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A survey on spherical designs and algebraic combinatorics on spheres

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ABSTRACT

This survey is mainly intended for non-specialists, though we try to include many recent developments that may interest the experts as well. We want to study "good" finite subsets of the unit sphere. To consider "what is good" is a part of our problem. We start with the definition of spherical t-designs on S^{n-1} in $\mathbb{R}^n$. After discussing some important examples, we focus on tight spherical t-designs on S^{n-1}. Tight t-designs have good combinatorial properties, but they rarely exist. So, we are interested in the finite subsets on S^{n-1}, which have properties similar to tight t-designs from the various viewpoints of algebraic combinatorics. For example, rigid t-designs, universally optimal t-codes (configurations), as well as finite sets which admit the structure of an association scheme, are among them. We will discuss various results on the existence and the non-existence of special spherical t-designs, as well as general spherical t-designs, and their constructions. We will discuss the relations between spherical t-designs and many other branches of mathematics. For example: by considering the spherical designs which are orbits of a finite group in the real orthogonal group $\mathrm{O}(n)$, we get many connections with group theory; by considering those which are shells of Euclidean lattices, we get many unexpected connections with number theory, such as modular forms and Lehmer's conjecture about the zeros of the Ramanujan τ function. Spherical t-designs and Euclidean t-designs are special cases of cubature formulas in approximation theory, and thus we get many connections with analysis and statistics, and in particular with orthogonal polynomials, and moment problems. Moreover, Delsarte's linear programming method and many recent generalizations, including

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the work of Musin and the subsequent progress in using semi-definite programming, have strong connections with geometry (in particular sphere packing problems) and the theory of optimizations. These various connections explain the reason of the charm of algebraic combinatorics on spheres. At the same time, these theories of spherical t-designs and related topics have strong roots in the developments of algebraic combinatorics in general, which was started as Delsarte theory of codes and designs in the framework of association schemes.

1. Introduction

The purpose of this paper is to give a survey on the study of spherical designs and codes, in particular from the viewpoint of "algebraic combinatorics".

Our aim is to study "good" finite sets of points on the unit sphere S^{n-1} in the Euclidean space $\mathbb{R}^n$. The natural question is what does "good" mean. There is yet no final answer known. (Also, it is unrealistic to expect a single good answer.) To consider this question of "what is good" is an important part of our problem.

Let X be a finite subset of S^{n-1}. Spherical codes and spherical designs are nothing but finite subsets of S^{n-1}. Roughly speaking, the code theoretical viewpoint is to try to find X, whose points are scattered on S^{n-1} as far as possible, i.e. the minimum distance $d_{\min}$ of X is as large as possible for a given size of X. (In some other cases, we impose some other conditions, e.g. there are only s kinds of distances between two distinct points of X, i.e. X is an s-distance subset, and then try to increase the size of X as large as possible.) On the other hand, the design theoretical viewpoint is to try to find X which globally approximates the sphere S^{n-1} very well. Of course, "What does approximate the sphere S^{n-1} well mean?" is also an interesting question. There is one very reasonable answer introduced by Delsarte–Goethals–Seidel [88] in 1977. Namely, a finite subset X on S^{n-1} is called a spherical t-design on S^{n-1}, if for any polynomial $f(\boldsymbol{x}) = f(x_1, x_2, \ldots, x_n)$ of degree at most t, the value of the integral of $f(\boldsymbol{x})$ on S^{n-1} (divided by the volume of S^{n-1}) is just the average value of $f(\boldsymbol{x})$ on the finite set X. As is obvious from the definition, a spherical t-design is better if t is larger, and usually a spherical t-design X is better if the cardinality $|X|$ is smaller. It seems that the concept of spherical t-designs caught the hearts of combinatorialists, as a natural analogue of the classical concept of combinatorial t-designs (t–(v, k, λ) designs) which appear in the traditional theory of designs in combinatorics. Actually there is a nice analogy between the theories of codes and designs in the frame work of association schemes as formulated in Delsarte [87] and the theory of spherical codes and designs as formulated in Delsarte–Goethals–Seidel [88]. These are called Delsarte theory on association schemes and Delsarte theory on spheres, or slightly more broadly, algebraic combinatorics on association schemes or algebraic combinatorics on spheres.

In Section 2, we consider algebraic combinatorics on spheres starting from the definition of spherical t-designs. Our main focus is on the interplay between the design theoretical viewpoint and the code theoretical viewpoint. We will discuss examples of spherical t-designs, including tight t-designs, i.e. t-designs whose size attain the natural lower bounds (called Fisher type bounds). Also, we see that tight t-designs have good extremal properties in the interplay of code theory and design theory viewpoints. (Tight $2s$-designs are s-distance sets, and tight $(2s-1)$-designs are antipodal s-distance sets, etc.) If $X \subset S^{n-1}$ is a t-design and an s-distance set, then we always have $t \leq 2s$, so the designs with t close to $2s$ are interesting. In particular if X is a tight t-design, or more generally if $t \geq 2s-2$, then X has the structure of a Q-polynomial association scheme [88]. Then we discuss topics, such as classification problems of tight t-designs, rigid spherical t-designs, the existence theorem (of Seymour–Zaslavsky [164]) of spherical t-designs on S^{n-1} for any t and n, and the explicit construction problems of spherical t-designs (and interval t-designs). Most of the material treated in Section 2 is standard and well known, and not particularly new to experts. But some of them are very recent, and may be new to them.

In Section 3, we give more examples of spherical designs. It seems that the most natural way to get a good distribution of finitely many points on S^{n-1} is to take a finite subgroup G of the real orthogonal group $O(n)$, and a point $\boldsymbol{u}$ on S^{n-1}, and take the orbit of $\boldsymbol{u}$ by G. Many good examples are obtained this way. For example, taking $G = W(H_4)$, the real reflection group of type H_4 and a suitable $\boldsymbol{u} \in S^3 \subset \mathbb{R}^4$, we can get a spherical 19-design on $S^3 \subset \mathbb{R}^4$. Many more examples of spherical t-designs are obtained by this method. However, it seems difficult to obtain t-designs for large t by this method for $n \geq 3$. In Section 3.1, we mention what kinds of conditions of the group G insure that such orbits are t-designs for certain t. For example, the irreducibility of certain representations of G is an important factor. Various related materials will also be surveyed. Somewhat older work in this direction are due to Sobolev [171], Goethals–Seidel [97,98], Bannai [22–24] and many others. Newer results include the work of: Sidelnikov [168,169], Nebe–Rains–Sloane [138], de la Harpe–Pache [84], and some others.

Another natural way to get examples of spherical t-designs is to take a shell (layer, i.e. the set of points in the lattice with a fixed distance from the origin) of a lattice in $\mathbb{R}^n$. For example, any shell of the E_8-lattice (Korkine–Zorotaleff lattice) in $\mathbb{R}^8$ is a spherical 7-design on $S^7 \subset \mathbb{R}^8$, and any shell of the Leech lattice in $\mathbb{R}^{24}$ is a spherical 11-design on $S^{23} \subset \mathbb{R}^{24}$. Many other examples are also obtained from other lattices. In Section 3.2, we first review the work of Venkov which says that any shell of an extremal even unimodular lattices in $\mathbb{R}^{8m}$ is a spherical 11-(respectively, 7-, 3-)design if m is congruent to 0 (respectively, 1, 2) modulo 3. (The E_8-lattice and the Leech lattice are extremal even unimodular lattices. The proof of this theorem uses modular forms.) This result of Venkov can be regarded as an analogue of the Assmus–Mattson theorem which guarantees that a certain combinatorial t-design is obtained as a shell of a code, that is, a subset of a code with a fixed Hamming weight. This work of Venkov was influential in obtaining new proofs of the Assmus–Mattson theorem in the classical design theory. It is very interesting to note that so far no spherical t-designs with $t \geq 12$ have been obtained as shells of a lattice. (The situation is also true for classical design theory, namely no combinatorial 6-designs have been obtained as shells of a code either in F_2 or in F_q.) It is an interesting open question, whether this is indeed the case. As for the E_8-lattice, it is known that all the shells are spherical 7-designs. Whether there is any among them which is a spherical 8-design is a very interesting and difficult open question. It is equivalent to the famous conjecture called D.H. Lehmer's Conjecture in number theory, as pointed out by Venkov, de la Harpe, and Pache [85,86,147]. The situation is the same for other extremal even unimodular lattices in $\mathbb{R}^{24k}$, and it is a very interesting problem to determine whether every shell can be a spherical 12-design in $\mathbb{R}^{24k}$ or not. We mention an approach to this question, although it is still at an unsatisfactory stage, by Bannai–Koike–Shinohara–Tagami [47].

In Section 3.3, we discuss some connections with the sphere packing problems and algebraic combinatorics on spheres. Recently, there were three major breakthroughs on sphere packing problems, they are (1) the proof of Kepler's Conjecture by Hales, (2) the determination of the kissing number in dimension 4 by Musin, and (3) the developments of sphere packing problems in 8- and 24-dimensional Euclidean spaces by Cohn and Elkies, in particular the optimality of the Leech lattice among lattice packings in the 24-dimensional Euclidean space. In this paper we discuss the last two topics. Musin used a method which is a generalization of the linear programming method by Delsarte (which is outlined in Sections 2.2 and 2.4 of this paper). In the meantime, there were several recent attempts to generalize the method of Delsarte in various directions. Schrijver, using the idea of Terwilliger algebras of association schemes, formulated semi-definite programming to improve the previously known bounds for binary codes (i.e. codes in binary Hamming schemes), and succeeded in obtaining very notable improvements. (Gijswijt–Schrijver–Tanaka [95] generalized it for q-ary codes.) Setting up semi-definite programming for the kissing numbers was successfully done, and again, notable improvements were obtained even for kissing numbers in dimensions 3 and 4. (So, a new proof for the result of Musin was obtained.) We briefly survey these developments. In the sphere packing problem, as well as in the kissing number problem, the dimensions 8 and 24 are very special. It is expected that the packing of spheres coming from the E_8-lattice and the Leech lattice in these dimensions give the best sphere packing among general (not necessarily lattice) packings. Cohn–Elkies first proved that these lattice packings are close to the best packings with respect to density. Moreover, Cohn–Kumar [75,76] proved in particular that the Leech lattice gives the best packing among all the lattice packings. In a part of their proof, the consideration of the association schemes attached to the 196 560 minimum vectors together with other techniques, which are a kind of generalization of the method of Delsarte, play an important role. Another important new concept is that of universally optimal codes (on S^{n-1}), due to Cohn–Kumar [77]. A universally optimal code is a subset of S^{n-1} which gives the minimum energy among all the subsets of the same size, with respect to any potential function in a very wide class. The concept of universally optimal codes is defined independently of the degree s or strength t of finite sets on S^{n-1}. However tight spherical t-designs as well as subsets of S^{n-1} satisfying $t \geq 2s - 1$ are universally optimal codes. We believe that the classification problem of universally optimal codes on S^{n-1} proposed by Cohn–Kumar will become a very important problem in the future. We conclude Section 3 by discussing related materials, e.g. the connection with real MUBs and Q-polynomial association schemes, etc.

In Section 4, we briefly survey various generalizations of spherical designs. In the first subsection, we discuss generalizations to compact symmetric spaces of rank one, i.e. projective spaces over real, complex, quaternion fields and Cayley octanion, and the designs on Grassmannian spaces, which are examples of compact symmetric spaces of bigger ranks. In Section 4.2, we will focus on the recent developments of the theory of Euclidean t-designs. The concept of Euclidean t-designs is a two step generalization of the theory of spherical t-designs: we allow the points to have different weights (that is, we consider cubature formulas) and we allow the points to have different norms (that is, we consider points on several concentric spheres). Our main emphasis is on the study of Euclidean t-designs which are tight t-designs (in the sense we define later in this paper), or close to tight t-designs. Analogous to the fact that good subsets on the sphere are related to association schemes (as we have seen before in the introduction of Section 2), good subsets in Euclidean spaces (i.e. on several concentric spheres) are related to coherent configurations, which is a generalization of association schemes. Then, in Section 4.3, we will discuss cubature formulas of degree t, i.e. weighted t-designs in various spaces. The theory of cubature formulas is far older than the theory of spherical designs in combinatorics, and includes the theory of spherical designs as a special case. We will discuss the similarities as well as the differences between the theory of cubature formulas and the theory of t-designs. In Section 4.4, we discuss various related materials, as much as time and space permit.

So far we have been giving a very brief description of the contents of this survey. The main objective of this survey is to look at "good" finite subsets of the sphere and also in some other spaces. We would like to emphasize that the charm of this direction of research is that the theory of spherical designs have many interesting strong ties with many different areas of mathematics, and at the same time, the theories we introduced in this survey have solid foundation in algebraic combinatorics, and association schemes play important roles in various places. This is the reason why this survey is included in this special volume whose main topic is on association schemes.

The authors would be extremely happy, if the reader recognizes both the diversity of spherical designs (more generally of algebraic combinatorics on spheres) and the strong backbone of the theory of association schemes (more generally of algebraic combinatorics in general) behind it.

2. Theory of spherical designs

In the first two Sections 2.1 and 2.2 we will very briefly discuss the basic theories of spherical t-designs and their connection with association schemes following Delsarte–Goethals–Seidel [88]. We will not repeat most of the materials treated in the fundamental paper [88]. Instead we encourage the reader to read the paper [88]. (See also [29] which is written in Japanese.) In [88], the so-called Delsarte theory on spheres is developed very much. The first part of Section 2.4 is also taken from [88].

Notation
$\mathbb{R}^n := \{\boldsymbol{u} = (u_1, u_2, \ldots, u_n) \mid u_i \in \mathbb{R}, 1 \le i \le n\}$, the n-dimensional Euclidean space.
$\boldsymbol{u} \cdot \boldsymbol{v}$:= usual inner product of $\mathbb{R}^n$.
$\|\boldsymbol{u}\| := \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}}$.
For a matrix $M = (m_{i,j})$, we define $\|M\|^2 := \sum_{i,j} m_{i,j}^2$.
$S^{n-1} := \{\boldsymbol{u} \in \mathbb{R}^n \mid \|\boldsymbol{u}\| = 1\}$, the unit sphere.
$\mathcal{P}(\mathbb{R}^n) := \mathbb{R}[x_1, x_2, \ldots, x_n]$, the vector space over the real number field $\mathbb{R}$ consisting of all the polynomials in n variables $x_1, x_2, \ldots, x_n$.
$\mathrm{Harm}(\mathbb{R}^n) := \{f \in \mathcal{P}(\mathbb{R}^n) \mid \Delta f = 0\}$, where $\Delta := \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator.
$\mathrm{Hom}_l(\mathbb{R}^n)$:= the subspace of $\mathcal{P}(\mathbb{R}^n)$ spanned by all the homogeneous polynomials of degree l.
$\mathrm{Harm}_l(\mathbb{R}^n) := \mathrm{Harm}(\mathbb{R}^n) \cap \mathrm{Hom}_l(\mathbb{R}^n)$.
$\mathcal{P}_l(\mathbb{R}^n) := \sum_{j=0}^l \mathrm{Hom}_j(\mathbb{R}^n)$.
$h_{l,n} := \dim(\mathrm{Harm}_l(\mathbb{R}^n)) = \binom{n+l-1}{l} - \binom{n+l-3}{l-2}$.
$\langle f, g\rangle := \frac{1}{|S^{n-1}|} \int_{\boldsymbol{x} \in S^{n-1}} f(\boldsymbol{x}) g(\boldsymbol{x}) \mathrm{d}\sigma(\boldsymbol{x})$ for $f, g \in \mathcal{P}(\mathbb{R}^n)$, with the usual integral on the unit sphere.
$\{\varphi_{l,1}, \varphi_{l,2}, \ldots, \varphi_{l,h_{l,n}}\}$:= an orthonormal basis of $\mathrm{Harm}_l(\mathbb{R}^n)$ with respect to the inner product $\langle\, ,\rangle$.
$Q_{l,n}(x)$:= the Gegenbauer polynomial of degree l which is normalized to $Q_{l,n}(1) = h_{l,n}$.
$A(X) := \{\boldsymbol{u} \cdot \boldsymbol{v} \mid \boldsymbol{u}, \boldsymbol{v} \in X, \boldsymbol{u} \ne \boldsymbol{v}\}$ for $X \subset S^{n-1}$ with $|X| < \infty$. $A'(X) := A(X) \cup \{1\}$.
H_l := lth characteristic matrix of X whose rows and columns are indexed by $X \times \{\varphi_{l,1}, \varphi_{l,2}, \ldots, \varphi_{l,h_{l,n}}\}$, $(\boldsymbol{u}, \varphi_{l,i})$-entry of H_l is defined by $H_l(\boldsymbol{u}, \varphi_{l,i}) = \varphi_{l,i}(\boldsymbol{u})$.

2.1. Definitions and basic properties of spherical designs

Spherical designs were defined by Delsarte–Goethals–Seidel in 1977 [88].

We consider a finite subset X on the unit sphere S^{n-1} in n-dimensional Euclidean space $\mathbb{R}^n$. The following is the definition of spherical t-designs.

Definition 2.1. Let t be a natural number. A finite subset $X \subset S^{n-1}$ is called a spherical t-design if

$$\frac{1}{|S^{n-1}|} \int_{\boldsymbol{x} \in S^{n-1}} f(\boldsymbol{x}) \mathrm{d}\sigma(\boldsymbol{x}) = \frac{1}{|X|} \sum_{\boldsymbol{u} \in X} f(\boldsymbol{u})$$

holds for any polynomial $f(\boldsymbol{x})$ $(=f(x_1, x_2, \ldots, x_n))$ of degree at most t, with the usual integral on the unit sphere.

To study spherical t-designs, it is convenient to use the conditions given in the following theorem.

Theorem 2.2. *Let $X \subset S^{n-1}$ be a non-empty finite set. Then the following conditions are equivalent to each other.*

(1) *X is a spherical t-design.*
(2) $\sum_{\boldsymbol{u} \in X} \varphi(\boldsymbol{u}) = 0$ *for any* $\varphi \in \mathrm{Harm}_l(\mathbb{R}^n)$ *with* $1 \le l \le t$.
(3) ${}^tH_l H_0 = 0$ *for* $l = 1, 2, \ldots, t$.
(4) ${}^tH_k H_l = |X| \Delta_{k,l}$, *for any k, l satisfying $0 \le k + l \le t$, where $\Delta_{k,k}$ denotes the identity matrix of size $h_{k,n}$ and $\Delta_{k,l}$ is a zero matrix of size $h_{k,n} \times h_{l,n}$.*
(5) *Any kind of moment of X of degree at most t is invariant under any orthogonal transformation.*
(6) *The following holds for any $\boldsymbol{a} \in \mathbb{R}^n$.*

$$\frac{1}{|X|} \sum_{\boldsymbol{u} \in X} (\boldsymbol{a} \cdot \boldsymbol{u})^k = \begin{cases} \dfrac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{n(n+2) \cdots (n+2k-2)} (\boldsymbol{a} \cdot \boldsymbol{a})^{\frac{k}{2}} & \text{if } k \text{ is even and } 0 \le k \le t, \\ 0 & \text{if } k \text{ is odd and } 0 \le k \le t. \end{cases}$$

Remark. The condition (6) in Theorem 2.2 is due to Venkov [179], which is called the fundamental equation of spherical t-designs.

2.2. Spherical designs and association schemes (Delsarte theory on spheres)

Let X be a finite set on S^{n-1}. Let $A(X) = \{\boldsymbol{u} \cdot \boldsymbol{v} \mid \boldsymbol{u}, \boldsymbol{v} \in X,\ \boldsymbol{u} \neq \boldsymbol{v}\}$. We say that X *is of degree* s if X is an s-distance set, i.e. $|A(X)| = s$. On the other hand we say X *has the strength* t, if X is a spherical t-design and X is not a spherical $(t+1)$-design. Delsarte–Goethals–Seidel proved that if a spherical t-design satisfies a certain condition, then X has a structure of a Q-polynomial association scheme. Before giving the precise statement of their result, we briefly give the definition of an association scheme. Now we consider $\mathfrak{X} = (X, \{R_i\}_{0\le i\le d})$, where X is an abstract finite set and $R_0, R_1, \ldots, R_d$ are subsets in $X \times X$.

Definition 2.3 (*[46,64,87]*). We say $\mathfrak{X} = (X, \{R_i\}_{0\le i\le d})$ is a commutative association scheme of class d if the following conditions are satisfied.

(1) $\{R_i\}_{0\le i\le d}$ is a partition of $X \times X$, i.e. $X \times X = R_0 \cup R_1 \cup \cdots \cup R_d$ and $R_i \cap R_j = \emptyset$ for $i \neq j$.
(2) $R_0 = \{(x, x) \mid x \in X\}$.
(3) For each $i \in \{0, 1, \ldots, d\}$, there exists $i' \in \{0, 1, \ldots, d\}$ satisfying ${}^tR_i = R_{i'}$, where ${}^tR_i = \{(x, y) \mid (y, x) \in R_i\}$.
(4) For each $i, j, k \in \{0, 1, \ldots, d\}$, the cardinality $|\{z \in X \mid (x, z) \in R_i,\ (z, y) \in R_j\}|$ is constant for any $(x, y) \in R_k$. We denote this cardinality by $p^k_{i,j}$.
(5) $p^k_{i,j} = p^k_{j,i}$ for any $i, j, k \in \{0, 1, \ldots, d\}$.

An association scheme is called *symmetric* if ${}^tR_i = R_i$ holds for any $i \in \{0, 1, \ldots, d\}$.

For each relation R_i of an association scheme, we define an *adjacency matrix* A_i which is indexed by X. More precisely for $(x, y) \in X \times X$, the (x, y)-entry of A_i is 1 if $(x, y) \in R_i$ and 0 if $(x, y) \notin R_i$. It is well known that the linear span $\mathfrak{A} = \langle A_0, A_1 \ldots, A_d\rangle$ is closed under matrix multiplications. $\mathfrak{A}$ is called the *Bose–Mesner algebra* of the association scheme $\mathfrak{X}$. An association scheme $\mathfrak{X}$ is called a *P-polynomial scheme* if there is an ordering of the relations $R_1, \ldots, R_d$ so that each A_i is expressed as a polynomial in A_1 of degree i. It is well known that $\mathfrak{X}$ is a P-polynomial scheme if and only if the graph (X, R_1) defined on X by the relation $R_1 \subset X \times X$ is a distance-regular graph.

The Bose–Mesner algebra $\mathfrak{A}$ is a semi-simple algebra. Hence it has a basis consisting of primitive idempotents $E_0, E_1, \ldots, E_d$. The Bose–Mesner algebra $\mathfrak{A}$, has another multiplication, so called the *Hadamard product*: for $M_1, M_2 \in \mathfrak{A}$, $M_1 \circ M_2$ is defined to have (x, y)-entry $M_1(x, y) \cdot M_2(x, y)$. With the basis $E_0, E_1, \ldots, E_d$ and the Hadamard product $\circ$, we define the notion of *Q-polynomial association schemes*. That is, if there exists an ordering of $E_0, E_1, \ldots, E_d$ so that each E_i is expressed as a polynomial in E_1 of degree i using the Hadamard product, then $\mathfrak{X}$ is called a *Q-polynomial association scheme*. Both P-polynomial association schemes and Q-polynomial association schemes are very important families of association schemes.

Delsarte–Goethals–Seidel proved Theorems 2.11 and 2.12 given in Section 2.4, which imply that $t \le 2s$ holds for any finite set $X \subset S^{n-1}$ of degree s and strength t. The following theorem is also proved by Delsarte–Goethals–Seidel [88], which shows that good structures on the unit sphere in the Euclidean space have good combinatorial structures. (We will not touch any more of the materials contained in [88], hoping that the reader can read the original paper [88]. Also refer to [29] which is written in Japanese.)

Theorem 2.4 (*[88]*). *Let X be a finite set on S^{n-1} of degree s and strength t. If $t \ge 2s - 2$, then the relations defined by the set $A'(X) = A(X) \cup \{1\}$ give the structure of a Q-polynomial association scheme on X.*

Proof. Let $A'(X) = \{\alpha_0\ (=1), \alpha_1, \ldots, \alpha_s\}$. Let A_i be a matrix whose rows and columns are indexed by X and whose $(\boldsymbol{u}, \boldsymbol{v})$-entry is defined by

$$A_i(\boldsymbol{u}, \boldsymbol{v}) = \begin{cases} 1 & \text{if } \boldsymbol{u} \cdot \boldsymbol{v} = \alpha_i \\ 0 & \text{otherwise.} \end{cases}$$

Let $p(\alpha_i, \alpha_j, \boldsymbol{u}, \boldsymbol{v}) = |\{\boldsymbol{a} \in X \mid \boldsymbol{u} \cdot \boldsymbol{a} = \alpha_i,\ \boldsymbol{a} \cdot \boldsymbol{v} = \alpha_j\}|$ for $\boldsymbol{u}, \boldsymbol{v} \in X$. On the other hand Theorem 2.2(4) implies

$$({H_k}^t H_k)({H_l}^t H_l) = |X| H_k \Delta_{k,l} {}^t H_l \tag{2.1}$$

for any k, j satisfying $0 \le k, l \le t,\ 0 \le k + l \le t$. By computations of the $(\boldsymbol{u}, \boldsymbol{v})$-entry of both sides of (2.1) we obtain the following.

$$\sum_{i=0}^{s} \sum_{j=0}^{s} Q_{k,n}(\alpha_i) Q_{l,n}(\alpha_j) p(\alpha_i, \alpha_j, \boldsymbol{u}, \boldsymbol{v}) = \delta_{k,l} |X| Q_{k,n}(\boldsymbol{u} \cdot \boldsymbol{v}). \tag{2.2}$$

Let $\boldsymbol{u} \cdot \boldsymbol{v} = \gamma$. Since $2s - 2 \le t$, (2.2) implies

$$\begin{aligned} &\sum_{i=1}^{s} \sum_{j=1}^{s} Q_{k,n}(\alpha_i) Q_{l,n}(\alpha_j) p(\alpha_i, \alpha_j, \boldsymbol{u}, \boldsymbol{v}) \\ &\quad = \delta_{k,l} |X| Q_{k,n}(\gamma) - Q_{k,n}(1) Q_{l,n}(1) \delta_{\gamma,1} - Q_{k,n}(1) Q_{l,n}(\gamma) - Q_{k,n}(\gamma) Q_{l,n}(1), \end{aligned} \tag{2.3}$$

for any k and l with $0 \le k, l \le s - 1$. Let W be a matrix of degree s whose (i, j)-entry is $Q_{i-1,n}(\alpha_j)$. Let $P_{\boldsymbol{u},\boldsymbol{v}}$ be the matrix of degree s whose (i, j)-entry equals $p(\alpha_i, \alpha_j, \boldsymbol{u}, \boldsymbol{v})$. Then for any $0 \le k, l \le s - 1$, the left hand side of (2.3) is the $(k + 1, l + 1)$-entry of $W P_{\boldsymbol{u},\boldsymbol{v}} W$. Since W is a non-singular matrix, the matrix $P_{\boldsymbol{u},\boldsymbol{v}}$ is uniquely determined by γ and does not depend on the choice of $\boldsymbol{u} \cdot \boldsymbol{v} = \gamma$. This implies that $(X, \{R_i\}_{0 \le i \le s})$ is an association scheme, where $R_i = \{(\boldsymbol{u}, \boldsymbol{v}) \in X \times X \mid \boldsymbol{u} \cdot \boldsymbol{v} = \alpha_i\}$ for $0 \le i \le s$. Next, let $E_k = \frac{1}{|X|} H_k {}^t H_k$ for $0 \le k \le s - 1$. The $(\boldsymbol{u}, \boldsymbol{v})$-entry of E_k is given by

$$E_k(\boldsymbol{u}, \boldsymbol{v}) = \frac{1}{|X|} \sum_{i=1}^{h_k} \varphi_{k,i}(\boldsymbol{u}) \varphi_{k,i}(\boldsymbol{v}) = \frac{1}{|X|} Q_{k,n}(\boldsymbol{u} \cdot \boldsymbol{v}).$$

Hence $E_k = \frac{1}{|X|} \sum_{i=0}^{s} Q_{k,n}(\alpha_i) A_i$ for any $k = 0, 1, \ldots, s - 1$. Let $E_s = I - \sum_{k=0}^{s-1} E_k$. Then (2.1) implies $E_k E_l = \delta_{k,l} E_k$ for any $0 \le k, l \le s$. Hence $E_0, E_1, \ldots, E_s$ is the basis consisting of primitive idempotents of the Bose–Mesner algebra of $(X, \{R_i\}_{0 \le i \le s})$. Thus we can describe the second eigenmatrix of the association scheme $(X, \{R_i\}_{0 \le i \le s})$ using Gegenbauer polynomials, and using the property of Gegenbauer polynomials we can show that it is a Q-polynomial association scheme. As for a detailed proof refer to the paper by Delsarte–Goethals–Seidel [88]. ■

If we assume that X is antipodal then we can prove the following theorem. As for the proof refer [30].

Theorem 2.5 (*[30]*). *Let X be an antipodal finite set on S^{n-1} of degree s and strength t. If $t \ge 2s - 3$, then the relations defined by the set $A'(X) = A(X) \cup \{1\}$ give the structure of a Q-polynomial association scheme on X.*

2.3. Examples of spherical designs

Example 2.6. Regular N-gon on $S^1 \subset \mathbb{R}^2$ is a t-design for $1 \le t \le N - 1$.

Example 2.7 (*Spherical Designs in $\mathbb{R}^3$*).

X	$\lvert X\rvert$	s, degree	t, strength
Regular tetrahedron	4	1	2
Cube	8	3	3
Regular octahedron	6	2	3
Regular dodecahedron	20	4	5
Regular icosahedron	12	3	5

Example 2.8 (*Regular Polytopes X in $\mathbb{R}^n$*).

n	Number of cells	$\|X\|$	s, degree	t, strength
4	24	24	4	5
4	120	600	30	11
4	600	120	8	11
n	$n+1$	$n+1$	1	2
n	$2n$	2^n	n	3
n	2^n	$2n$	2	3

Example 2.9 (*Root Systems in $\mathbb{R}^n$*).

n	Type	$\|X\|$	s, degree	t, strength
$n \geq 1$	A_n	$n(n+1)$	4	3
4	D_4	24	4	5
$n \geq 5$	D_n	$2n(n-1)$	4	3
6	E_6	72	4	5
7	E_7	126	4	5
8	E_8	240	4	7

Example 2.10 (*The Set of Minimal Vectors of the Leech Lattice*).

n	$\|X\|$	s, degree	t, strength
24	196 560	6	11

2.4. More on Delsarte theory on spheres and tight spherical designs

In a sense, a spherical t-design X approximates the unit sphere with respect to the integrations of polynomials of degree at most t. In this sense it is natural to ask how small the cardinality $|X|$ can be. Delsarte–Goethals–Seidel found similar theorems for designs and codes on the unit sphere S^{n-1} in the context of classical design and code theory. First they proved the following very natural upper bounds for the cardinalities of s-distance sets on S^{n-1}.

Theorem 2.11 ([88]). *Let $X \subset S^{n-1}$ be a finite set of degree s and let X^* be a maximal subset of X satisfying $X^* \cap (-X^*) = \emptyset$. Then the following hold.*

(1) $|X| \leq \binom{n+s-1}{s} + \binom{n+s-2}{s-1}$.

(2) *If X is antipodal, then $|X| = 2|X^*|$ and $|X| \leq 2\binom{n+s-2}{s-1}$.*

Proof. Let $A(X) = \{\alpha_i \mid 1 \leq i \leq s\}$. (1) For each $\boldsymbol{u} \in X$ we define a polynomial $f_{\boldsymbol{u}}(\boldsymbol{x}) = \prod_{i=1}^{s}(\boldsymbol{x} \cdot \boldsymbol{u} - \alpha_i)$. Then $\{f_{\boldsymbol{u}} \mid \boldsymbol{u} \in X\}$ is a linearly independent set of polynomials of degree s. Hence $|X| \leq \sum_{i=0}^{s} \dim(\mathrm{Hom}_i(S^{n-1})) = \sum_{i=0}^{s} \dim(\mathrm{Harm}_i(\mathbb{R}^n)) = \binom{n+s-1}{s} + \binom{n+s-2}{s-1}$. (2) Let $A^2(X^*) = \{(\boldsymbol{u}\cdot\boldsymbol{v})^2 \mid \boldsymbol{u}, \boldsymbol{v} \in X^*,\ \boldsymbol{u} \neq \boldsymbol{v}\} = \{\beta_i \mid 1 \leq i \leq s^*\}$ and $f_{\boldsymbol{u}}^*(\boldsymbol{x}) = \prod_{\beta_i \in A^2(X^*)}((\boldsymbol{x}\cdot\boldsymbol{u})^{2-\delta_{\beta_i,0}} - \beta_i)$. Then $\deg(f_{\boldsymbol{u}}^*) = s-1$ and $f_{\boldsymbol{u}}^*$ is a linear combination of monomials whose degrees are equal to $s-1$ modulo 2 for any $\boldsymbol{u} \in X^*$. Since $\{f_{\boldsymbol{u}}^* \mid \boldsymbol{u} \in X^*\}$ is a linearly independent set of polynomials, similar arguments imply $|X| = 2|X^*| \leq 2\sum_{i=0}^{[\frac{s-1}{2}]} \dim(\mathrm{Harm}_{s-1-2i}(\mathbb{R}^{n-1})) = 2\binom{n+s-2}{s-1}$. ∎

Delsarte–Goethals–Seidel found that the lower bounds for the cardinalities of spherical t-designs are very much related to the upper bounds given in Theorem 2.11.

Theorem 2.12 ([88]).

(1) *Let X be a spherical $2e$-design, then $|X| \geq \binom{n+e-1}{e} + \binom{n+e-2}{e-1}$ holds.*

(2) *Let X be a spherical $(2e+1)$-design, then $|X| \geq 2\binom{n+e-1}{e}$ holds.*

Definition 2.13 (*Tight Design*). Let X be a spherical t-design on S^{n-1}. If equality holds in one of the inequalities given above, then we say X is a tight spherical t-design.

A tight spherical t-design, if it exists, has a good combinatorial structure. The following theorem was also proved by Delsarte–Goethals–Seidel.

Theorem 2.14 ([88]).

(1) *Let X be a spherical $2e$-design. Then X is a tight spherical $2e$-design if and only if X is of degree e.*
(2) *Let X be a spherical $(2e+1)$-design. Then X is a tight spherical $(2e+1)$-design if and only if X is of degree $e+1$ and is antipodal.*

Actually the theorems given above are proved in a much stronger manner. In the following we introduce the method given by Delsarte–Goethals–Seidel. Let $X \subset S^{n-1}$ be a finite set of degree s and strength t. Let $A'(X) = \{\alpha_0(=1), \alpha_1, \ldots, \alpha_s\}$. For each i, we define a matrix A_i whose entries are indexed by $X \times X$ in the following way.

$$A_i(\boldsymbol{u}, \boldsymbol{v}) = \begin{cases} 1 & \text{for } \boldsymbol{u} \cdot \boldsymbol{v} = \alpha_i \\ 0 & \text{otherwise.} \end{cases}$$

Let $F(x)$ be an any polynomial and

$$F(x) = \sum_{l=0}^{\infty} f_l Q_{l,n}(x), \quad f_l \in \mathbb{R}, l = 0, 1, \ldots,$$

be the Gegenbauer expansion of $F(x)$. Then they showed the following lemma.

Lemma 2.15 ([88]). *Let $X \subset S^{n-1}$ be a non-empty finite set of degree s. Then*

$$(f_0|X| - F(1))\,|X| = \sum_{i=1}^{s} F(\alpha_i) d_i - \sum_{l=1}^{\infty} f_l \, \|{}^t H_l H_0\|^2$$

holds, where $d_i = \sum_{\boldsymbol{u},\boldsymbol{v} \in X} A_i(\boldsymbol{u}, \boldsymbol{v})$ for $i = 1, 2, \ldots, s$.

Then apply Lemma 2.15 to non-empty finite sets $X \subset S^{n-1}$ of degree s and strength t with particular polynomials $F(x)$. In this case Theorem 2.2(3) implies $\|{}^t H_l H_0\|^2 = 0$ for any $1 \leq l \leq t$.

Case $t = 2e$

In this case, they used the polynomial $F(x) = f(x)^2$, where $f(x) = \sum_{l=0}^{e} Q_{l,n}(x)$. Then the formula for the products of Gegenbauer polynomials implies $\frac{F(1)}{f_0} = \binom{n+e-1}{e} + \binom{n+e-2}{e-1}$, $F(\alpha_i) \geq 0$ for $i = 1, \ldots, s$, $f_l = 0$ for any $l \geq 2e+1$. This implies $|X| \geq \binom{n+e-1}{e} + \binom{n+e-2}{e-1}$ and the equality holds if and only if $f(\alpha_i) = 0$ holds for $i = 1, 2, \ldots s$. Since $f(x)$ is a polynomial of degree exactly e with simple zeros, we must have $s \leq e$. On the other hand since X is of degree s, Theorem 2.11(1) implies $|X| \leq \binom{n+s-1}{s} + \binom{n+s-2}{s-1}$. Hence we must have $s = e$. This implies Theorem 2.14(1). We note that for tight spherical $2e$-design X, $A(X)$ must coincide exactly with the set of zeros of $\sum_{l=0}^{e} Q_{l,n}(x)$, which is a Jacobi polynomial.

Case $t = 2e + 1$

In this case, they used the polynomial $F(x) = (x+1)g(x)^2$, where $g(x) = \sum_{l=0}^{[\frac{e}{2}]} Q_{e-2l,n}(x)$. Then $\frac{F(1)}{f_0} = 2\binom{n+e-1}{e}$, $F(\alpha_i) \geq 0$ holds for $i = 1, 2, \ldots, s$, $f_l = 0$ holds for any $l \geq 2e+2$. This implies $|X| \geq 2\binom{n+e-1}{e}$ and the equality $|X| = 2\binom{n+e-1}{e}$ holds if and only if $F(\alpha_i) = 0$ for $i = 1, 2, \ldots, s$. Hence X is a tight spherical $(2e+1)$-design, if and only if $A(X)$ coincides with the set of zeros of $(x+1)g(x)$. It is known that $(x+1)g(x)$ is a polynomial of degree exactly $e+1$ and every root is simple. Hence we must have $s \leq e+1$. Moreover it is well known that $g(\alpha) = 0$ holds if and only if $g(-\alpha) = 0$. Let $Y = X \cup (-X)$ and $s' = |A(Y)|$. Then Y is an antipodal set and $s \leq s' \leq e+1$ holds. Hence Theorem 2.11(2) implies $2\binom{n+e-1}{e} \leq |X| \leq |Y| \leq 2\binom{n+s'-2}{s'-1} \leq 2\binom{n+e-1}{e}$. This implies Theorem 2.14(2). We note that for a tight spherical $(2e+1)$-design X, $A(X)$ must coincide with the set of zeros of the polynomial $(x+1)\sum_{l=0}^{[\frac{e}{2}]} Q_{e-2l,n}(x)$.

Classification of tight spherical designs

Later we will introduce the existence theorems for spherical t-designs. It is known that for fixed n and t, there always exists a spherical t-design X on S^{n-1} if the cardinality $|X|$ is large enough. However it is still a difficult problem to construct them explicitly. We are also interested in the classification of tight spherical t-designs. It is well known that the image of a spherical t-design under an orthogonal transformation is also a spherical t-design. So we want to classify tight spherical designs up to orthogonal transformations. In the following we explain the classification of tight spherical t-designs done so far.

Theorem 2.16 (*Bannai–Damerell [42,43]*). *Assume $n \geq 3$. If a tight spherical t-design exists on S^{n-1}, then t is in* $\{1, 2, 3, 4, 5, 7, 11\}$. *Moreover, if $t = 11$, then $n = 24$ and hence $|X| = 196\,560$.*

Let X be a tight spherical t-design on S^{n-1} with $t \geq 4$ and $n \geq 3$. The first crucial step of the proof of Theorem 2.16 is to show that all the elements $\alpha_i \in A(X)$ $(i = 0, 1, \ldots, [\frac{t+1}{2}])$ must be rational numbers, "except for the case $(t, n) = (5, 3)$", if $t \geq 4$ and $n \geq 3$. The statements in [42, 43] are incorrect, since it failed to mention this exception. But it is easily shown that this is the only exception. (See Lemma 8.3.7 in [29] for the details.) The second step is to show that (if $(t, n) \neq (5, 3)$ and) if $\alpha_i \neq 0$, then it must be the reciprocal of an integer. The rest of the proof in [42,43] is to show that this does not happen if $t = 6, 8$, and ≥ 9 unless $(t, n) = (11, 24)$. The proofs in [42, 43] showing this use quite different techniques for t even and t odd, and are fairly involved. The last case actually corresponds to the existence of a tight 11-design on S^{23}, and its uniqueness is proved by Bannai–Sloane [49].

Theorem 2.17 (*Bannai–Sloane [49]*). *There is (up to orthogonal transformations) a unique tight spherical* 11*-design on $S^{23} \subset \mathbb{R}^{24}$ is unique, namely the* 196 560 *minimal vectors of the Leech lattice.*

- A tight 1-design consists of 2 points on S^{n-1} which are antipodal to each other.
- A tight 2-design is a regular simplex on S^{n-1} consisting of $n+1$ points.
- A tight 3-design is a cross polytope on S^{n-1}, that is, $\{\pm \boldsymbol{e}_i \mid 1 \leq i \leq n\}$, where $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ is a orthonormal basis of $\mathbb{R}^n$.
- The classification of tight spherical t-designs for $t = 4, 5, 7$ is still an open problem. The following are the known example of spherical tight designs on S^{n-1} $(n \geq 3)$.

$t = 4$: 27 points on S^5 related to the E_6 root system. 275 points on S^{21}.

$t = 5$: 12 vertices of an icosahedron on S^2. 126 vectors of the E_7 root system on S^6. 552 points on S^{22}.

$t = 7$: 240 points of the E_8 root system on S^7. 4600 points on S^{22} which is a section of the Leech lattice.

There is a work by Bannai–Munemasa–Venkov [48] that shows the non-existence of certain tight 4-, 5-, and 7-designs. It is shown in [88] that the existence of a tight 4-design on S^{n-2} implies the existence of a tight 5-design on S^{n-1} (and vice versa). So, we consider the cases $t = 5$ and $t = 7$. It is shown by the same arguments as in Theorem 2.16, that if there exists a tight 5-design X on S^{n-1}, then either $n = 3$ or $n = (2m+1)^2 - 2$ for a positive integer m. The case $n = 3$ actually corresponds to the 12 vertices of a regular icosahedron. Examples for $m = 1, 2$ are known, as just mentioned above. [48] shows the non-existence for $m = 3, 4$ and for infinitely many values of m. The first open case for $t = 5$ is $m = 5$ ($n = 119$). Similarly, if X is a tight 7-design on S^{n-1}, then $n = 3d^2 - 4$ for a positive integer $d \geq 2$. For $d = 2, 3$, the examples are known, as just mentioned above. [48] shows the non-existence for $d = 4, 5$ and for infinitely many values of d. The first open case for $t = 7$ is $d = 6$ ($n = 104$). These proofs for both $t = 5$ and $t = 7$ are very involved, and the considerations of the Euclidean lattices generated by the points of X play important roles. The reader is referred to [48] for the details.

2.5. Rigid spherical designs

Let $X \subset S^{n-1}$ be a spherical t-design. Let σ be an orthogonal transformation of $\mathbb{R}^n$. It is well known that X^σ is also a spherical t-design. The following definition is given by Bannai in [25].

Definition 2.18. Let $X = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_N\} (\subset S^{n-1})$ be a spherical t-design. X is called non-rigid, if for any positive real number ε, there exists a spherical t-design $X' = \{\boldsymbol{u}'_1, \boldsymbol{u}'_2, \ldots, \boldsymbol{u}'_N\}$ satisfies the following two conditions.

(1) $\|\boldsymbol{u}_i - \boldsymbol{u}'_i\| < \varepsilon$ for $i = 1, 2, \ldots, N$.
(2) There is no orthogonal transformation σ satisfying $X' = X^\sigma$.

If X is not non-rigid, then X is called rigid.

The definition of rigid t-designs implies that they cannot be deformed in the class of t-designs. So, this rigidity may have some physical meaning, though we do not know what they are in a rigorous sense.

Tight t-designs are rigid t-designs. Besides tight t-designs, not many rigid t-designs are known. Bannai [25] (cf. [107]) proved the following theorem which gives the classification of rigid t-designs on $S^1 \subset \mathbb{R}^2$.

Theorem 2.19 (*[25]*). *Let X be a rigid t-design on S^1. Then X is a regular $(k+1)$-gon, with $t \leq k \leq 2t$.*

The only rigid t-designs on S^{n-1} currently known (with $n \geq 3$) which are not tight t-designs are the following:

(1) $(n+2)$-point sets on S^{n-1} with $n =$ even. (See Sali [156].) (These point sets first appeared in the old work of Seidel [159] on 2-distance sets in $\mathbb{R}^n$.)

(2) 120 points of a 600-cell on $S^3 \subset \mathbb{R}^4$. (See Boyvalenkov [59], and Nozaki [143].)

It is interesting to note that each of these examples has the structure of an association scheme. It seems that it is an interesting open question to classify rigid t-designs X on S^{n-1}, even for some special values of t, n, and $|X|$. Bannai proposed the following conjectures:

Conjecture 1. *There exists a function $f(n, t)$ such that, if X is a spherical t-design on S^{n-1} with $|X| > f(n, t)$, then X is non-rigid.*

Conjecture 2. *For each given pair of n and t, there are only finitely many rigid spherical t-designs on S^{n-1}, up to orthogonal transformations.*

(Obviously, Conjecture 2 implies Conjecture 1.)

Lyubich–Vaserstein [127] proved that Conjecture 1 implies Conjecture 2, and so these two conjectures are equivalent. It seems that we do not know the answer to this conjecture even for the simplest case $(n, t) = (3, 2)$. There are many results which prove the non-rigidity of some particular t-designs. (See, Seki [162], Sali [156,157], Cohn–Conway–Elkies–Kumar [73], Nozaki [143].) In particular, Sali studied which spherical t-designs on S^{n-1} that are orbits of a real reflection group G in $O(n)$ are rigid as spherical m_2-design on S^{n-1}. (See Section 3.1, for the undefined terminology.) The two cases of H_4 and E_8 were left open in Sali [157], and these cases were settled by Nozaki [143]. (See also Boyvalenkov [59] for the case H_4.)

2.6. Existence theorem of spherical designs

In this section we discuss the existence of spherical t-designs in general. Seymour–Zaslavsky [164] proved a theorem in a very general situation including the case of spherical t-designs. There is also a proof for the existence of interval t-designs on an interval of the real line $\mathbb{R}$ by Juan Arias de Reyna [2]. Her proof is comparatively easier to read even for non-experts. Here we state her theorem first.

Theorem 2.20 (*Juan Arias de Reyna, [2]*). *Given real valued continuous functions $f_1, f_2, \ldots, f_m$ defined on the interval $[0, 1] \subset \mathbb{R}$, there exist points $x_1, x_2, \ldots, x_N$ in $[0, 1]$ satisfying*

$$\int_0^1 f_j(x)\mathrm{d}x = \frac{1}{N}\sum_{i=1}^{N} f_j(x_i)$$

for any j with $1 \le j \le m$. Moreover N may be any number with a finite number of exceptions.

Next we introduce the theorem proved by Seymour–Zaslavsky [164]. Let Ω be a path-connected topological space provided with a positive finite measure μ which satisfies $\mu(S) \ge 0$ for any measurable set and $\mu(U) > 0$ for any non-empty open set. Let $f_1, f_2, \ldots, f_m$ be continuous integrable functions: $f_i : \Omega \longrightarrow \mathbb{R}^p$. An *averaging set* for $f_1, f_2, \ldots, f_m$ is a finite subset $X \subset \Omega$ satisfying

$$\frac{1}{\mu(\Omega)}\int_\Omega f_j \mathrm{d}\mu = \frac{1}{|X|}\sum_{\boldsymbol{u}\in X} f_j(\boldsymbol{u})$$

for any f_j, $1 \le j \le m$.

Theorem 2.21 (*Main Theorem in [164], 1984*). *Given Ω, μ and $f_1, f_2, \ldots, f_m$ as above, there exists an averaging set X. The size of X may be any number, with a finite number of exceptions. Moreover X may be chosen so that the vectors $\{f_1(\boldsymbol{u}), f_2(\boldsymbol{u}), \ldots, f_m(\boldsymbol{u})\}$ for $\boldsymbol{u} \in X$ are all distinct.*

If $\Omega = S^{n-1}$, with the usual Haar measure σ, and for a basis $\{f_1, f_2, \ldots, f_m\}$ of $\mathcal{P}_t(S^{n-1})$, then an averaging set X is a spherical t-design. As a corollary of Theorem 2.21 we have the following existence theorem for spherical t-designs.

Theorem 2.22 (*[164]*). *For each pair of integers n and $t > 0$, and for any sufficiently large integer N, there exists a spherical t-design $X \subset S^{n-1}$ whose cardinality is N.*

These theorems show that for given n and t, there exists an integer $g(n, t)$ depending on n and t, and for any $N \ge g(n, t)$ there exists a spherical t-design of cardinality N. There are several works estimating the lower bound of such an integer $g(n, t)$. For more information on this material refer the following papers (and also the Section 2.7): Wagner [182], Rabau–Bajnok [151], Korevaar–Meyers [113], Kuijlaars [114].

2.7. Explicit construction of spherical designs

Spherical t-designs with small parameters have been constructed explicitly. For $t = 2$, Mimura [131] proved that if n is even and $|X| = n + 2$, spherical 2-designs on S^{n-1} do not exist. Except for these cases, spherical 2-designs on S^{n-1} with $|X| \ge n + 1$ do exist.

For $t = 3$. Bajnok [13] proved the following:

If $n = 3$, then for $|X| = 6, 8, \ge 10$, spherical 3-designs on S^{n-1} exist. For other values of $|X|$, it is conjectured that spherical 3-designs on S^{n-1} do not exist.

If $n = 4$, then for $|X| = 8 \ge 10$, spherical 3-designs on S^{n-1} exist. For other values of $|X|$, it is conjectured that spherical 3-designs on S^{n-1} do not exist.

If $n = 5$, then for $|X| = 10, 12, \ge 14$, spherical 3-designs on S^{n-1} exist. For other values of $|X|$, it is conjectured that spherical 3-designs on S^{n-1} do not exist.

If $n \geq 6$, then for $|X|$: even number with $\geq 2n$, or $|X|$: odd number with $\geq \frac{5n}{2}$, spherical 3-design on S^{n-1} exist. For other values of $|X|$, it is conjectured that spherical 3-designs on S^{n-1} do not exist. (For some further developments, see [62].)

For $t = 4$, Hardin–Sloane [101] proved that if $n = 3$, then those with $|X| = 12, 14, \geq 16$ do exist, and conjecture that they do not exist for other values of $|X|$.

For $t = 5$, Hardin–Sloane [102] and Reznick [152] proved that if $n = 3$, then those with $|X| = 12, 14, 16, 18, 20, \geq 22$ do exist, and conjecture that they do not exist for other values of $|X|$.

Further results are available in Hardin–Sloane [102] and Reznick [152]. More detailed and updated information will be available in the home page of N.J.A. Sloane.: http://www.research.att.com/~njas/sphdesigns/index.html. For related topics, see Bajnok [14–17], Reznick [153].

For $n = 3$, Hardin–Sloane [102] constructed putative t-designs with size $N = 12m$ ($m \geq 2$) with $N = (t^2/2)(1 + o(1))$, numerically for $t \leq 21$. They conjecture that there exist spherical t-designs on S^2 with $N = (t^2/2)(1 + o(1))$. (For some further discussion on related topics, see Chen–Womersley [72]. Also numerically computed spherical t-designs with $(t+1)^2$ points for degree up to 50 are available at http://web.maths.unsw.edu.au/~rsw/Sphere.

(See Sloan and Womersley [170].)

Concerning the existence of spherical t-designs on S^{n-1}, there is a method to reduce it to the existence of interval t-designs with certain (i.e. Gegenbauer) weight functions for the integral, by using the separation of variables as follows.

Let $\omega_\alpha(x) := (1-x^2)^\alpha$ with $\alpha = \frac{n-3}{2}$, be the weight function (Gegenbauer weight) on the interval $[-1, 1]$. We say that a subset $X = \{\xi_1, \xi_2, \ldots, \xi_M\}$ is an interval t-design on $[-1, 1]$ with respect to the weight function $\omega_\alpha(x)$. Namely,

$$\frac{1}{\int_{-1}^{1} \omega_\alpha(x)\mathrm{d}x} \int_{-1}^{1} \omega_\alpha(x) f(x)\mathrm{d}x = \frac{1}{|X|} \sum_{x \in X} f(x)$$

for any polynomial $f(x)$ of degree at most t. Let S_i ($i = 1, 2, \ldots, M$) be the sphere in $\mathbb{R}^{n-1}$ of radius $\sqrt{1 - \xi_i^2}$ with the center $(\xi_i, 0, \ldots, 0) \in \mathbb{R}^n$ whose first abscissa is ξ_i. Then all the S_i's are on S^{n-1}. Let X_i ($i = 1, 2, \ldots, M$) be any spherical t-design on S_i ($i = 1, 2, \ldots, M$). We assume that all $|X_i| = K$ ($i = 1, 2, \ldots, M$). Then $\cup_{i=1}^{M} X_i$ becomes a spherical t-design of size $K \times M$ on S^{n-1}. (See, e.g. Wagner [182] Rabau–Bajnok [151].)

If we apply this method of separation of variables to construct spherical t-designs with $n = 3$, then ω_α is a constant function. Therefore, if there exists an ordinary interval t-design $X = \{\xi_1, \xi_2, \ldots, \xi_M\}$ on the interval $[-1, 1]$, then taking X_i as a regular $(t + 1)$-gon on S_i for $i = 1, 2, \ldots, M$, we have a spherical t-design on S^2 of size $(t+1)M$. As is explained in the next paragraph, combining the result of Bernstein [52] and Kuijlaars [114] (or just by Kuijlaars) we get an ordinary interval t-design on $[-1, 1]$ of size $O(t^2)$. Therefore, we get the existence of spherical t-designs of size $O(t^3)$ on S^2. (See also [112, 113].)

The main theorem of Kuijlaars [114] (Theorem 2.25 in the following) shows the existence of interval t-designs of size $O(t^{2+2\alpha})$ on $[-1, 1]$ with respect to the weight function $\omega_\alpha(x)$. Therefore, there are spherical t-designs on S^{n-1} of size $O(t^{n(n+1)/2})$. Then what is the smallest cardinality of spherical t-designs actually existing? This still seems to be an open problem. It seems that many people think $O(t^{n-1})$ should be possible cf. Remark 3.3.4 in [112] (see Seidel [160]). In [183] (page 1062 in English version), Yudin mentioned that the conjecture $N_n(t) \ll t^{n-1}$ has been made by many authors, where $N_n(t)$ is the smallest cardinality of a t-designs on S^{n-1} and $A \ll B$ means that $A \leq cB$ for some constant $c > 0$ that depends only on n. Yudin [184] gives an interesting new kind of bound for the size of spherical t-designs.)

Kuijlaars explains the work of Bernstein [53,52] in his paper published in 1993 [114].

Theorem 2.23 (*Bernstein [53]*). *Let*

$$\int_{-1}^{1} f(x)w(x)\mathrm{d}x = \sum_{i=1}^{n} p_i f(x_i)$$

be a quadrature formula of degree $t = 2m - 1$. Let $X = \{x_1, x_2, \dots, x_N\}$ be a Chebyshev type quadrature formula, i.e. the p_i's are all equal. Then,

$$N \geq \frac{1}{\lambda_{1,m}}$$

holds, where $\lambda_{1,m}$ is the Christoffel number appearing in the m-point Gauss quadrature formula

$$\int_{-1}^{1} f(x)w(x)\mathrm{d}x = \sum_{i=1}^{m} \lambda_{i,m} f(\xi_{i,m})$$

of degree $2m - 1$ with respect to $w(x)$ and $1 > \xi_{1,m} > \xi_{2,m} \cdots > \xi_{m,m} > -1$. (This implies $N \geq O(t^2)$ for $w(x) = $ constant, and $N \geq O(t^{2+2\alpha})$ for $w_\alpha(x) = (1 - x^2)^\alpha$.)

Theorem 2.24 (*Bernstein [52]*). *For $w(x) = \frac{1}{2}$, and for every m, there exists a Chebyshev type quadrature formula of degree $2m - 1$ with N nodes, where*

$$N \approx 4\sqrt{2}(m + 1)(m + 4).$$

(Note that, here we do not assume that N nodes $x_1, x_2, \dots, x_N$ are mutually distinct.)

Kuijlaars explains that these results of Bernstein are not well known, and are stronger than the estimate $O(t^3)$ for the minimal number of nodes in a Chebyshev quadrature formula of degree t, mentioned in Rabau–Bajnok [151].

The main theorem of Kuijlaars's paper [114] is stated as follows.

Theorem 2.25 (*The Main Theorem of Kuijlaars [114]*). *Let $w_\alpha(x) = C_\alpha(1 - x^2)^\alpha$, where C_α is a positive constant and $\alpha \geq 0$. There is a constant $K = K_\alpha > 0$, depending only on α such that, for every t, there exists a Chebyshev type quadrature formula of degree t for the weight function $w_\alpha(x)$ having size $N < Kt^{2+2\alpha}$. Moreover, it is possible to use either a large number of multiple nodes: only $\approx \frac{t+1}{2}$ nodes, or N distinct nodes.*

The last part of the claim in Theorem 2.25 is obtained by using the following splitting theorem, which may be of interest itself.

Theorem 2.26 (*Theorem 2.1 in [114]*). *Let $1 > y_1 \geq y_2 \geq \cdots \geq y_m > -1$ be m points. Assume that $y_i = y_{i+1}$ and $y_j = y_{j+1}$ imply $j - i$ is even. Let $p_i > 0$, $i = 1, 2, \dots, m$. Then for $t > 0$ sufficiently small, there exist points $x_i(t)$, $i = 1, 2, \dots, m$, satisfying the following* (i)–(iii) .

(i) $1 > x_1(t) > x_2(t) > \cdots > x_m(t) > -1$.
(ii) $\lim_{t\downarrow 0} x_i(t) = y_i$, *for* $i = 1, 2, \dots, m$.
(iii) $\sum_{i=1}^{m} p_i x_i(t)^l = \sum_{i=1}^{m} p_i y_i^l$, *for* $l = 1, 2, \dots, m$.

Using this separation lemma, Kuijlaars proves that if there are enough, say $N_0 > m - 1$, distinct nodes $x_i \in (-1, 1)$ in the original Chebyshev type quadrature formula of degree $m - 1$, then we can get a new Chebyshev type quadrature formula of degree $m - 1$ with $N_0 + 1$ nodes. Repeating this process, we can take the N nodes all distinct in the main theorem, Theorem 2.25.

As is shown in Wagner [182], Rabau–Bajnok [151], it is easily shown that if we have an interval t-design on $[-1, 1]$ with the Gegenbauer weight function w_α with $\alpha = \frac{n-3}{2}$ and a spherical t-design on S^{n-2}, then we can construct a spherical t-design on S^{n-1}, using the separation of variables of integral on the sphere. So, it is very interesting to find explicit constructions of interval t-designs with Gegenbauer weight functions. As we discussed in Section 2.6, the existence is guaranteed by the theorem of Seymour–Zaslavsky and there are many different proofs available. Note that the combination of Bernstein [52] and the main theorem of Kuijlaars [114] (Theorem 2.25) gives yet another existence proof of spherical t-designs in S^{n-1}. It seems that all these results for $n \geq 3$ and large t are existence theorems. So, explicit constructions are very desirable. We remark that Kuperberg [115] proved that 2^s points

$$\pm z_1 \pm z_2 \pm \cdots \pm z_s$$

form a Chebychev type quadrature formula of degree $(2s+1)$ on $[-1, 1]$ with constant weight if and only if the z_i's are the zeros of the polynomial

$$Q(x) = x^s - \frac{x^{s-1}}{3} + \frac{x^{s-2}}{45} - \cdots + \frac{(-1)^s}{1 \cdot 3 \cdot 15 \cdot \cdots \cdot (4^s - 1)}.$$

Moreover, all the roots of Q are real and the resulting quadrature formula is interior, i.e. all these 2^s points are in the open interval $(-1, 1)$. Moreover, these 2^s points are distinct. This result can be regarded as giving a kind of explicit construction for an interval design on $[-1, 1]$ with $w_\alpha = \frac{1}{2}$, hence the explicit construction of spherical designs for $S^2 \subset \mathbb{R}^3$. It is interesting if we could find similar explicit constructions for $n \geq 4$, i.e. for interval t-design on $[-1, 1]$ with the Gegenbauer weight function $w_\alpha(x) = (1 - x^2)^\alpha$ with $\alpha = \frac{n-3}{2}$. We expect it is probably not difficult. We propose this as an open problem for the reader to challenge. This would give an explicit construction of spherical t-designs on S^{n-1} for any t and n.

3. Further examples of spherical configurations

3.1. Spherical designs which are orbits of finite subgroups of O(n)

Let G be a finite subgroup of the real orthogonal group $\mathrm{O}(n)$, and let $\boldsymbol{u}$ be an element on S^{n-1}. Let

$$\boldsymbol{u}^G = \{\boldsymbol{u}^g \mid g \in \mathrm{O}(n)\} \subset S^{n-1}.$$

(Namely, $\boldsymbol{u}^G$ is obtained as an orbit of the group G.) Note that all the examples of spherical designs mentioned in Section 2.3 are obtained in this way. Actually, this is one of the easiest ways to obtain spherical t-designs. For $n = 2$, we can obtain spherical t-designs with arbitrary (large) t, since regular $(t+1)$-gons are t-designs obtained in this way. Our basic question is whether we can obtain good spherical t-designs for $n \geq 3$ by this method. The answer is yes and no. We do get many good examples of t-designs with relatively big t, but we cannot get very good ones (i.e. those with arbitrary big t) by this method. In this section we will discuss these facts as well as many related topics.

It seems that Sobolev [171] was the first who considered spherical t-designs which are orbits of finite subgroups of $\mathrm{O}(n)$. Sobolev and his school, or more generally people in analysis of approximation theory, were interested in these problems from the viewpoint of cubature formulas. They were the true pioneers in the study of spherical designs. (Cf. also [137].)

It was in 1977 (Delsarte–Goethals–Seidel [88]) that the spherical t-designs were defined as finite subsets (of points) on S^{n-1}, without considering weights in cubature formulas (or, with constant weight). The spherical t-designs which are orbits of finite groups of $\mathrm{O}(n)$ were studied in Goethals–Seidel [97,98], Bannai [22] (in 1970's), and then Bannai [23,24] (in 1980's), and more recently in Sidelnikov [168,169], de la Harpe–Pache [84], Bajnok [18], Victoir [181], and others. Also, there are several related works which classify good finite subgroups of $\mathrm{O}(n)$ by using the classification of finite simple groups, e.g. Lempken–Schroder–Tiep [121], Tiep [178], and so on. Some of the basic results are summarized as follows. (We refer the reader to de la Harpe–Pache [84] for details.)

Let G be a subgroup of $\mathrm{O}(n)$. For any non-negative integer k, let $\pi^{(k)}$ be the real irreducible representation of $\mathrm{O}(n)$ on the space $\mathrm{Harm}_k(\mathbb{R}^n)$ of the homogeneous harmonic polynomials of degree k. $\pi_G^{(k)}$ denotes the restriction of the representation to the subgroup G. (It is well known that the degree of $\pi^{(k)}$ is equal to $\binom{n+k-1}{k} - \binom{n+k-3}{k-2}$.) We call a subgroup G of $\mathrm{O}(n)$ *t-homogeneous*, if for an arbitrary $\boldsymbol{u} \in S^{n-1}$, the orbit $\boldsymbol{u}^G$ is a spherical t-design. Also, for representations ρ and σ of G, if ρ is a subrepresentation of σ, then we denote $\rho < \sigma$. Let 1_G denote the identity representation of G.

Theorem 3.1 (*Combination of Many Works, cf. [22,97,84]*). *Let G be a finite subgroup of $\mathrm{O}(n)$ and let s, t be positive integers.*

(1) $1_G \not< \pi_G^{(k)}$ *for $1 \leq k \leq t$ if and only if G is t-homogeneous.*
(Note that $1_G \not< \pi_G^{(k)}$ is equivalent to $\mathrm{Harm}_k(\mathbb{R}^n)^G = \{0\}$.*)*

(2) *If $n \geq 3$ and if $\pi_G^{(k)}$ is irreducible for $1 \leq k \leq s$, then G is $2s$-homogeneous.*
(3) *If $n \geq 3$, if $\pi_G^{(k)}$ is irreducible for $1 \leq k \leq s$, and if $\pi_G^{(s)} \not\prec \pi_G^{(s+1)}$, then G is $(2s+1)$-homogeneous.*

Remark. It was later obtained by Pache (see [84], Appendix) that in (2), the irreducibility of $\pi_G^{(s)}$ is enough to ensure that G is $2s$-homogeneous.

We also mention that the harmonic Molien series for G is given as follows:

$$\Phi_G = \sum_{i\geq 0} \dim \left(\mathrm{Harm}_i(\mathbb{R}^n)^G\right) t^i = \sum_{i \geq 0} a_i t^i.$$

(Here, $\mathrm{Harm}_i(\mathbb{R}^n)^G$ means the subspace of G-invariant homogeneous harmonic polynomials of degree i in $\mathbb{R}^n$.)

Then G is t-homogeneous if and only if $a_1 = a_2 = \cdots = a_t = 0$. (Moreover, the following assertion holds: let $\boldsymbol{u}_0 \in S^{n-1}$. Then $\boldsymbol{u}_0^G$ is a spherical t-design if $f(\boldsymbol{u}_0) = 0$, for any $f \in \oplus_{i=1}^t \mathrm{Harm}_i(\mathbb{R}^n)^G$.

As a special case of this assertion, for any irreducible real reflection group W in $O(n)$, we have the following results.

Let W be an irreducible real reflection group in $\mathrm{O}(n)$, and let $1 = m_1 \leq m_2 \leq m_3 \leq \cdots \leq m_n$ be the exponents of W. (See Bourbaki [56] for the definition of exponents.) Then the following assertions hold. (Here, note that $\Phi_W = \prod_{i=2}^n \frac{1}{(1-t^{m_i+1})}$, for W.)

(1) For any $\boldsymbol{u} \in S^{n-1}$, $\boldsymbol{u}^W$ is a spherical m_2-design.
(2) For some $\boldsymbol{u}_0 \in S^{n-1}$, $\boldsymbol{u}_0^W$ is a spherical m_3-design.

(Note that $W(E_8)$ has exponents $m_1 = 1, m_2 = 7, m_3 = 11, \ldots$. Therefore, any orbit $\boldsymbol{u}^{W(E_8)}$ is a spherical 7-design, and some orbit $\boldsymbol{u}_0^{W(E_8)}$ is a spherical 11-design.)

Theorem 3.2 (*Bannai [23]*). *Let $\boldsymbol{u}_1, \boldsymbol{u}_2 \in S^{n-1}$. Suppose that $\boldsymbol{u}_1^G$ is a t_1-design but not a (t_1+1)-design, and also that $\boldsymbol{u}_2^G$ is a t_2-design but not a (t_2+1)-design. Then $t_2 \leq 2t_1 + 1$.*

Remark. This theorem gives a reason to why considering t-homogeneous subgroups of $\mathrm{O}(n)$ is important.

Theorem 3.3 (*Bannai [24]*). *Assume $n \geq 3$. Then, there is a function $f(n)$ such that if $\boldsymbol{u}^G$ is a spherical t-design for $\boldsymbol{u} \in S^{n-1}$ and for a finite subgroup G of $O(n)$, then $t \leq f(n)$ holds.*

It is conjectured that there is an absolute constant t_0 such that if $\boldsymbol{u} \in S^{n-1}$ and if $\boldsymbol{u}^G$ is a spherical t-design, then $t \leq t_0$, though we have not yet succeeded in proving this. The largest example we have is $t = 19$, when $G = W(H_4)$. (We can get examples of $t = 15$ for $G = \cdot 0$ (Conway group, i.e. the automorphism group of the Leech lattice fixing the origin) and so, t_0 might be 19.) Also, it is interesting to remark that no 12-homogeneous finite subgroup G of $\mathrm{O}(n)$ is known. (It would be natural to expect that these problems might be solvable by using the classification of finite simple groups, and extending the approach of Lempken–Schroder–Tiep [121], and Tiep [178].)

We remark that Sidelnikov [168] proved that the subgroups $G = 2_+^{1+2m}\mathrm{O}_{2m}^+(2)$ with $m \geq 3$ are 7-homogeneous subgroups of $\mathrm{O}(2^m)$, where G are called real Clifford groups. Furthermore, Sidelnikov [169] proved that $\dim \mathrm{Harm}_j(\mathbb{R}^{2^m})^G = 0$, for $j = 9, 10, 11$. This implies that for some appropriate point $\boldsymbol{u}_0 \in S^{2^m-1}$, the orbit of $\boldsymbol{u}_0$ by G becomes an 11-design. (Nebe–Rains–Sloane [138] explains that this information also follows from Runge [154].) It is very interesting that these groups are famous in group theory, and they appear in many different areas: quantum error correcting codes [67,68], construction of packings in real Grassmannian spaces [66,79,166], and construction of orthogonal spreads and Kerdock sets (Calderbank–Cameron–Kantor–Seidel [65]). Also, these groups, or more exactly speaking the complex Clifford groups $Z_8 * 2^{1+2m}Sp_{2m} \cong Z_8 * 2^{1+2m}\mathrm{O}_{2m+1}(2)$, which are subgroups of the unitary group $U(2^m)$ are closely connected to the Siegel modular forms of genus m (Runge [154]). The reader will find these (and further) interesting information in Chapter 6 of Nebe–Rains–Sloane [138].

Although the authors have not digested the details yet, they found several further work by Sidelnikov and others on spherical designs which are orbits of finite groups, see e.g. Dorofeev–Kazarin–Sidelnikov–Tuzhilin [93], and many other papers written in Russian.

3.2. Spherical designs which are shells of Euclidean lattices

Another natural method to obtain spherical t-designs is to consider the shells of a lattice in Euclidean space.

(a) Modular forms and a theorem of Venkov

Let L be a lattice in $\mathbb{R}^n$. This means that $L = \mathbb{Z}\boldsymbol{x}_1 + \mathbb{Z}\boldsymbol{x}_2 + \cdots + \mathbb{Z}\boldsymbol{x}_n$ is a free abelian subgroup of the additive group $\mathbb{R}^n$ and $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_n\}$ is a basis of the $\mathbb{R}$-vector space $\mathbb{R}^n$. As we discussed in the previous section, if the automorphism group $G \subset O(n)$ of L is t-homogeneous, then each orbit of any $\boldsymbol{x} \in L$ by G is a spherical t-design. Therefore, if we assume $L_m = \{\boldsymbol{x} \in L \mid \|\boldsymbol{x}\|^2 = m\}$, then L_m is a union of some orbits of G, and so, it is a spherical t-design. We call L_m *a shell of the lattice L*. L_m is a finite set on $S^{n-1}(\sqrt{m})$, the sphere of radius $\sqrt{m}$. (Note that $X \subset S^{n-1}(\sqrt{m})$ is called a spherical t-design if $\frac{1}{\sqrt{m}}X$ is a spherical t-design on the unit sphere S^{n-1}.)

Venkov [180] proved that for extremal even unimodular lattices, the shells are spherical t-designs, even if their automorphism groups are not t-homogeneous. This result is stated as follows. Let us recall some terminologies. A lattice $L \subset \mathbb{R}^n$ is called *unimodular* if the *dual lattice* L^* coincides with L, where $L^* = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} \cdot \boldsymbol{y} \in \mathbb{Z},\ {}^\forall\boldsymbol{y} \in L\}$. If $\|\boldsymbol{x}\|$ is an even integer for any $\boldsymbol{x} \in L$, then L is called *even*. If L is even unimodular, then n must be a multiple of 8. For an even unimodular lattice L, it is known that the minimum squared norm satisfies $\min\{\|\boldsymbol{x}\|^2 \mid \boldsymbol{x} \in L,\ \boldsymbol{x} \neq 0\} \leq 2[\frac{n}{24}] + 2$. An even unimodular lattice L is called *extremal*, if the minimum squared norm of L is $2[\frac{n}{24}] + 2$, i.e. $L_{2m} = 0$ holds for $m = 1, 2, \dots, [\frac{n}{24}]$.

Theorem 3.4 (*Venkov [180]*). *Let Λ be an extremal even unimodular lattice in $\mathbb{R}^n$ with $n = 24\mu$. Then any shell Λ_{2m} with $2m \geq 2\mu + 2$ is a spherical 11-design. If $n = 24\mu + 8$, then Λ_{2m}, $2m \geq 2\mu + 2$, is a spherical 7-design. Also if $n = 24\mu + 16$, then Λ_{2m}, $2m \geq 2\mu + 2$, is a spherical 3-design.*

Examples. The E_8-lattice in $\mathbb{R}^8$, is an extremal even unimodular lattice and so any of its shells is a 7-design. The Leech lattice in $\mathbb{R}^{24}$ is an extremal even unimodular lattice and so any of its shells is an 11-design.

The proof of Theorem 3.4 uses the following two well known results in modular forms, and it is immediate. That is, just show that, if $n = 24\mu$, then $\sum_{\boldsymbol{x} \in L_{2m}} P(\boldsymbol{x}) = 0$, for any $P \in \mathrm{Harm}_j(\mathbb{R}^n)$, $j = 1, 2, \dots, 11$.

Theorem 3.5 (*Theorem of Hecke and Schoeneberg*). *Let $P \in \mathrm{Harm}_j(\mathbb{R}^n)$, and let Λ be an even unimodular lattice in $\mathbb{R}^n$. Then*

$$\Theta_{\Lambda,P} := \sum_{\boldsymbol{x} \in \Lambda} P(\boldsymbol{x}) q^{\frac{\boldsymbol{x}\cdot\boldsymbol{x}}{2}}$$

is a modular form of weight $j + \frac{n}{2}$ for the full modular group $SL(2, \mathbb{Z})$, where $q = e^{2\pi\sqrt{-1}z}$, $z \in \mathbb{H}$ (upper half plane). Moreover, if $j \geq 1$, then $\Theta_{\Lambda,P}$ is a cusp form.

Theorem 3.6 (*Well known*). *The space of cusp forms of weight ≤ 10 and 14 are of dimension 0.*

We remark that Theorem 3.4 is regarded as an analogue of the Assmus–Mattson theorem, which ensures that any shell (the set of elements of a fixed weight) of a certain code becomes a combinatorial t-design.

Currently, there are many different proofs of the Assmus–Mattson theorem, and also many generalizations in various contexts. See Assmus–Mattson [3], Koch [110,111], Bachoc [4], Janusz [108], etc. The reader is referred to a recent paper of Tanaka [175] about this topic.

We remark that there is a generalization of Theorem 3.4 for modular lattices. (Note that a lattice L is p-modular if the integral dual L^* satisfies $L = pL^*$. (See Bachoc–Venkov [12].) Let us call all those generalizations of the Assmus–Mattson theorem in both code, lattice theories, etc. *Assmus–Mattson type theorems*.

De la Harpe–Pache–Venkov [86] consider obtaining cubature formulas by taking the union of shells of a lattice as the support of the cubature formulas. This topic will be discussed in Section 4, in connection with cubature formula and Euclidean designs. (For further connections between lattices and spherical designs, see [149,81,99], etc.)

Question. We wonder whether there is an analogue of Theorem 3.2 for spherical t-designs obtained as shells of a lattice. Namely, let L be an integral lattice, and let L_1 and L_2 be shells of it. If L_1 is a t_1-design but not a (t_1+1)-design, and if L_2 is a t_2-design but not a (t_2+1)-design, then does there exist any restriction between t_1 and t_2 ?

(b) *D. H. Lehmer's conjecture in Number theory, and more on Assmus–Mattson type theorems*

Let $\eta(q) = q^{\frac{1}{24}} \prod_{i=1}^{\infty}(1-q^i) = q^{\frac{1}{24}}(1-q-q^2+q^5+\cdots)$, and let

$$\begin{aligned}\Delta_{24}(q) &= \eta(q)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - 16744q^7 + \cdots \\ &= \sum_{m\geq 1} \tau(m) q^m,\end{aligned}$$

where $q = e^{2\pi\sqrt{-1}z}$ and Δ_{24} is the cusp form of weight 12 with respect to the full modular group $SL(2,\mathbb{Z})$.

It is known that $|\tau(p)| < 2\,p^{\frac{11}{2}}$ for all prime p. (This was conjectured by Ramanujan, and proved by Deligne.)

(D.H.) Lehmer's Conjecture. $\tau(m) \neq 0$ *for any* $m \geq 1$.

It is known that it is true for $m \leq 3316799$ (Lehmer [120], 1947), and true for $m \leq 10^{15}$ (Serre [163], 1985). Why are we interested in Lehmer's conjecture? (It seems that a reason is partly because, this is another extreme of the conjecture of Ramanujan mentioned above.) The following very interesting observation was known to Venkov for many years and is stated in [85,86].

Observation (*Venkov, de la Harpe–Pache, Pache*). *Let Λ be the E_8-lattice in $\mathbb{R}^8$. Then Λ_{2m} is an 8-design, if and only if $\tau(m)=0$.*

(So, if Lehmer's conjecture is true then Λ_{2m} will never be an 8-design for any m.)

Proof of Observation. Note that for any $P \in \mathrm{Harm}_8(\mathbb{R}^8)$, $\Theta_{\Lambda,P} = \sum_{m=0}^{\infty}(\sum_{\boldsymbol{x}\in\Lambda_{2m}} P(\boldsymbol{x}))q^m = c(P)\Delta_{24}$ for some constant $c(P)$, since the space of the cusp forms of weight 12 is one-dimensional. Hence, if Λ_{2m} is not a spherical 8-design then there exists a polynomial $P \in \mathrm{Harm}_8(\mathbb{R}^8)$ satisfying $\sum_{\boldsymbol{x}\in\Lambda_{2m}} P(\boldsymbol{x}) \neq 0$. Hence $\Theta_{\Lambda,P} \neq 0$ and $c(P)\neq 0$. Hence we have $\tau(m)\neq 0$. The converse is also true because it is well known that Λ_2 is not a spherical 8-design ($|\Lambda_2| = 240$ and Λ_2 is a spherical tight 7-design). ■

We note that there are many similar situations and conjectures. For example, let Λ be the Leech lattice in $\mathbb{R}^{24}$. Then Λ_{2m} is a 12-design, if and only if $a_m = 0$. (Note that here $a_m = \sum_{i+j=m}\tau(i)\tau(j)$.)

Suppose $n = 24\mu$. Then, $\Theta_{\Lambda,P} \equiv 0$ holds for all $P \in \mathrm{Harm}_{2j}(\mathbb{R}^n)$ and $j \in \{1,2,3,4,5,7\}$. That is, Λ_{2m} is a $(11\frac{1}{2})$-design (i.e. $\sum_{\boldsymbol{x}\in\Lambda_{2m}} f(\boldsymbol{x}) = 0$ for any $f \in \mathrm{Harm}_{2j}(\mathbb{R}^n)$ for $j \in \{1,2,3,4,5,7\}$) in the sense of Venkov [180]. Moreover,

$$\Theta_{\Lambda,P} = c(P)\cdot \Delta_{24}^{1+\mu}$$

holds for $P \in \mathrm{Harm}_{12}(\mathbb{R}^n)$, where $c(P)$ satisfies one of the following two conditions:

Case 1: $c(P) = 0$, for any $P \in \mathrm{Harm}_{12}(\mathbb{R}^n)$.

(Then Λ_{2m} is a 12-design for all m.)

Case 2: $c(P) \neq 0$, for some $P \in \mathrm{Harm}_{12}(\mathbb{R}^n)$.

Theorem 3.7 (*Bannai–Koike–Shinohara–Tagami [47]*). *Let Λ be an extremal even unimodular lattice in $\mathbb{R}^n$ with $n = 24\mu$. If $\mu \leq 150$ and μ is* ***not*** *in B, where*

$$\begin{aligned}B = \{&5, 10, 15, 17, 20, 25, 28, 30, 39, 40, 45, 50, 52, 55, 61, 65, 70, 72, 75, 80, 83, 90, 94, 95,\\ &100, 103, 115, 116, 120, 125, 127, 128, 130, 135, 138, 140, 145, 147, 149, 150\},\end{aligned}$$

then Case 2 holds (Case 1 does not hold.), i.e. $c(P) \neq 0$, for some $P \in \mathrm{Harm}_{12}(\mathbb{R}^n)$.

Note that it is known (by Mallows–Odlyzko–Sloane [128]) that extremal even unimodular lattices in $\mathbb{R}^n$ exist only for n up to about 41 000. So we may assume that $\mu \leq 1800$, say.

The proof of Theorem 3.7 is outlined as follows. We use the fundamental equation for spherical designs, that is, Theorem 2.2(6) in Section 2.2, due to Venkov [179]. For each $\mu \leq 150$ ($\mu \neq 5, 6$) which is not in the set B, we can find an odd prime p and m, which satisfy the following conditions for some $k \in \{1, 2, 3, 4, 5, 7\}$:

(1) $p | n(n+2) \cdots (n+2k-2)$.
(2) $p \nmid 1 \cdot 3 \cdot 5 \cdots (2k-1)$.
(3) $p \nmid |\Lambda_{2m}|$.
(4) $p \nmid m$.

This implies that Theorem 2.2(6) does not hold for Λ_{2m}. Hence Λ_{2m} is not a $2k$-design. The cases $\mu = 5$ and $\mu = 6$ were taken care of by ad hoc arguments. In fact, the case $\mu = 5$ was later removed from the set B, by a personal communication from G. Nebe, Oct. 29, 2007.

The following conjecture is known in number theory.

Generalizations of Lehmer's conjecture due to Serre and Atkin (**cf.** [163])

Let r be even and let

$$\eta(q)^r = q^{\frac{r}{24}} \sum_{n \geq 0} p_r(n) q^n.$$

Then if $r \neq 2, 4, 6, 8, 10, 14, 26$ *then* $p_r(n) \neq 0$ *holds for any n.* (*Note that* $\tau(n) = p_{24}(n-1)$.)

Many numerical confirmations for this conjecture are obtained in number theory!! So, if this conjecture is true then there are no 12-designs among Λ_{2m} for $n = 24\mu$ and $\mu \leq 154$ and μ **not** in B.

In [47] we also obtained a result similar to Theorem 3.7 for extremal Type II codes, i.e. self-dual doubly even binary codes. Let C be a Type II code of length n over F_2. Then n is a multiple of 8. The weight of elements of C are all multiples of 4. We say C is extremal if the minimum weight (of non-zero elements) of C is equal to $4\mu + 4$. Here we assume that C is an extremal type II code, and additionally that $n = 24\mu$ for simplicity. Let C_i be the shell of C of weight i. The Assmus–Mattson theorem for codes guarantees that all the C_i ($4\mu + 4 \leq i \leq n - (4\mu + 4)$, $i \equiv$ (mod 4)) become combinatorial 5-designs. Our question is whether any of them can become a combinatorial 6-design. We note that by studying the relative invariants of the finite group (the complex reflection group No. 9 of order 192), Bachoc [4] obtained the result corresponding to the Hecke–Schoeneberg Theorem (Theorem 3.5). By a similar method we used to prove Theorem 3.7, we obtained the following result (Bannai–Koike–Shinohara–Tagami): In the assumptions given above, if μ is **not** in the following set B:

$$B = \{8, 15, 19, 35, 40, 41, 42, 50, 51, 52, 55, 57, 59, 60, 63, 65, 74, 75, 76, 80, 86, 90, 93, \\ 100, 101, 104, 105, 107, 118, 125, 127, 129, 130, 135, 143, 144, 150, 151\},$$

then any of the C_i cannot be a 6-design. (Note that it is known that extremal Type II codes do not exist for $n \geq 3720$, hence for $\mu \geq 155$, cf. Conway–Sloane [80] page 194.) It is not known whether any of the C_i for μ in the set B given above can become a 6-design or not. There is one notable difference with the case of the lattice. Namely, we can show that in the case of codes, we have one of the following exclusive two alternatives. Namely, for a given μ, (1) all the C_i are 6-designs, (2) none of the C_i is a 6-design. (So, the Lehmer type question does not occur for codes.)

3.3. Sphere packing problems

In this subsection we survey the works on algebraic combinatorics on spheres related to sphere packing problems.

There were three major breakthroughs in the area of sphere packing recently:

(1) The proof of the Kepler Conjecture by Hales [100].
(2) The determination of the kissing number in 4 dimensions by Musin [133].

(3) The developments of the sphere packing problem in 8- and 24-dimensional Euclidean spaces by Cohn–Elkies [74], and in particular the optimality of the Leech lattice among the lattice packings in 24-dimensional Euclidean space by Cohn–Kumar [75,76].

Here, we will discuss the last two topics of the three mentioned.

(a) Kissing numbers

For a given unit sphere in $\mathbb{R}^n$, how many unit spheres can touch the sphere without overlapping each other? The maximum number of such spheres is called the kissing number in $\mathbb{R}^n$, and is denoted by $k(n)$. Clearly, we have $k(2) = 6$. Then how about $k(3)$? This question was started in the Newton–Gregory dispute in 1694, whether it is 12 or 13. Now it is known that $k(3) = 12$, but the first rigorous and acceptable proof was obtained by Shütte–van der Waerden [167] only in 1952. Since then, the problem of determining the kissing number $k(n)$ for other values of n was studied by many mathematicians. The exact values of $k(n)$ for $n = 8$ and $n = 24$ were determined by Odlyzko–Sloane [146] and Levenshtein [122] independently, both in 1979. Namely, $k(8) = 240$ and $k(24) = 196\,560$. The full paper [133] of Musin which proves $k(4) = 24$ was published only recently, in 2008. (The announcement was in 2003.) For other cases, i.e. $n \geq 5$ and $n \neq 8, 24$, exact values for $k(n)$ are still unknown.

Let us quickly recall the idea of Odlyzko–Sloane [146] and Levenshtein [122] which proved $k(8) = 240$ and $k(24) = 196\,560$. The basic idea of the proof is due to Delsarte. Lemma 2.15 implies the following lemma.

Lemma 3.8 (*[88]*). *Let $X \subset S^{n-1}$ be a finite subset. Let $F(x)$ be a polynomial of finite degree and let $F(x) = \sum_{i=0}^{\infty} f_i Q_{i,n}(x)$ be the Gegenbauer expansion of $F(x)$. If $f_0 > 0$ and $f_l \geq 0$ for any $l \geq 1$ and $F(\alpha) \leq 0$ for any $\alpha \in A(X)$, then*

$$|X| \leq \frac{F(1)}{f_0}$$

holds.

As for $k(8)$, they use $F(x) = (x+1)(x+\frac{1}{2})^2x^2(x-\frac{1}{2})$. If X is a kissing configuration, then $A(x) \subset [-1, \frac{1}{2}]$ holds, and this polynomial satisfies the condition of Lemma 3.8, and they obtained $k(8) \leq 240$. For $k(24)$, they use $F(x) = (x+1)(x+\frac{1}{2})^2x^2(x-\frac{1}{2})(x+\frac{1}{4})^2(x-\frac{1}{4})^2$ and proved $k(24) \leq 196\,560$. Since we have examples with 240 kissing configuration in $\mathbb{R}^8$ and 196 560 kissing configuration in $\mathbb{R}^{24}$, we obtain $k(8) = 240$ and $k(24) = 196\,560$.

The idea of the proof of Musin [133] was to generalize the idea of Delsarte given above. Roughly speaking, the basic idea is as follows:

It was rigorously proved (by Arestov and Babenko) that as long as one tries to use Lemma 3.8 and tries to find a good polynomial $F(x)$ of arbitrary large degree satisfying the assumptions of Lemma 3.8 for the interval $[-1, 1/2]$, one cannot get better estimates than $k(4) \leq 25$. So, instead of considering the function $F(x)$ which satisfies $F(\alpha) \geq 0$ for all $\alpha \in [-1, 1)$, Musin consider a function $F(x)$ which can be positive for α on a small subinterval close to -1, and then use some geometric considerations which are very delicate. (See Musin [134,133], for the proofs of $k(3) = 12$ and $k(4) = 24$, respectively. The proof for $n = 3$ is quite accessible and quite convincing.)

A new proof of $k(3) = 12$ was also given by Musin [134], following the line of proving $k(4) = 24$. This proof is very transparent and very convincing. Computers were used extensively to find a good evaluation polynomial $F(x)$, but once it was obtained, the use of computer was limited to the use of very standard software, like Maple and/or Mathematica, and it was used only to calculate the extrema of certain polynomials of one variable in some interval. (See the good expository paper of Pfender-Zigler [149]. Also, for different kind of generalization of Delsarte's method, see [148].)

After the success of Musin, another very important generalization of the method of Delsarte was obtained by Schrijver [158] in 2005. Using the idea of Terwilliger algebras of association schemes, Schrijver succeeded in formulating semi-definite programming improving Delsarte's linear programming method. Then, using the actual computer calculation of semi-definite programming, Schrijver succeeded in improving the actual bounds for binary codes (i.e. codes in binary Hamming schemes) with small parameters. (It seems at this stage the amount of calculation in semi-definite programming is huge in general, and so it is difficult to get asymptotic good new bounds, if we understand the situation correctly.) Gijswijt–Schrijver–Tanaka [95] generalized this to non-binary codes.

The method of semi-definite programming is now rapidly going to be applied to wider areas. Bachoc–Vallentin [9] succeeded in formulating semi-definite programming for the kissing number problems. In particular, they gave a new proof of $k(3) = 12$ and $k(4) = 24$ in a very convincing way. Also, they improved many of the previously known upper bounds of $k(n)$ considerably for some small values of n. This method is also applied to show the optimality of $(4, 10, \frac{1}{6})$ spherical codes, as well as to show the maximal cardinality of packings of spheres on spherical caps (for example, for one sided kissing numbers), see [10,11,50], etc. We remark that Musin [136] also set up semi-definite programming in a general context, but the actual use of it is not easy in general (it seems).

(b) Universally optimal configurations (Cohn–Kumar)

We say that $X \subset S^{n-1}$ *is optimal* (or is an *optimal code*), if its minimum distance is the largest among all the subsets $Y \subset S^{n-1}$ with $|X| = |Y|$. An optimal code exists for any given n and size $|X|$. Generally, they may not be unique up to orthogonal transformations. Optimal codes on S^2 are classified completely for $|X| \leq 12$ and $|X| = 24$, by various mathematicians. (This problem is also known as Tamme's problem which originated in botany. Namely, how to plant the fixed number of trees so that they are separated from each other most.) For other values of $|X|$ the classification is still open. The reader is referred to the book of Ericson–Zinoviev [94], for the details of this classification for $n = 3$. The classification of optimal codes for a given pair $(n, |X|)$ is usually difficult, and known only for very special cases:

For $n \geq 4$, except for the case $(n, |X|) = (4, 10)$, all the cases so far solved are the cases where there is a *universally optimal code* (this concept will be explained soon). The most interesting open case is $(n, |X|) = (4, 24)$, since this case is related to the uniqueness of the kissing configurations in dimension 4. On the other hand, many conjectured (putative) optimal codes (of small dimensions and relatively small sizes) are produced by computer simulations. (See, e.g. the home page of N.J.A. Sloane.)

Good configurations, such as optimal codes, sometimes have good extremal properties. Yudin, Kolushov and Andreev considered the extremal property, given in the following, and proved that some of the known examples having good combinatorial properties satisfy the extremal condition (see the reference of Cohn–Kumar [77]):

Let $f(r)$ be a potential function, $f(r) = \frac{1}{r^{\frac{n}{2}-1}}$ (harmonic potential law). Consider a finite subset X of S^{n-1} satisfying the following condition:

$$\sum_{\boldsymbol{u},\boldsymbol{v}\in X,\boldsymbol{u}\neq\boldsymbol{v}} f(\|\boldsymbol{u}-\boldsymbol{v}\|^2) \leq \sum_{\boldsymbol{u},\boldsymbol{v}\in Y,\boldsymbol{u}\neq\boldsymbol{v}} f(\|\boldsymbol{u}-\boldsymbol{v}\|^2)$$

for all the subset $Y \subset S^{n-1}$ with $|X| = |Y|$.

Cohn–Kumar [77] formulated the concept of universally optimal codes (configurations), which satisfy the optimality for a very wide class of potential functions in the following sense.

Definition 3.9 (*Cohn–Kumar [77]*). We call $X \subset S^{n-1}$ universally optimal if it minimizes $\sum_{\boldsymbol{u},\boldsymbol{v}\in X,\boldsymbol{u}\neq\boldsymbol{v}} f(\|\boldsymbol{u}-\boldsymbol{v}\|^2)$, among the finite sets of cardinality with a fixed cardinality $|X|$, for any completely monotonic function $f : (0, 4] \longrightarrow \mathbb{R}$, (i.e. f is in C^∞, and $(-1)^k f^{(k)} \geq 0$, for all k, where $f^{(k)}$ denotes the kth derivative of f).

This definition is equivalent to the following definition.

Definition 3.10 (*Cohn–Kumar [77]*). We call $X \subset S^{n-1}$ universally optimal if it minimizes $\sum_{\boldsymbol{x},\boldsymbol{y}\in X,\boldsymbol{x}\neq\boldsymbol{y}} \alpha(\boldsymbol{x}\cdot\boldsymbol{y})$, among the finite sets of cardinality with a fixed cardinality $|X|$, for any absolutely monotonic function $\alpha : [-1, 1) \longrightarrow \mathbb{R}$, i.e. α is in C^∞, and $\alpha^{(k)} \geq 0$, for all k.

Note: A universally optimal code is an optimal code. We can see this by considering the function $f(r) = 1/r^m$ with $m \to \infty$.

Cohn–Kumar [77] proved the following theorem.

Theorem 3.11 (*Cohn–Kumar [77]*). *If $X \subset S^{n-1}$ is a t-design and an s-distance set, and if $t \geq 2s-1$, then X is universally optimal.*

Theorem 3.11 implies that tight t-designs are universally optimal.

Cohn–Kumar [77] also conjecture that for each given $n \geq 3$, there are only finitely many universally optimal codes on S^{n-1} (up to orthogonal transformations.) Moreover, they point out that, using an old work of Leech [119], there are only three universally optimal codes on S^2, that is, the vertices of the regular tetrahedron, the regular octahedron (cross polytope), and the regular icosahedron. (They are all tight designs.) They point out that for any other value of $n \geq 4$, the classification of universally optimal codes is open, and propose this classification problem. All the known examples are listed in Table 1 in Cohn–Kumar [77] (cf. also Table 3 in [123], see also [124,125]). All of the known examples satisfy the condition $t \geq 2s-1$, except for the example of 120 points in a 600-cell in $\mathbb{R}^4$. (Note that this example already appeared before several times $t = 11$, and $s = 8$.)

Cohn–Kumar [77] and Ballinger, et al. [20] search for other candidates of universally optimal codes of small dimension and size systematically by using computer simulations. In the range of $n \leq 32$ and $|X| \leq 100$, they found two new candidates with $(n, |X|) = (10, 40)$, and $(n, |X|) = (14, 64)$, and conjecture that there are no more new ones. Interestingly enough, they both have good structures of association schemes. They produced these point sets just by using a machine, and also proved that they form association schemes by the machine, and then they identified these association schemes to already known association schemes. See Section 4.1 in Ballinger et al. [20] for the first one. (Exactly speaking, [20] says that this 40 point subset in $\mathbb{R}^{10}$ appears in Sloane's online tables. It seems that the fact it is an association scheme was first noticed in [20].) See de Caen–van Dam [83] for the second one. Cohn personally asked us whether these association schemes are characterized by their parameters, and Bannai–Bannai–Bannai [37] answered that indeed they are. Whether they are actually universally optimal or not is still an open question.

Ballinger et al. [20] extended the area of their search, and found 2 more candidates. Namely, those with the parameters: $(n, |X|) = (7, 182), (15, 128)$. It is very delicate whether any of them are actually universally optimal or not, and this question still remains open.

Abdukhalikov–Bannai–Suda [1] tried to find generalizations of some of these examples for higher dimension. Actually, they find a series of candidates generalizing those with $(14, 64), (15, 128), (16, 288)$. They showed that if a possible maximum sized real MUB in $\mathbb{R}^N$ exists, then it gives candidates of universally optimal codes for $(n, |X|) = (N, N^2+2N), (N-1, N^2/2), (N-2, N^2/4)$. It is known (cf. [70,65]) such an MUB exists for any N which is a power of 4, and that if such an MUB exists then either $N = 4$ or N must be a square which is a multiple of 16. It seems that for $N > 4$ only the codes in the first family of the three parameters given above are the strong candidates of universally optimal codes. But whether any codes of the three families are actually universally optimal or not is an open problem. (See, [1,31] for more details on this topic.) As we mentioned before in Section 2.2, the uniqueness of certain spherical codes (or designs) are important. The uniqueness of tight 11-design on S^{23} was strongly relevant. We say that $X \subset S^{n-1}$ is an $(n, |X|, a)$-*code*, if $\boldsymbol{x} \cdot \boldsymbol{y} \leq a$ for all $\boldsymbol{x}, \boldsymbol{y} \in X$ with $\boldsymbol{x} \neq \boldsymbol{y}$. The maximum size $|X|$ of all $(n, |X|, 1/2)$-codes on S^{n-1} is nothing but the kissing number $k(n)$.

Let X be an $(n, |X|, 1/k)$-code. Take $\boldsymbol{x} \in X$, and let $X_{\boldsymbol{x}}$ be the set of $\boldsymbol{y} \in X$ with the inner product $\boldsymbol{x} \cdot \boldsymbol{y} = 1/k$. Then the set $X_{\boldsymbol{x}}$ forms an $(n-1, |X_{\boldsymbol{x}}|, 1/(k+1))$-code on S^{n-2} after normalization. Thus, starting from the kissing configuration in $\mathbb{R}^8$, one gets a sequence of spherical codes with parameters $(8, 240, 1/2), (7, 56, 1/3), (6, 27, 1/4), (5, 16, 1/5), (4, 10, 1/6), (3, 6, 1/7)$. Starting from the kissing configuration in $\mathbb{R}^{24}$, one gets sequences of spherical codes with parameters $(24, 196\,560, 1/2), (23, 4600, 1/3), (22, 891, 1/4), (21, 336, 1/5), (20, 170, 1/6)$. The uniqueness of these configurations for each of the cases $(8, 240, 1/2), (7, 56, 1/3), (24, 196\,560, 1/2), (23, 4600, 1/3)$ was proved in [49]. The cases $(6, 27, 1/4), (5, 16, 1/5)$ are treated in Cohn–Kumar [77], but also follows from the uniqueness of the corresponding association scheme. The case $(22, 891, 1/4)$ was treated by Cuypers [82] and independently by Cohn–Kumar [78]. ([78] corrects a minor error in the proof of the case $(23, 4600, 1/3)$ by Bannai–Sloane [49]. Whether each of

(21, 336, 1/5), (20, 170, 1/6) codes is optimal, or unique are open problems. The case (3, 6, 1/7) is obviously not optimal and not unique since the octahedron is a (3, 6, 0) code. The interesting remaining case (4, 10, 1/6) was solved by Bachoc–Vallentin [10], by using the method of semi-definite programming. This is so far the only case where the method of linear programming does not work but the method of semi-definite programming works.

Another very interesting and important open case is (4, 24, 1/2). This case is related to the uniqueness of the kissing configuration in dimension 4. This case was shown not to be universally optimal by Cohn–Conway–Elkies–Kumar [73]. It is still open whether this code is optimal or not. (We expect that it is optimal.)

(c) Work of Levenshtein

Here we survey the main result by Levenshtein [124,125,123]. The idea is to find good test functions.

Let $P_i^{a+\frac{n-3}{2},b+\frac{n-3}{2}}(x)$ be the classical Jacobi polynomials with $a, b \in \{0, 1\}$. Set

$$T_k^{1,\epsilon}(x, y) = \sum_{i=0}^{k} r_i^{1,\epsilon} P_i^{\frac{n-1}{2},\epsilon+\frac{n-3}{2}}(x) P_i^{\frac{n-1}{2},\epsilon+\frac{n-3}{2}}(y),$$

where $r_i^{1,\epsilon} = (\frac{n+2i-1+\epsilon}{n+\epsilon-1})^{2-\epsilon}\binom{n+i-2-\epsilon}{i}$ and $\epsilon \in \{0, 1\}$. Let $t_k^{1,1}$ and $t_k^{1,0}$ be the largest zero of the polynomials $P_k^{\frac{n-1}{2},\frac{n-1}{2}}(x)$ and $P_k^{\frac{n-1}{2},\frac{n-3}{2}}(x)$, respectively. Let $P_k^n(x) = P_k^{\frac{n-1}{2},\frac{n-1}{2}}(x)$. Then $P_k^n(x) = \frac{Q_{k,n}(x)}{Q_{k,n}(1)}$ holds and $P_k^n(x)$ is the Gegenbauer polynomial which is normalized to satisfy $P_k^n(1) = 1$. Levenshtein's polynomials are defined to be

$$f_m^{(y)}(x) = (x + 1)^{\epsilon}(x - y)(T_{k-1}^{1,\epsilon}(x, y))^2$$

where $m = 2k - 1 + \epsilon$ and $t_{k-1+\epsilon}^{1,1-\epsilon} \le y < t_k^{1,\epsilon}$.

Theorem 3.12 (*Levenshtein [124,125,123]*). *If X is an (n, M, y) code, then we have*

$$M \le \begin{cases} L_{2k-1}(n, y), & \text{for } t_{k-1}^{1,1} \le y < t_k^{1,0}, \\ L_{2k}(n, y), & \text{for } t_k^{1,0} \le y < t_k^{1,1}, \end{cases}$$

where

$$L_{2k-1}(n, y) = \binom{k+n-3}{k-1}\left(\frac{2k+n-3}{n-1} - \frac{P_{k-1}^{(n)}(y) - P_k^{(n)}(y)}{(1-y)P_k^{(n)}(y)}\right),$$

$$L_{2k}(n, y) = \binom{k+n-2}{k}\left(\frac{2k+n-1}{n-1} - \frac{(1+y)(P_k^{(n)}(y) - P_{k+1}^{(n)}(y))}{(1-y)(P_k^{(n)}(y) + P_{k+1}^{(n)}(y))}\right).$$

Moreover, if $M = L_m(n, y)$, then X is a spherical m-design and all the inner products $\boldsymbol{x}\cdot\boldsymbol{y}$ ($\boldsymbol{x}, \boldsymbol{y} \in X, \boldsymbol{x} \neq \boldsymbol{y}$) are zeros of $f_m^{(y)}(t)$.

We point out that the converse also holds. Namely, if X is a spherical t-design and an s-distance set on S^{n-1}, and if $t \ge 2s - 1$, then $|X| = L_t(n, \alpha)$ holds, where α is the largest number in $A(X)$.

The work of Levenshtein mentioned above was to find good polynomial test functions $F(x)$ to use Lemma 3.8 or Lemma 2.15 for spherical codes and designs using Delsarte's linear programming method. Further improvements by finding better test functions for some particular n and t (say for t-designs) were obtained by Boyvalenkov [57] and his associates. (See papers [57,60,62,142,61], etc.) Yudin [183] extended these results further by considering test functions $F(x)$ which are not necessarily polynomial functions. His bound for the size $|X|$ of spherical t-designs X on S^{n-1} is far better than the Fisher type bounds (mentioned in Theorem 2.12), if the dimension n is fixed and if t goes to ∞. (On the other hand, if n and t are related in some way, it seems difficult to obtain substantial improvements of the Fisher type bounds.) The method of Yudin is used in the work of Cohn–Kumar [77] on universally optimal codes.

(d) Spherical embeddings of association schemes and Q-polynomial association schemes

In this section, for simplicity, we consider symmetric association schemes. Let $\mathfrak{A} = \langle A_0, A_1, \ldots, A_d \rangle = \langle E_0, E_1, \ldots, E_d \rangle$ be the Bose–Mesner algebra (over the real number field) of the association scheme where $E_0, E_1, \ldots, E_d$ is the set of primitive idempotents. Then, for each $i \geq 1$, we can consider the embedding of the association scheme on the unit sphere S^{m_i-1}, where $m_i = \text{rank } E_i$. As usual let $\{\boldsymbol{e}_x \mid x \in X\}$ be the canonical basis of the $|X|$-dimensional vector space V indexed by X, i.e. for each $x \in X$, $\boldsymbol{e}_x(y) = \delta_{x,y}$ for $y \in X$.

$$X \ni x \longmapsto \sqrt{\frac{|X|}{m_i}} \boldsymbol{e}_x E_i \in S^{m_i-1}.$$

(Sometimes this map is not injective, but is always injective if the association scheme is primitive.)

We remark that in Bannai–Bannai [32], the primitive association scheme with $m_1 = 3$ is uniquely determined, i.e. a tetrahedron in $\mathbb{R}^3$. It is an interesting open problem to study this problem for $m_1 = 4$, say. You will find that this problem is not so easy to solve, because there are infinitely many examples with $m_1 = 4$. We believe that it is interesting to study the case when X are Q-polynomial schemes. (It seems that this problem is still open.) Note that the balanced condition in Terwilliger [177] is closely connected with the Q-polynomial property.

The condition that the association scheme X is Q-polynomial means that the dual intersection matrix B_1^* is a tridiagonal matrix of size $d + 1$. The condition that X is embedded on $S^{m_1-1} \subset \mathbb{R}^{m_1}$ as a finite set of strength t (spherical t-design) and degree s (s-distance set) implies $s = d$ and the first few columns (about half of t) of B_1^* take certain specific values. So, in particular t is close to $2s$, and the freedom of parameters is very much restricted.

Very recently, Martin–Williford [130] proved that there are finitely many Q-polynomial association schemes with a given first multiplicity (i.e. m_1) at least three. (Still explicit determination even for small values of $m_1 \geq 4$ seems to be relatively difficult.) Note that this is an analogue of the P-polynomial version that there are only finitely many distance-regular graphs with given valency $k \geq 3$. (Bang–Koolen–Moulton [21] recently announced that the final solution to this problem was obtained, but the paper is not yet available at the time of writing this survey: Oct 2008.)

The condition that $X \subset S^{m_1-1}$ is equivalent to the condition that the dual intersection number $q_{1,1}^1 = 0$. (See Cameron–Goethals–Seidel [69], Cameron–van Lint [71], etc.) Such X (of strength $t = 3$ and degree $s = 2$) are strongly regular graphs whose subconstituents are also strongly regular graphs, and are related to Margaret Smith graphs. Their parameters are classified as families (see [71]). The construction and the classification problems of such graphs are very interesting. This can be regarded as a very special case of the classification of universally optimal spherical codes.

In passing, we mention that a very short paper Bannai–Bannai [33] considers the spherical embeddings of strongly regular graphs, and interpret the values of their character tables from the viewpoint of 2-distance sets. (Larman–Rogers–Seidel [118] gives a nice result for 2-distance sets in $\mathbb{R}^n$ namely, for a finite 2-distance set X in $\mathbb{R}^n$ satisfying $|X| > 2n + 3$, there exists a natural number $k \geq 2$ such that the ratio of the two distances of X is given by $\sqrt{k} : \sqrt{k-1}$ with an integer satisfying $k \leq \sqrt{\frac{n}{2}} + \frac{1}{2}$.)

It would be very interesting if some generalization of this result of Larman–Rogers–Seidel could be obtained, in particular for symmetric association schemes of class $d \geq 3$ embedded on the unit sphere S^{m_1-1}.

Remark. We mention the papers Martin–Muzychuk–Williford [129] and Abdukhalikov–Bannai–Suda [1], as papers discussing Q-polynomial association schemes and spherical embeddings.

4. Generalizations

4.1. Designs in compact symmetric spaces

In this subsection, we will discuss generalizations of spherical codes and designs to other spaces such as projective spaces, Grassmannian spaces, Euclidean spaces, and possibly hyperbolic spaces, mostly just by referring the existing literature. The case of Euclidean spaces will be discussed more carefully in the next section, and so in this subsection, we will discuss the generalizations to other spaces.

(a) Projective spaces

There are many attempts to try to generalize the theory of spherical codes and designs to other spaces. Before Delsarte–Goethals–Seidel studied spherical designs and codes in [88], they considered systems of lines in [89] in $\mathbb{R}^n$, and $\mathbb{C}^n$. This is essentially equivalent to considering points in real and complex projective spaces. Jacobi polynomials appear there instead of Gegenbauer polynomials.

The attempts try to find a common framework to study finite subsets in these projective spaces (compact symmetric spaces of rank 1) and subsets in certain association schemes. They introduced several concepts, like Delsarte spaces (see Neumaier [139,140], Godsil [96]), polynomial spaces (see Levenshtein [124,125,123]), etc. The similarities of the theories between these continuous spaces and association schemes (which are discrete) were evident from the earlier works of Delsarte [87], Delsarte–Goethals–Seidel [89,88]. The most natural set up (for the continuous case) is in the compact symmetric spaces of rank 1. (These spaces are classified, and they consist of projective spaces over real, complex, quaternion fields, and projective places over the Cayley octanion.) The theory was developed systematically by Hoggar [104] in that framework. Theory of codes and designs, the concept of tight t-designs, and the classification problems of tight t-designs (i.e. Delsarte theory, or algebraic combinatorics on projective spaces) were developed (cf. [44,45,105,106]). As it was mentioned already, Gegenbauer polynomials in the case of spheres are replaced by certain classical Jacobi polynomials. A similar theory was also developed in Russia, somewhat independently since the communication between East and West was not very good at that time, see e.g., Levenshtein [124,125, 123], Kabatiansky–Levenshtein [109]. The work of Levenshtein is very complete and thorough. (We will not discuss these topics here, mainly because of the lack of both time and space, and the reader is referred to the References [124,125,123,109,58,63,142], etc.)

(b) Grassmannian spaces

The Grassmannian spaces are the set of fixed-dimensional subspaces of a vector space. So this is a generalization of projective spaces, where one-dimensional subspaces are considered. We can consider Grassmannian spaces $G(n, k)$ as the set of k-dimensional subspaces of an n-dimensional vector space V over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$. These are examples of general compact symmetric spaces of higher ranks. (In what follows, we mostly consider real Grassmannian spaces over real field $\mathbb{R}$.)

Interesting enough, sphere packing problems appear (in mid 1990) in connection with quantum error correcting codes. (See [66,79,166], etc.)

The concept of t-designs in Grassmannian spaces was introduced in Bachoc–Coulangeon–Nebe [8], and basic properties were studied. Kinds of Delsarte theory, as well as the formulation of tight t-designs was further studied in Bachoc–Bannai–Coulangeon [7], but the classification problems of tight t-designs are still open. There are further studies on algebraic combinatorics of Grassmannian spaces, although we will not go into the details. See, e.g., Bachoc [5,6], Barg–Nogin [51], Meyer (Ph.D. thesis, Bordeaux, 2008), etc. Also, there are studies of t-designs in real Grassmannian spaces which are orbits of finite groups of $O(n)$. (See Tiep [178], etc.) We will not go into the details. It is expected that similar theories exist in the framework of general (irreducible) compact symmetric spaces of arbitrary rank. (They are completely classified, see e.g., Helgason [103], etc.)

(c) *s*-distance sets in various spaces

There is a natural upper bound for the cardinality of an s-distance subset $X \subset S^{n-1}$ as we mentioned it in Section 2.4. (If this upper bound is attained, X becomes a tight $2s$-design. So, they are basically classified except for some small values of s. The situation is also true for s-distance sets in compact symmetric spaces of rank 1.)

For s-distance subsets in $\mathbb{R}^n$, Blokhuis [54], Bannai–Bannai–Stanton [39], proved an upper bound $|X| \leq \binom{n+s}{s}$. No non-trivial example attaining this upper bound was known at that time. Earlier attempts to try to find or try to show the non-existence of such examples of s-distance set which attain thus upper bound was unsuccessful. Finally, Lisoněk [126] found a non-trivial example for $(n, s) = (8, 2)$ with $|X| = 45$. (So far, this is the only known non-trivial example for $s \geq 2$ and $n \geq 2$.) An attempt to try to understand this example in the frame work of Delsarte theory on $\mathbb{R}^n$

is unsuccessful yet. But it seems that this motivated the study of finite configurations in $\mathbb{R}^n$, and the recent study of Euclidean t-designs.

Deza–Frankl [92] consider s-inner product sets in $\mathbb{R}^n$ and obtain the same upper bound $|X| \leq \binom{n+s}{s}$. The existence and the classification problems seem to be open yet. There are recent works by Nozaki [144], Nozaki–Shinoharaa [145], Musin [135], Shinohara [165] try to classify large sized s-distance sets on S^{n-1} or $\mathbb{R}^n$ for small s and n.

Finally, we mention that for the real hyperbolic space $\mathbb{H}^{n-1}$, we have addition formulas on $\mathbb{H}^{n-1}$, and the same upper bound $|X| \leq \binom{n+s}{s}$ for an s-distance subset X in $\mathbb{H}^{n-1}$, (see Bannai–Blokhuis–Delsarte–Seidel [41], also Blokhuis–Seidel [55]). Again, it is open whether there is any example which attains the upper bound. The concept of t-designs in $\mathbb{R}^n$ is defined. The concept seems to be an appropriate one, although it may not be completely convincing, since they are rather t-designs on several concentric spheres with the same center. The authors have been trying to define the concept of t-designs in $\mathbb{H}^{n-1}$, in some reasonable sense. We have to say our attempts have been unsuccessful so far. So, we would like to challenge the reader to find it, if it is at all possible.

4.2. Euclidean designs

As a generalization of spherical t-designs, Neumaier and J.J. Seidel defined Euclidean t-designs [141]. Let X be a finite set in $\mathbb{R}^n$. Let $\{\|\boldsymbol{u}\| \mid \boldsymbol{u} \in X\} = \{r_1, r_2, \ldots, r_p\}$. Let $S_i = \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x}\| = r_i\}$ and $X_i = X \cap S_i$ for $i = 1, 2, \ldots, p$. Let $A(X_i, X_j) = \{\boldsymbol{u}\cdot\boldsymbol{v} \mid \boldsymbol{u} \in X_i,\ \boldsymbol{v} \in X_j\}$, and $s_{i,j}(= s_{j,i}) = |A(X_i, X_j)|$ for $1 \leq i, j \leq p$. We consider a positive weight function $w : X \longrightarrow \mathbb{R}_{>0}$. Let $w(X_i) = \sum_{\boldsymbol{u}\in X_i} w(\boldsymbol{u})$ for $i = 1, 2, \ldots, p$. Let $\sigma,\ \sigma_1, \ldots, \sigma_p$ be the usual Haar measures on $S^{n-1}, S_1, \ldots, S_p$ respectively satisfying $|S_i| = r_i^{n-1}|S^{n-1}|$ for $i = 1, 2, \ldots, p$. Let $S = S_1 \cup S_2 \cup \cdots \cup S_p$. Let $\mathcal{P}_l(\mathbb{R}^n) = \oplus_{i=0}^{l} \mathrm{Hom}_i(\mathbb{R}^n)$, $\mathcal{P}_l^*(\mathbb{R}^n) = \oplus_{i=0}^{[\frac{l}{2}]} \mathrm{Hom}_{l-2i}(\mathbb{R}^n)$, $\mathcal{P}_l(S) = \{f|_S \mid f \in \mathcal{P}_l(\mathbb{R}^n)\}$ and $\mathcal{P}_l^*(S) = \{f|_S \mid f \in \mathcal{P}_l^*(\mathbb{R}^n)\}$.

Definition 4.1. A finite weighted set (X, w) is a Euclidean t-design if

$$\sum_{i=1}^{p} \frac{w(X_i)}{|S_i|} \int_{S_i} f(\boldsymbol{x})\mathrm{d}\sigma_i(\boldsymbol{x}) = \sum_{\boldsymbol{x}\in X} w(\boldsymbol{x})f(\boldsymbol{x})$$

holds for any polynomial $f(\boldsymbol{x})$ of degree at most t.

The following theorem gives a condition which is equivalent to the definition of Euclidean t-designs.

Theorem 4.2 (*Neumaier–Seidel*). *Let (X, w) be a weighted finite set in $\mathbb{R}^n$. Then the following conditions are equivalent.*

(1) *(X, w) is a Euclidean t-design.*
(2) *$\sum_{\boldsymbol{x}\in X} w(\boldsymbol{x})\|\boldsymbol{x}\|^{2j}\varphi(\boldsymbol{x}) = 0$ holds for any integers l and j satisfying $1 \leq l \leq t$, $0 \leq j \leq [\frac{t-l}{2}]$ and $\varphi \in \mathrm{Harm}_l(\mathbb{R}^n)$.*
(3) *Any kind of moment of X of degree at most t is invariant under any orthogonal transformation.*

Natural lower bounds for the cardinalities of Euclidean t-designs are known. The following theorem is proved by Möller.

Theorem 4.3 (*Möller*). *Let (X, w) be a Euclidean t-design. Then the following hold.*

(1) *If $t = 2e$, then $|X| \geq \dim(\mathcal{P}_e(S))$.*
(2) *If $t = 2e + 1$, then* $|X| \geq \begin{cases} 2\dim(P_e^*(S)) - 1 & \text{if } e \text{ is even and } 0 \in X, \\ 2\dim(P_e^*(S)) & \text{otherwise.} \end{cases}$

Definition 4.4. Let (X, w) be a Euclidean t-design. If equality holds in one of the inequalities given in Theorem 4.3, we say (X, w) is a tight t-design on p concentric spheres. Moreover if $\dim(\mathcal{P}_e(S)) = \dim(\mathcal{P}_e(\mathbb{R}^n))$ or $\dim(\mathcal{P}_e^*(S)) = \dim(\mathcal{P}_e^*(\mathbb{R}^n))$ holds, then (X, w) is called a Euclidean tight t-design.

For basic properties on Euclidean designs, refer [141,91,34,26]. Interesting examples of Euclidean tight designs or tight designs on p concentric spheres are found (see [19,18,31,26,27,38,40]).

In Section 2.2, we mentioned that Delsarte–Goethals–Seidel showed that "good" spherical designs have the structure of Q-polynomial association schemes. As a generalization of association schemes, Higman defined coherent configurations. We investigated all the known examples of Euclidean tight designs and found out that they have the structure of coherent configurations. Recently we proved the following theorem which gives a connection between Euclidean designs and coherent configurations [35]. (Cf. [142,61], etc. for coherent configurations.)

Theorem 4.5 (*Bannai–Bannai*). *Let (X, w) be a Euclidean t-design. Assume $w(x)$ is constant on each X_k, $(1 \le k \le p)$ and $s_{i,k} + s_{k,j} \le t - 2(p-2)$ for any $1 \le i, j, k \le p$. Then X has the structure of a coherent configuration.*

Theorem 4.6 (*Bannai–Bannai*). *Let (X, w) be an antipodal Euclidean t-design. Assume $w(x)$ is constant on each X_k, $(1 \le k \le p)$ and $s_{i,k} + s_{k,j} - \delta_{i,k} - \delta_{k,j} \le t - 2(p-2)$ for any $1 \le i, j, k \le p$. Then X has the structure of a coherent configuration.*

Remark. • It is proved that the weight functions of a tight t-design on p concentric spheres are constant on each X_i and in particular if $p = 2$ then tight t-designs (X, w) on 2 concentric spheres satisfy the condition of Theorems 4.5 and 4.6 [33,26].

• On the other hand, Bajnok [18] proved that in $\mathbb{R}^3$, the union of an octahedron, a cube–octahedron, and a cube with appropriate weights, radii, and configurations, forms a tight 7-design of $\mathbb{R}^3$ which is antipodal. We checked that it has a structure of the coherent configuration of type [3, 3, 2; 3, 5, 3; 2, 3, 4], however it does not satisfy the condition of Theorem 4.5.

It is known that if (X, w) is a Euclidean t-design, then (X', w') is also a Euclidean t-design, where $X' = \{a\boldsymbol{x} \mid \boldsymbol{x} \in X\}$ and $w'(a\boldsymbol{x}) = \lambda w(\boldsymbol{x})$ for $\boldsymbol{x} \in X$ with any positive real numbers a and λ. We call Euclidean t-designs with constant weights on a sphere $S(r) \subset \mathbb{R}^n$ of radius r (i.e. $p = 1$) also spherical t-designs when $r \ne 1$. We want to classify tight Euclidean t-designs or tight t-designs on p concentric spheres up to similarities (including scaling of the weight functions). In the following we list all known Euclidean tight t-designs or t-designs on p concentric spheres so far.

(1) In $\mathbb{R}^2$, Bajnok constructed examples of tight Euclidean t-designs or tight t-designs on p concentric spheres for any t (see [19]). It is proved in [38] that if p is at most $[\frac{t}{4}] + 1$, then it is similar to one of the examples given by Bajnok [19].
(2) (X, w) is a tight Euclidean 2-design in $\mathbb{R}^n$ if and only if X is a $(n+1)$-point 1-innerproduct set with a negative inner product, i.e. $\{\boldsymbol{u} \cdot \boldsymbol{v} \mid \boldsymbol{u}, \boldsymbol{v} \in X, \boldsymbol{u} \ne \boldsymbol{v}\} = \{\alpha\}$ for a real number $\alpha < 0$ (see [40]).
(3) Tight Euclidean 3-designs (X, w) in $\mathbb{R}^n$ are similar to $X = \{\pm r_i \boldsymbol{e}_i \mid 1 \le i \le n\}$ with $w(r_i \boldsymbol{e}_i) = \frac{1}{n r_i^2}$, $1 \le i \le n$ (see [26]).
(4) Let (X, w) be a tight Euclidean 4-design with $p = 2$. Let $X = X_1 \cup X_2$, $|X_1| \le |X_2|$.
If $X_1 = \{0\}$, then X_2 is a tight spherical 4-design.
If $|X_1| = n + 1$, then $n = 2, 4, 5, 6, 22$. X_2 has the structure of a tight 4-design in Johnson scheme $J(n+1, 2)$.
If $|X_1| = n + 2$, then $n = 4$ and X_2 has the structure of the Hamming scheme $H(2, 3)$.
If $|X_1| \ge n + 3$ and $n < 78$, then $n = 22$ and $|X_1| = 33$. In this case X_2 has the structure of the tight 4-design in the Hamming scheme $H(11, 3)$.
For $|X_1| \ge n + 3$, $n \ge 78$, the classification is still an open problem. For more information see [27].
(5) Let (X, w) be a tight Euclidean 5-design with $p = 2$.
If $X_1 = \{0\}$, then X_2 is a tight spherical 5-design.
If $0 \notin X$, then $n = 2, 3, 5, 6$. For more information see [26].
(6) Let (X, w) be a tight Euclidean 7-design with $p = 2$.
Then $0 \notin X$ and $n = 2, 4, 7$. For more information see [31].
(7) A tight Euclidean 7-design on 3 concentric spheres was constructed by Bajnok [18].

Rigidity of the Euclidean *t*-designs

All the known examples of tight Euclidean t-designs are non-rigid except those given in (4) with $n \neq 2$.

For Euclidean t-designs we give the following definition.

Definition 4.7 (*Strong Non-rigidity [40]*). Let $\left(\{\boldsymbol{u}_i\}_{i=1}^N, w\right)$ be a Euclidean t-design in $\mathbb{R}^n$. If the following condition holds, then we say $\left(\{\boldsymbol{u}_i\}_{i=1}^N, w\right)$ is *strongly non-rigid* in $\mathbb{R}^n$:

For any $\varepsilon > 0$ there exists a Euclidean t-design $\left(\{\boldsymbol{u}'_i\}_{i=1}^N, w'\right)$ such that the following two conditions hold:

(1) $\|\boldsymbol{u}_i - \boldsymbol{u}'_i\| < \varepsilon$ and $|w(\boldsymbol{u}_i) - w'(\boldsymbol{u}'_i)| < \varepsilon$, for any $1 \le i \le N$; and
(2) There exist distinct i, j satisfying $\|\boldsymbol{u}_i\| = \|\boldsymbol{u}_j\|$ and $\|\boldsymbol{u}'_i\| \neq \|\boldsymbol{u}'_j\|$.

It is known that Euclidean tight 4 and 5-designs in $\mathbb{R}^2$ are strongly non-rigid [40]. Tight spherical 2 and 3-designs on S^{n-1} are strongly non-rigid as Euclidean designs. (Euclidean tight 2 and 3-designs of $\mathbb{R}^n$ are strongly non-rigid.)

4.3. Cubature formulas

In this section we consider cubature formulas and introduce the result obtained by Möller [132]. There is a long history of research on this subject.

First we give the definition. Let Ω be a subset of $\mathbb{R}^n$. We consider an integral

$$\int_\Omega f(\boldsymbol{x})\mu(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$$

where μ is a positive weight function on Ω and we assume all polynomials of up to sufficiently large degrees are integrable. Usually the weight function $\mu(\boldsymbol{x})$ is normalized so that $\int_\Omega \mu(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 1$ holds.

Definition 4.8 (*Cubature Formula of Degree t*). Let $X = \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N\}$ be a finite set in Ω. Then the following equation is called a cubature formula of degree t with N points:

$$\int_\Omega f(\boldsymbol{x})\mu(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \sum_{i=1}^N \lambda_i f(\boldsymbol{u}_i)$$

for any polynomial $f(\boldsymbol{x}) \in \mathcal{P}_t(\mathbb{R}^n)$, where $\lambda_1, \ldots, \lambda_N$ are positive real numbers which are independent of the choice of the polynomial $f(\boldsymbol{x})$.

If Ω is radially symmetric, i.e. Ω is a union of finite or infinite number of spheres centered at the origin, and the weight function $\mu(\boldsymbol{x})$ is a radial function, i.e. depends only on $\|\boldsymbol{x}\|$, then the integral is center symmetric, i.e. $\int_\Omega f(\boldsymbol{x})\mu(\boldsymbol{x})\mathrm{d}(\boldsymbol{x}) = 0$ holds for any polynomial of odd degree. In such a case $\sum_{i=1}^N \lambda_i \|\boldsymbol{x}_i\|^{2j}\varphi(\boldsymbol{x}_i) = 0$ holds for any integers l and j satisfying $1 \le l \le t$, $0 \le j \le [\frac{t-l}{2}]$ and for any $\varphi \in \mathrm{Harm}_l(\mathbb{R}^n)$. Hence Theorem 4.2 implies that the weighted finite set (X, w) is a Euclidean t-design with the weight $w(\boldsymbol{x}_i) = \lambda_i$, $1 \le i \le N$.

For the number N of the points of a cubature formula of degree t, the following lower bounds are known.

Theorem 4.9 (*Möller [132]*). *Let N be the number of the points of a cubature formula of degree t on Ω. Then the following hold.*

(1) *If $t = 2e$, then $N \ge \dim(\mathcal{P}_e(\Omega))$.*
(2) *If $t = 2e + 1$, then* $N \ge \begin{cases} 2\dim(\mathcal{P}^*_e(\Omega)) - 1 \text{ for } e \text{ even and } 0 \in \Omega \\ 2\dim(\mathcal{P}^*_e(\Omega)) \text{ otherwise} \end{cases}$

Also the following theorem is known.

Theorem 4.10. *Let* $X = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N\} \subset \Omega$ *be the points of a cubature formula of degree* $2e + 1$. *Let* $w(\boldsymbol{x}_i) = \lambda_i$, $1 \le i \le N$. *Assume N attains one of the lower bounds given in Theorem 4.9. Then the following hold.*

(1) *(Möller) If e is odd, or e is even and* $0 \in X$, *then X is antipodal and the weight function* $w(\boldsymbol{x})$ *on X is center symmetric.*
(2) *If e is even and* $0 \notin X$, *assume* $|Y| \le \frac{e}{2} + 1$ *holds for any subset* $Y \subset X \cap l$ *satisfying* $\boldsymbol{x} \neq -\boldsymbol{y}$ *for any* $\boldsymbol{x}, \boldsymbol{y} \in Y$, *where l is any line passing through the origin. Then X is antipodal and the weight function* $w(\boldsymbol{x})$ *on X is center symmetric.*

Remark. Theorem 4.10(1) was proved in [132], Theorem 4.10(2) was proved in [38] using a similar method as in the proof of Möller.

A cubature formula of degree t of $\mathbb{R}^n$ with the exponential weight function $\mu(\boldsymbol{x}) = \mathrm{e}^{-\alpha\|\boldsymbol{x}\|^2}$ on $\mathbb{R}^n$ is called a Gaussian t-design when α is a positive real number (see [36]). Tight Euclidean 4-design with $n = 2$ in the example (1) and the tight Euclidean 5-designs with $n = 3, 5, 6$ in the example (5) given in the Section 4.2 are Gaussian designs with appropriately chosen $\alpha > 0$ and $w(\boldsymbol{x})$. For some of the Euclidean tight t-design (X, w) with $0 \notin X$ in the list we gave, $X \cup \{0\}$ become Gaussian t-designs for an appropriately chosen $\alpha > 0$ and $w(\boldsymbol{x})$. For more information on this subject refer [36,38].

In Section 2.7, we briefly mentioned cubature formulas. (A one-dimensional cubature formula is called quadrature formula.) The theory of cubature formulas has a tremendously rich history in analysis, starting from Gauss, Jacobi, Tchebycheff, etc. and has many connections with other branches of mathematics. We are not in a position to be able to summarize these deep and vast theories. Here, we just mention that Stroud [174], Sobolev [172] are among the basic references on this topic.

When we discussed spherical t-designs X on S^{n-1}, we were interested in the two kinds of bounds, (1): bounds for the value of $N_n(t)$ (the minimum size of $|X|$; see Section 2.7), lower bound of $|X|$, and (2): bounds for the value of $g(n, t)$ (the minimum value for which a spherical t-design on S^{n-1} exists for every size $|X| \ge g(n, t)$; see Section 2.6). We can consider these two kinds of bounds for cubature formulas. (We have briefly discussed the lower bound in this subsection already.) We remark that the bound given below is a very general upper bound (for general domains with a general measure) and known as Tchakaloff's theorem [176].

Theorem 4.11 (*Tchakaloff's Theorem*). *Let* μ *be a positive measure with compact support K in* $\mathbb{R}^n$ *and let t be a fixed positive integer. Then there are N points,* $\boldsymbol{x}_j \in K$ $(1 \le j \le N)$, *and positive real numbers* λ_j $(1 \le j \le N)$ *for some* $N \le \dim(\mathcal{P}_t(\mathbb{R}^n))$ *such that:*

$$\int_{\mathbb{R}^n} f\,\mathrm{d}\mu(\boldsymbol{x}) = \sum_{j=1}^{N} \lambda_j f(\boldsymbol{x}_j)$$

for all $f \in \mathcal{P}_t(\mathbb{R}^n)$.

An alternative accessible (and giving a slightly more general result) proof is available in Putinar [150].

The following theorem is an equivariant version of Tchakaloff's theorem obtained by Victoir [181], and for a finite group G in $O(n)$, it gives an upper bound of the number of nodes in the cubature formula for a G-invariant measure on $\mathbb{R}^n$.

Theorem 4.12 (*Corollary 4.3 in [181]*). *Let n and t be positive integers. Let G be a finite subgroup of* $O(n)$ *and* μ *be a positive G-invariant measure on* $\mathbb{R}^n$ *with the property that* $\int |f(\boldsymbol{x})|\mathrm{d}\mu(\boldsymbol{x}) < \infty$ *for all* $f \in \mathcal{P}_t(\mathbb{R}^n)$. *Then we can find N orbits* $\boldsymbol{x}_1^G, \ldots, \boldsymbol{x}_N^G$ *in the support of* μ *and weights* $\lambda_1, \ldots, \lambda_N$ *that generate a G-invariant cubature formula of degree m with respect to* μ *with*

$$N \le \dim(\mathcal{P}_t(\mathbb{R}^n)^G).$$

(Here the theorem of Tchakaloff and theorem of Sobolev are used.) Moreover, Victoir [181] used orthogonal arrays (designs in Hamming association schemes) to obtain cubature formulas of relatively small number of nodes for a hyperoctahedral group ($W(B_n)$)-invariant measure. Essentially the same result is also obtained by Kuperberg [116]. Bajnok [18] also considered explicit cubature formulas (Euclidean designs) whose nodes are orbits of the hyperoctahedral groups. It seems interesting to consider similar questions for other groups. (For further references on distributions of points on the spheres, cf. [155,173,117], etc.)

4.4. Suggestions for further readings

For algebraic combinatorics in general, the books: Bannai–Ito [46], Brouwer–Cohen–Neumaier [64] and Godsil [96] are the standard references. Delsarte [87] is a historical paper, which started Delsarte theory (algebraic combinatorics) in the framework of association schemes.

The concept of spherical t-designs was introduced in Delsarte–Goethals–Seidel [88]. The paper [88] covers many interesting materials and we believe that it is relatively easy even for non-experts to read. The book Bannai–Bannai [29] treat spherical t-designs, and more generally algebraic combinatorics on spheres, following the paper Delsarte–Goethals–Seidel [88] and expands the treatments in various directions. Unfortunately, this book is written in Japanese, and there is no English translation. The reader will find that the present survey is a kind of short digest version of the book [29] with many new updates.

There are already many survey articles which cover spherical codes and designs. Among them, de la Harpe–Pache [84] gives a good survey from very broad viewpoints. There are many overlaps with the book [29] and the present survey paper. Among our recent surveys, [38] discusses the topics of cubature formulas and spherical and Euclidean t-designs more specifically. Of course, there are so many books and survey papers on cubature formulas.

The papers Goethals–Seidel [97,98] are good references in early stages. Seidel [160] gives a short survey on spherical designs in general. (See also [28,90,161], etc.) We mention the recent survey papers [38,31] by the authors, including the discussions on Euclidean designs, in particular with connection to Sections 4.2 and 4.3.

Conway–Sloane [80] is of course a comprehensive volume which deals with many related topics. Ericson–Zinoviev [94] give many explicit examples of spherical codes. Levenshtein [123] gives a very in-depth and comprehensive treatment on the various generalizations of Delsarte theory. Venkov [179] of Enseignment Math explains the in-depth research of Venkov, as well as related developments, in particular in connection with Section 3.2. We recommend Nebe–Rains–Sloane [138], in particular in connection with Section 3.1 and a part of Section 4.1. In connection with Section 4.3, we recommend the reader to read Cohn–Kumar [77] and Ballinger et al. [20]. We are afraid that many important references are still missing here.

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