first order methods for computing spherical t -designs are rarely developed. In this paper, we numerically construct spherical t -designs by using Barzilai-Borwein method (BB method). The BB method [5] is a gradient method with modified step sizes, which is motivated by Newton’s method but not involving any Hessian. Further investigations [10] showed that BB method is locally R -linear convergent for general objective functions.

In the next section, we present the required techniques, definitions and first order conditions for spherical t -designs. The BB method for computing spherical t -designs and its convergence analysis are presented in Section 3. Numerical results for point sets which $t+1$ up to 127 and $N = (t+2)^2 = 16384$ are included in Section 4. Section 5 ends this paper with a brief conclusion.

2. First order conditions for spherical t -design

$\{Y_0^1, Y_1^1, \dots, Y_t^{2t+1}\}$ for degree $n = 0, \dots, t$ and order $k = 1, \dots, 2n+1$ is a complete set of orthonormal real spherical harmonics basis for Π_t , where orthogonality with respect to the L_2 inner product [12],

$$\langle f, g \rangle_{L_2} := \int_{\mathbb{S}^2} f(\mathbf{x})g(\mathbf{x}) d\omega(\mathbf{x}), \quad f, g \in L_2(\mathbb{S}^2). \tag{2}$$

Note that $Y_0^1 = \frac{1}{\sqrt{4\pi}}$. It is well known that the addition theorem [12] for spherical harmonics on \mathbb{S}^2 gives

$$\sum_{k=1}^{2n+1} Y_n^k(\mathbf{x})Y_n^k(\mathbf{y}) = \frac{2n+1}{4\pi} P_n(\langle \mathbf{x}, \mathbf{y} \rangle) \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{S}^2, \tag{3}$$

where $P_n : [-1, 1] \rightarrow \mathbb{R}$ is Legendre polynomial and $\langle \mathbf{x}, \mathbf{y} \rangle := \mathbf{x}^\top \mathbf{y}$ is the inner product in \mathbb{R}^3 . Sloan and Womersley [16] introduced a variational characterization of spherical t -designs

$$A_{N,t}(X_N) := \frac{4\pi}{N^2} \sum_{n=1}^t \sum_{k=1}^{2n+1} \left(\sum_{i=1}^N Y_n^k(\mathbf{x}_i) \right)^2 = \frac{4\pi}{N^2} \sum_{j=1}^N \sum_{i=1}^N \sum_{n=1}^t \frac{2n+1}{4\pi} P_n(\langle \mathbf{x}_j, \mathbf{x}_i \rangle). \tag{4}$$

Theorem 2.1 ([16]). *Let $t \geq 1$, and $X_N \subset \mathbb{S}^2$. Then*

$$0 \leq A_{N,t}(X_N) \leq (t+1)^2 - 1, \tag{5}$$

and X_N is a spherical t -design if and only if

$$A_{N,t}(X_N) = 0.$$

It is known that X_N is a spherical t -design if and only if $A_{N,t}(X_N)$ vanished. Naturally, one might consider the first order condition to check the global minimizer of $A_{N,t}(X_N)$.

Definition 2.1. A point \mathbf{x} is a stationary point of $f \in C^1(\mathbb{S}^2)$ if $\nabla^* f(\mathbf{x}) = 0$, where $\nabla^* := \nabla_{\mathbb{S}^2}^*$ is the spherical gradient (or surface gradient [12]) of f .

Let the basis matrix be $\mathbf{Y}_t(X_N) := \begin{bmatrix} \frac{1}{\sqrt{4\pi}} \mathbf{e}^\top \\ \mathbf{Y}_t^0(X_N) \end{bmatrix} \in \mathbb{R}^{(t+1)^2 \times N}$, where $\mathbf{e} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^N$ and

$$\mathbf{Y}_t^0(X_N) = \begin{bmatrix} Y_1^1(\mathbf{x}_1) & \cdots & Y_1^1(\mathbf{x}_N) \\ \vdots & \ddots & \vdots \\ Y_t^{2t+1}(\mathbf{x}_1) & \cdots & Y_t^{2t+1}(\mathbf{x}_N) \end{bmatrix} \in \mathbb{R}^{(t^2+2t) \times N}.$$

Definition 2.2. A finite set $X_N := \{\mathbf{x}_1, \dots, \mathbf{x}_N\} \subset \mathbb{S}^2$ is called a fundamental system for Π_t if the zero polynomial is the only element of Π_t that vanishes at each point in X_N .

An et al. [3] described the fundamental system in finding spherical t -designs.

Lemma 2.2 ([3]). A set $X_N \subset \mathbb{S}^2$ is a fundamental system for Π_t if and only if $\mathbf{Y}_t(X_N)$ is of full row rank $(t + 1)^2$.

Lemma 2.3 ([3]). Let $t \geq 2$ and $N \geq (t + 2)^2$. Assume $X_N \subset \mathbb{S}^2$ is a stationary point set of $A_{N,t}$ and X_N is a fundamental system for Π_{t+1} . Then X_N is a spherical t -design.

Based on these results, we have the applicable first order condition for spherical t -designs as follows.

Theorem 2.4. Let $t \geq 2$ and $N \geq (t + 2)^2$. Assume $X_N \subset \mathbb{S}^2$ is a stationary point set of $A_{N,t}$ and the minimal singular value of basis matrix $\mathbf{Y}_{t+1}(X_N)$ is positive. Then X_N is a spherical t -design.

Proof. Suppose that the minimal singular value of $\mathbf{Y}_{t+1}(X_N)$ is positive, then we have all the singular values of $\mathbf{Y}_{t+1}(X_N)$ are positive immediately. We know that the number of non-zero singular values of $\mathbf{Y}_{t+1}(X_N)$ equals the rank of $\mathbf{Y}_{t+1}(X_N)$, so $\mathbf{Y}_{t+1}(X_N)$ is of full rank, which means X_N is a fundamental system of Π_{t+1} by Lemma 2.2. And then suppose that X_N is a stationary point set, then X_N is a spherical t -design by Lemma 2.3. Hence, we complete the proof. \square

Theorem 2.4 is useful in first order optimization method, which provides a simple way to verify the global minimizer to the objective function.

3. Iterative methods for finding spherical t -designs

3.1. Algorithm design

Fix N and t , for objective function $A_{N,t} : \mathbb{S}^{2 \times N} \rightarrow \mathbb{R}$, we consider the optimization problem

$$\min_{X_N \subset \mathbb{S}^2} A_{N,t}(X_N). \tag{6}$$

Apparently, $A_{N,t}$ is a non-convex and differentiable function. For computing X_N conveniently, we assume the first point $\mathbf{x}_1 = (0, 0, 1)^\top$ is the north pole point and the second point $\mathbf{x}_2 = (x_2, 0, z_2)^\top$ is on the primer meridian. Then we can define coordinates convert functions $\eta : \mathbb{R}^{3 \times N} \rightarrow \mathbb{R}^{2N-3}$ which can convert Cartesian coordinates into spherical coordinates as a vector, and $\mu : \mathbb{R}^{2N-3} \rightarrow \mathbb{R}^{3 \times N}$ which can convert a vector form spherical coordinates into Cartesian coordinates as a matrix. So for $(\theta, \phi) \in [0, \pi] \times [0, 2\pi)$ we have

$$\eta(X_N) = \eta \begin{bmatrix} \sin \theta_1 \cos \phi_1 & \cdots & \sin \theta_N \cos \phi_N \\ \sin \theta_1 \sin \phi_1 & \cdots & \sin \theta_N \sin \phi_N \\ \cos \theta_1 & \cdots & \cos \theta_N \end{bmatrix} = (\Theta, \Phi),$$

$$\mu(\eta(X_N)) = \mu(\Theta, \Phi) = \begin{bmatrix} x_1 & \cdots & x_N \\ y_1 & \cdots & y_N \\ z_1 & \cdots & z_N \end{bmatrix},$$

where vector $\Theta := (\theta_2, \dots, \theta_N)^\top \in \mathbb{R}^{N-1}$ and vector $\Phi := (\phi_3, \dots, \phi_N)^\top \in \mathbb{R}^{N-2}$.

We apply BB method (with the step size (6) in [5]) to construct Algorithm 1 for seeking an efficient way to compute $A_{N,t}$, that x achieves the local minimum. Due to the universality of quasi-Newton method [16], we also apply quasi-Newton method for comparing the efficiency. And then we try to use Theorem 2.4 to prove the local minimum we found is the global minimum, that is, we find the real numerical spherical t -design.

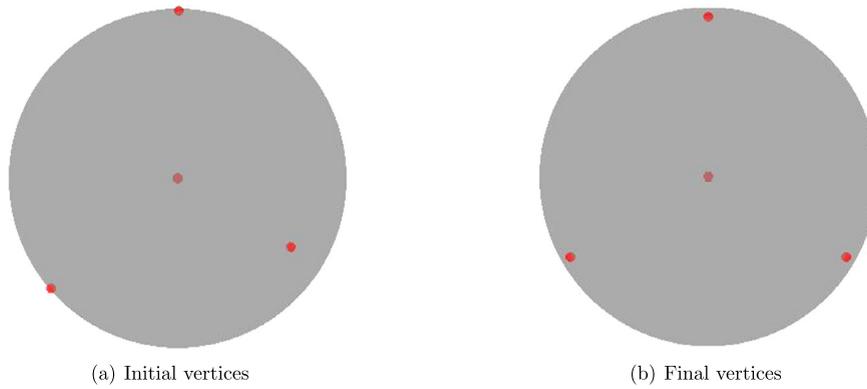


Fig. 1. Numerical simulation of regular tetrahedron vertices on S^2 by using Algorithm 1.

To make sure that objective function $f(x_k)$ is sufficient to descend and approximate to ε which is as near as 0, we use Armijo-Goldstein rule [18] and backtracking line search [18] to lead BB method in a proper way to find local minimum x^* .

Algorithm 1 Barzilai-Borwein method for computing spherical t -designs.

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Input:  $t$ : spherical polynomial degree;  $N$ : number of points;  $X_N$ : distributing  $N$  points on unit sphere  $S^2$ ;  $K_{max}$ : maximum iterations;  $\varepsilon_1$ : termination tolerance on the first-order optimality;  $\varepsilon_2$ : termination tolerance on progress in terms of function or parameter changes.
Initialize  $k = 1, x_0 = x_1 = \eta(X_N), f_0 = f_1 = A_{N,t}(\mu(x_0)), g_0 = g_1 = \eta(\nabla^* A_{N,t}(\mu(x_0)))$  and  $\alpha_1 = 1$ .
1: while  $k \leq K_{max}$  and  $\|g_{k+1} - g_k\| > \varepsilon_1, \|f_{k+1} - f_k\| > \varepsilon_2$  or  $\|x_{k+1} - x_k\| > \varepsilon_2$  do
2:    $s_k = x_k - x_{k-1}, y_k = g_k - g_{k-1}$ 
3:   compute step size  $\alpha_k = (s_k^T s_k)(s_k^T y_k)^{-1}$ 
4:   if  $\alpha_k \leq 10^{-10}$  or  $\alpha_k \geq 10^{10}$  then
5:      $\alpha_k = 1$ 
6:   end if
7:   if  $f(x_k - \alpha_k g_k) \leq f(x_k) - \alpha_k \rho g_k^T g_k$  and
8:    $f(x_k - \alpha_k g_k) \geq f(x_k) - \alpha_k (1 - \rho) g_k^T g_k, (0 < \rho < \frac{1}{2})$  then
9:      $\alpha_k = \alpha_k$  (Armijo-Goldstein rule)
10:  else
11:     $\alpha_k = \tau \alpha_{k-1}, \tau \in (0, 1)$  (backtracking line search)
12:  end if
13:   $x_{k+1} = x_k - \alpha_k g_k$ 
14:  compute  $f_{k+1} = A_{N,t}(\mu(x_{k+1}))$  and search direction  $g_{k+1} = \eta(\nabla^* A_{N,t}(\mu(x_{k+1})))$ 
15: end while
Output: numerical spherical  $t$ -designs  $x^* \subset S^2$ .

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Now we give a small numerical example by using Algorithm 1, which is used to illustrate the numerical construction of spherical t -design.

Example 3.1. We generate spiral points $X_4 = \{x_1, x_2, x_3, x_4\} \subset S^2$ from [6],

$$x_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} 0.9872 \\ 0 \\ -0.1595 \end{bmatrix}, x_3 = \begin{bmatrix} -0.3977 \\ 0.6727 \\ -0.6239 \end{bmatrix}, x_4 = \begin{bmatrix} -0.6533 \\ -0.7455 \\ -0.1318 \end{bmatrix}.$$

By using Algorithm 1, we obtain the termination output: $k = 25, |A_{N,t}(X_4^*)| = 2.775558 \times 10^{-17}, \|\nabla^* A_{N,t}(X_4^*)\| = 1.0446 \times 10^{-8}$ and X_4^* ends with value

$$x_1^* = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, x_2^* = \begin{bmatrix} 0.9428 \\ 0 \\ -0.3333 \end{bmatrix}, x_3^* = \begin{bmatrix} -0.4714 \\ 0.8165 \\ -0.3333 \end{bmatrix}, x_4^* = \begin{bmatrix} -0.4714 \\ -0.8165 \\ -0.3333 \end{bmatrix}.$$

In fact, X_4^* is a set of regular tetrahedron vertices, which is known as a spherical 2-design. As a result, Algorithm 1 reaches the global minimum X_4^* , thus the numerical solution for spherical 2-design was found. We can see the explicit change of X_4 by using Algorithm 1 from Fig. 1 and the behavior of objective function from Fig. 2.

3.2. Convergence analysis

From the view of (4), we know $A_{N,t} \in C^t(S^2)$ for $t \geq 2$. We assume that

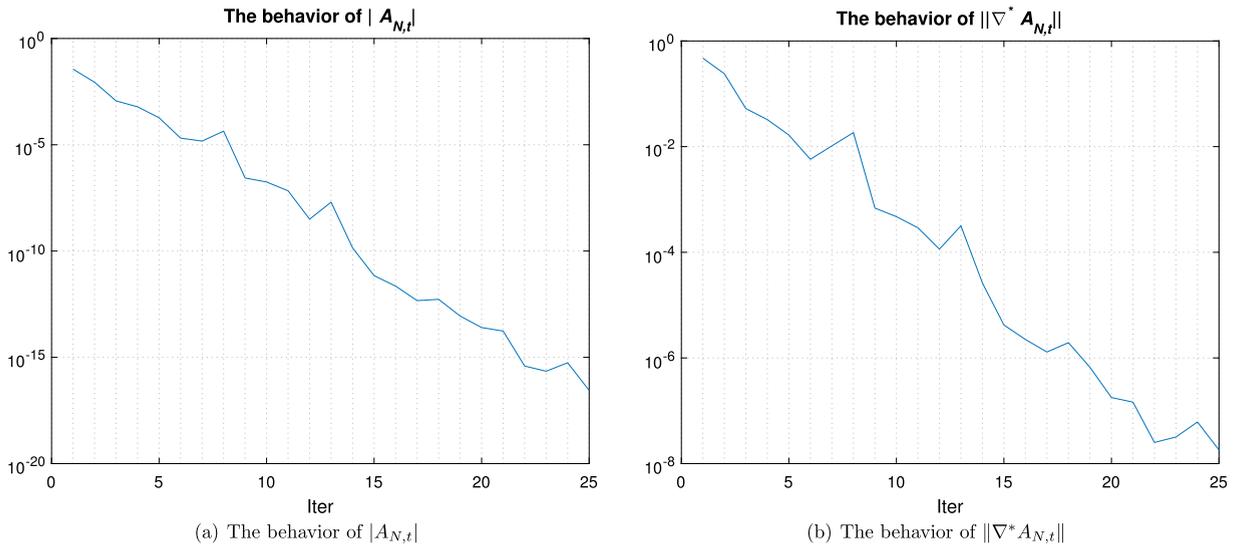


Fig. 2. Numerical behavior of $|A_{N,t}|$ and $\|\nabla^* A_{N,t}\|$ with $t = 2, N = 4$ by using Algorithm 1.

Assumption 3.1. The level set $D := \{x \in \mathbb{R}^n | f(x) \leq f(x_1)\}$ is bounded, and there exists $M > 0$ such that $\|\nabla^2 f(x)\| \leq M$, where $\nabla^2 f$ is the Hessian matrix of $f(x)$.

Now we present the convergence result of Algorithm 1. We shall mention that the idea of proof originated in [19,13].

Theorem 3.1. Let $x_1 = \eta(X_N)$ be an initial point and $g_1 = \eta(\nabla^* A_{N,t}(\mu(x_1)))$ and assume Assumption 3.1 holds. Suppose that x_k is generated by Algorithm 1, then $\liminf_{k \rightarrow \infty} \|g_k\| = 0$.

Proof. By using the Armijo rule (mark 8) from Algorithm 1 and mean value theorem, we have

$$f(x_k) - f(x_k - \alpha_k g_k) = \alpha_k \nabla f(x_k - \kappa \alpha_k g_k)^\top g_k \leq \alpha_k (1 - \rho) g_k^\top g_k, \tag{7}$$

where ∇f is a gradient of f and $\kappa \in (0, 1)$, then

$$\rho g_k^\top g_k \leq (\nabla f(x_k) - \nabla f(x_k - \kappa \alpha_k g_k))^\top g_k. \tag{8}$$

According to Cauchy inequality, we obtain

$$\rho g_k^\top g_k \leq (\nabla f(x_k) - \nabla f(x_k - \kappa \alpha_k g_k))^\top g_k \leq \|\nabla f(x_k) - \nabla f(x_k - \kappa \alpha_k g_k)\| \|g_k\|, \tag{9}$$

moreover, by using the mean value theorem

$$\|\nabla f(x_k) - \nabla f(x_k - \kappa \alpha_k g_k)\| = \left\| \int_0^1 F(x_k - \xi \kappa \alpha_k g_k) \kappa \alpha_k g_k d\xi \right\| \leq M \kappa \alpha_k \|g_k\|. \tag{10}$$

Combine (9) and (10), we know

$$\frac{\rho g_k^\top g_k}{\|g_k\|} \leq M \kappa \alpha_k \|g_k\|. \tag{11}$$

Therefore

$$\alpha_k \|g_k\| \geq \frac{\rho g_k^\top g_k}{M \|g_k\|}. \tag{12}$$

By Armijo rule (mark 7) from Algorithm 1 and (12), we have

$$f(x_{k+1}) \leq f(x_k) - \alpha_k \rho \|g_k\| \frac{g_k^\top g_k}{\|g_k\|} \leq f(x_k) - \frac{\rho^2}{M} \left(\frac{g_k^\top g_k}{\|g_k\|} \right)^2, \tag{13}$$

thus

Table 1
Computing of spherical t -designs by BB method.

$t + 1$	N	Iteration	$A_{N,t}(X_N)$	$\ \nabla^* A_{N,t}(X_N)\ _\infty$	Time	$\min(\sigma_i)$
10	121	100	7.796661e-16	1.8478e-09	1.049909s	1.3270
50	2601	335	1.879594e-12	6.3307e-09	336.663545s	2.3394
96	9409	715	8.237123e-10	7.3212e-08	22724.767736s	2.1647
127	16384	803	8.229142e-10	2.7122e-08	84579.358811s	1.9673

Table 2
Computing of spherical t -designs by quasi-Newton method.

$t + 1$	N	Iteration	$A_{N,t}(X_N)$	$\ \nabla^* A_{N,t}(X_N)\ _\infty$	Time	$\min(\sigma_i)$
10	121	81	1.054716e-15	2.1856e-09	1.062004s	1.3232
50	2601	278	8.455657e-15	2.4398e-09	397.480675s	2.3380
96	9409	465	1.418436e-14	1.4049e-09	40101.809512s	2.1637
127	16384	543	3.709079e-13	2.0536e-09	143631.176093s	1.9656

$$\sum_{j=1}^k \left(\frac{g_j^\top g_j}{\|g_j\|} \right)^2 \leq \frac{M}{\rho^2} (f(x_1) - f(x_{k+1})). \tag{14}$$

Since D is bounded, we know $\lim_{k \rightarrow \infty} f(x_{k+1})$ exists, then

$$\sum_{j=1}^k \left(\frac{g_j^\top g_j}{\|g_j\|} \right)^2 < +\infty, \tag{15}$$

hence

$$\lim_{k \rightarrow \infty} \frac{g_k^\top g_k}{\|g_k\|} = 0. \tag{16}$$

Now we assume that $\limsup_{k \rightarrow \infty} \|g_k\| \neq 0$. We can find a set of $\{k_n\}$ ($n \in \mathbb{Z}^+$), when $n \rightarrow \infty$, $k_n \rightarrow \infty$, and there exist $\epsilon > 0$ such that $\|g_{k_n}\| > \epsilon$. Therefore, (15) can not be hold, which contradicts. Thus, $\liminf_{k \rightarrow \infty} \|g_k\| = \limsup_{k \rightarrow \infty} \|g_k\| = 0$, we complete the proof. \square

Theorem 3.1 shows the convergence of Algorithm 1. Based on the above theorem, we summarize the following results.

Remark 1. Let x_k be the starting point set of Algorithm 1, by Theorem 3.1, then there exist $\epsilon > 0$ such that $\lim_{k \rightarrow \infty} \nabla^* A_{N,t}(\mu(x_k)) = \mathbf{0}$ when $k > \epsilon$, where $\mathbf{0} \in \mathbb{R}^N$ is a zero vector. Therefore, X_N is a stationary point set of $A_{N,t}$.

Remark 2. Let x_k be the starting point set of Algorithm 1, then we have $\lim_{k \rightarrow \infty} A_{N,t}(\mu(x_k)) = 0$. If Theorem 2.4 is established in x^* , then x^* is a spherical t -design.

4. Numerical results

Based on the code in [16,14], we present the feasibility of Algorithm 1 to compute spherical t -design with the point set X_N where $N = (t + 2)^2$ for $t + 1$ up to 127. As an initial point set X_N to solve the optimization problem of minimizing $A_{N,t}(X_N)$ from (4), we use the extremal systems from [15] without any additional constraint. To make sure BB method is meaningful in spherical t -designs, we compare BB method with quasi-Newton method (QN). These methods are implemented in Matlab R2015b and tested on an Intel Core i7 4710MQ CPU with 16 GB DDR3L memory and a 64 Bit Windows 10 Education.

We present the results in Table 1 and Table 2, and these numerical spherical t -designs can be founded in [2]. We observe that BB method cost less time than quasi-Newton method, especially in large X_N . Furthermore, all point sets X_N are verified to be fundamental systems. In fact, we use singular value decomposition (SVD) [18] to obtain all singular values of $\mathbf{Y}_{t+1}(X_N)$, which are defined as $\{\sigma_i\}$ for $i = 1, \dots, (t + 2)^2$. As a result, the $\min(\sigma_i) > 0$, then $\mathbf{Y}_{t+1}(X_N)$ is of full rank, thus X_N is a fundamental system. This is a strong numerical support to Theorem 2.4. Here we set $\epsilon_1 = \epsilon_2 = 10^{-16}$.

Fig. 3(a) and Fig. 4(a) are well exhibited the locally R -linear convergence [10] of BB method by numerical computation of $A_{N,t}$ with $t = 50, N = 2601$. We can see that $A_{N,t}$ converges to 0 with iteration increase.

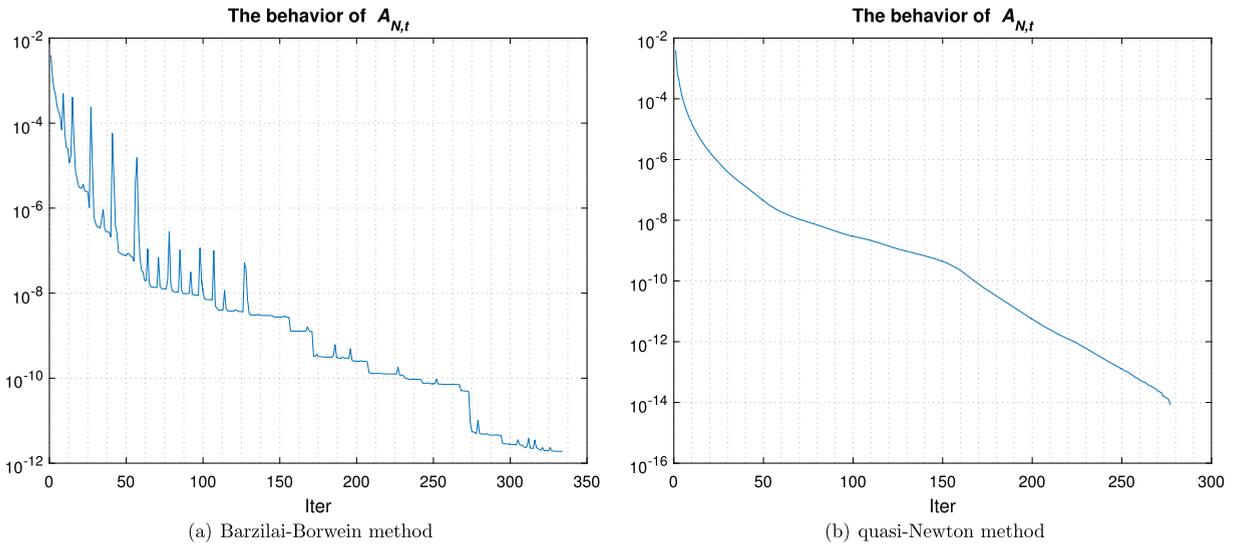


Fig. 3. The behavior of $A_{N,t}$ for $t = 50, N = (t + 1)^2$ on \mathbb{S}^2 in each iteration.

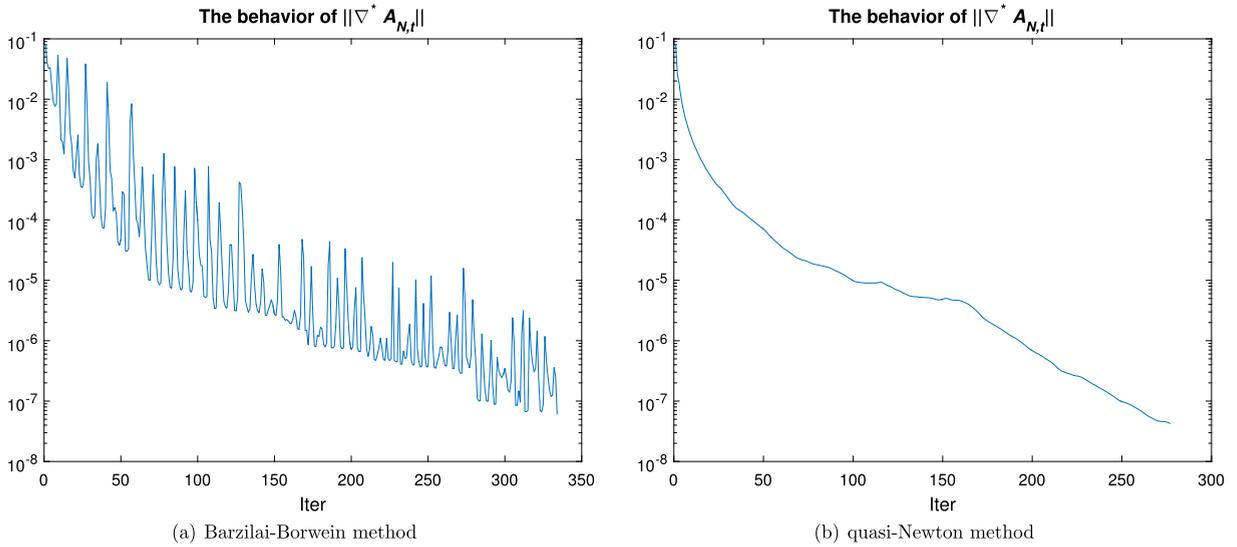


Fig. 4. The behavior of $\|\nabla^* A_{N,t}\|$ for $t = 50, N = (t + 1)^2$ on \mathbb{S}^2 in each iteration.

5. Conclusion

In this paper, we employ Barzilai-Borwein method for finding numerical spherical t -designs with $N = (t + 2)^2$ points for $t + 1$ up to 127. This method performs high efficiency and accuracy. Moreover, we check numerical solution as global minimizer with positivity of minimal singular value of basis matrix. Numerical experiments show that Barzilai-Borwein method is better than quasi-Newton method in time efficiency for solving large scale spherical t -designs. These numerical results are interesting and inspiring. The numerical construction of higher order spherical t -designs are expected in future study.

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Appendix A. Supplementary material

Supplementary material related to this article can be found online at <https://doi.org/10.1016/j.apnum.2019.10.008>.

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