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SPHERICAL CODES AND DESIGNS

1. Introduction

A finite non-empty set X of unit vectors in Euclidean space $\mathbb{R}^d$ has several characteristics, such as the dimension $d(X)$ of the space spanned by X, its cardinality $n = |X|$, its degree $s(X)$ and its strength $t(X)$.

The *degree* $s(X)$ is the number of values assumed by the inner product between distinct vectors in X; that is,

$$s(X) = |A(X)|, \qquad A(X) = \{\langle \xi, \eta \rangle; \xi \neq \eta \in X\}.$$

We shall consider sets X having the property that $A(X)$ is contained in a prescribed subset A of the interval $[-1, 1[$. Such sets are called *spherical A-codes*. We are interested in upper bounds for $n = |X|$, and in the structure of spherical A-codes which are extremal with respect to such bounds. For $A = [-1, \beta]$, this problem is equivalent to the classical problem of non-overlapping spherical caps of angular radius $\frac{1}{2} \arccos \beta$; for $\beta \leqslant 0$ exact formulae have been obtained by Rankin [21]. In Section 4 of the present paper we derive bounds for the cardinality of spherical A-codes in terms of the Gegenbauer coefficients of polynomials compatible with A. Apart from these we find, for $s = |A| < \infty$,

$$n \leqslant \binom{d+s-1}{d-1} + \binom{d+s-2}{d-1}.$$

Several examples are mentioned of sets X for which any of these bounds is attained. As an application, a non-existence theorem for graphs is proved.

The notion (and the name) of a *spherical t-design* is explained in Section 5. It serves to measure certain regularity properties of sets X on the unit sphere Ω_d. A spherical 1-design has its centre of mass in the centre of the sphere. A spherical 2-design is what Schläfli called a eutactic star, essentially the projection into $\mathbb{R}^d$ of n orthogonal vectors, cf. [6]. A spherical t-design X is defined by requiring that, for $k = 0, 1, \ldots, t$, the kth moments of X are constants with respect to orthogonal transformations of $\mathbb{R}^d$ or, equivalently, that $\sum_{\xi \in X} W(\xi) = 0$ for all homogeneous harmonic polynomials $W(\xi)$ in $\mathbb{R}^d$ of degree $1, 2, \ldots, t$. The *strength* $t(X)$ is the maximum value of t for which X is a t-design. We explain how spherical t-designs on the sphere, with the orthogonal group, correspond to classical t-designs on the discrete sphere,

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with the symmetric group. The cardinality of a spherical t-design X is bounded from below, and we find

$$n \geqslant \binom{d+e-1}{d-1} + \binom{d+e-2}{d-1}, \qquad n \geqslant 2\binom{d+e-1}{d-1},$$

for $t = 2e$ and $t = 2e+1$, respectively. The spherical t-design is called *tight* if any one of these bounds is attained.

Combining the notions introduced above, we consider in Section 6 *spherical* (d, n, s, t)-*configurations* X. These are sets X of cardinality n on the unit sphere Ω_d, which are spherical t-designs and spherical A-codes with $|A| = s$; in other words, the strength $t(X)$ is at least t and the degree $s(X)$ is at most s. A condition is given for a spherical A-code to be a spherical t-design, in terms of the Gegenbauer coefficients of an annihilator of the set A. This yields $t \leqslant 2s$, and $t \leqslant 2s-1$ if $A \cup \{1\}$ is symmetric with respect to 0, as well as inequalities for $|X|$. The cases of equality are discussed. In Section 7 it is shown that $t \geqslant 2s-2$ implies that X carries an $s(X)$-class association scheme. It is proved that there exist no tight spherical 6-designs, apart from the regular heptagon in $\mathbb{R}^2$.

Sections 8 and 9 contain many examples of spherical (d, n, s, t)-configurations; there exist tight spherical t-designs with $t = 2, 3, 4, 5, 7, 11$, and non-tight spherical $(2s-1)$-designs. The constructions of these examples use sets of lines with few angles, and association schemes, respectively. In Section 8 the derived of a spherical t-design of degree s on Ω_{d+1} is shown to be a spherical $(t+1-s)$-design on Ω_d, and a spherical $(t+2-s)$-design on Ω_d in the antipodal case. Example 9.2 mentions a relation to the Krein condition.

As in [11], matrix techniques and the addition formula for Gegenbauer polynomials are our basic tools. These orthogonal polynomials are reviewed in Section 2. In Section 3 the inequalities of later sections are prepared, in terms of the Gegenbauer coefficients of certain polynomials, and of the characteristic matrices H_k defined on the set X and on an orthogonal basis of harmonic polynomials of degree k.

We use $\mathbb{N}$ for the set of the natural numbers including 0, and J for the all-one matrix.

2. Gegenbauer polynomials

We shall need the following family $\{Q_k(x); k \in \mathbb{N}\}$ of polynomials $Q_k(x)$ in one indeterminate x, defined for a fixed $d \geqslant 2$.

DEFINITION 2.1 The *Gegenbauer polynomial* $Q_k(x)$ of degree k is defined by

$$\lambda_{k+1}Q_{k+1}(x) = xQ_k(x) - (1-\lambda_{k-1})Q_{k-1}(x),$$
$$\lambda_k = k/(d+2k-2), \qquad Q_0(x) = 1, \qquad Q_1(x) = dx.$$

The first few polynomials are:

$$\begin{aligned}2Q_2(x) &= (d+2)(dx^2-1),\\ 6Q_3(x) &= d(d+4)((d+2)x^3-3x),\\ 24Q_4(x) &= d(d+6)((d+2)(d+4)x^4-6(d+2)x^2+3),\\ 120Q_5(x) &= d(d+2)(d+8)((d+4)(d+6)x^5-10(d+4)x^3+15x).\end{aligned}$$

Remark 2.2. For $d \geqslant 3$ the present polynomials are related to the usual [1] Gegenbauer polynomials $C_k^m(x)$ by

$$Q_k(x) = \frac{d+2k-2}{d-2} C_k^{(d-2)/2}(x).$$

For $d = 2$, $k \geqslant 1$, they are related to the Chebyshev polynomials of the first kind $T_k(x)$ by

$$Q_k(x) = kC_k^0(x) = 2T_k(x).$$

DEFINITION 2.3.

$$R_k(x) := \sum_{i=0}^{k} Q_i(x),$$

$$C_k(x) := \sum_{i=0}^{\lfloor k/2 \rfloor} Q_{k-2i}(x).$$

Apart from a constant, $R_k(x)$ is the Jacobi polynomial $P_k^{(\mu+1,\mu)}(x)$, with $\mu = \frac{1}{2}(d-3)$, whereas $C_k(x)$ is the usual Gegenbauer polynomial $C_k^{d/2}(x)$. From the definitions the following theorem is easily proved.

THEOREM 2.4.

$$Q_k(1) = \binom{d+k-1}{d-1} - \binom{d+k-3}{d-1},$$

$$R_k(1) = \binom{d+k-1}{d-1} + \binom{d+k-2}{d-1},$$

$$C_k(1) = \binom{d+k-1}{d-1}, \qquad \textit{for} \quad k \geqslant 1.$$

The Gegenbauer polynomials are orthogonal polynomials, that is,

$$\int_{-1}^{1} Q_k(x)Q_l(x)(1-x^2)^{(d-3)/2}\,dx = a_d Q_k(1)\delta_{k,l},$$

where a_d is some positive constant, and $\delta_{k,l}$ is the Kronecker symbol. To any polynomials $F(x)$, $G(x) \in \mathbb{R}[x]$ we associate their *Gegenbauer expansions*

$$F(x) = \sum_{k=0}^{\infty} f_k Q_k(x), \qquad G(x) = \sum_{k=0}^{\infty} g_k Q_k(x),$$

for well-defined Gegenbauer coefficients f_k and g_k. We shall need the following lemmas, which readily follow from the definitions and the well-known properties of Gegenbauer polynomials, cf. [1], [2].

LEMMA 2.5. *Let* $Q_i(x)Q_j(x) = \sum_{k=0}^{i+j} q_k(i,j)Q_k(x)$. *Then*

$$q_0(i,j) = Q_i(1)\delta_{i,j} \qquad \textit{and} \qquad q_k(i,j) \geqslant 0$$

for all i, j, k, *with* $q_k(i,j) > 0$ *if and only if* $|i-j| \leqslant k \leqslant i+j$ *and* $k \equiv i+j \pmod 2$.

LEMMA 2.6. *Let* $G(x) = Q_l(x)F(x)/Q_l(1)$ *for some* $l \in \mathbb{N}$. *Then*

$$g_0 = f_l, \qquad (\forall_{k\in\mathbb{N}}(f_k \geqslant 0)) \Rightarrow (\forall_{k\in\mathbb{N}}(g_k \geqslant 0)).$$

LEMMA 2.7. *Let* $G(x) = x^l F(x)$ *for some* $l \in \mathbb{N}$. *Then, for each* $k \in \mathbb{N}$, *the number* g_k *is a convex linear combination, with strictly positive coefficients, of the numbers* f_{k+l-2i}, *for* $i = 0, 1, \ldots, \min(l, \lfloor \frac{1}{2}(k+l) \rfloor)$.

Proof. By induction with respect to l. For $l = 1$ we have

$$g_k = \lambda_k f_{k-1} + (1-\lambda_k) f_{k+1}.$$

3. Harmonic polynomials

Let Ω_d, with measure ω_d, denote the unit sphere in the Euclidean space $\mathbb{R}^d$ of dimension d, endowed with the inner product $\langle\ ,\ \rangle$. For any $k \geqslant 0$, let $\mathrm{Hom}(k) = \mathrm{Hom}_d(k)$ denote the linear space of all functions $V\colon \Omega_d \to \mathbb{R}$ which are represented by polynomials $V(\zeta) = V(\zeta_1, \ldots, \zeta_d)$, homogeneous of total degree k in the d variables ζ_i. Let $\mathrm{Harm}(k)$ denote the subspace of $\mathrm{Hom}(k)$ consisting of all functions represented by harmonic polynomials of degree k. Then $\mathrm{Harm}(k)$ is invariant under the orthogonal group $O(d)$ of $\mathbb{R}^d$. Any function $V \in \mathrm{Hom}(k)$ can be uniquely written as

$$V(\zeta) = \sum_{i=0}^{\lfloor k/2 \rfloor} \langle \zeta, \zeta\rangle^i W_{k-2i}(\zeta), \qquad W_l \in \mathrm{Harm}(l).$$

Therefore, the following direct sum decompositions hold, cf. [2], [15].

THEOREM 3.1.

$$\mathrm{Hom}(k) = \sum_{i=0}^{\lfloor k/2 \rfloor} \mathrm{Harm}(k-2i),$$

$$\mathrm{Hom}(k) \oplus \mathrm{Hom}(k-1) = \sum_{i=0}^{k} \mathrm{Harm}(i).$$

The linear space of the second line consists of all functions on Ω_d represented by (not necessarily homogeneous) polynomials of total degree $\leqslant k$ in d variables. For the dimensions we have, cf. [15] and Theorem 2.4,

THEOREM 3.2.

$$\dim \mathrm{Hom}(k) = C_k(1), \qquad \dim \mathrm{Harm}(k) = Q_k(1),$$
$$\dim \mathrm{Hom}(k) \oplus \mathrm{Hom}(k-1) = R_k(1).$$

The *addition formula* relates the Gegenbauer polynomial $Q_k(x)$, and any orthogonal basis $\{W_{k,i};\, i = 1, 2, \ldots, Q_k(1)\}$ of Harm(k), with norm $W_{k,i} = \omega_d^{1/2}$, as follows, cf. [1], [2], [15].

THEOREM 3.3.

$$\sum_{i=1}^{Q_k(1)} W_{k,i}(\xi) W_{k,i}(\eta) = Q_k(\langle \xi, \eta \rangle); \qquad \xi, \eta \in \Omega_d.$$

DEFINITION 3.4. For any finite non-empty set $X \subset \Omega_d$ of size n, for any orthogonal basis $\{W_{k,i}\}$ of Harm(k), with norm $W_{k,i} = \omega_d^{1/2}$, and for any fixed numbering of these, the $n \times Q_k(1)$ matrix

$$H_k := [W_{k,i}(\xi)], \qquad \xi \in X,\, i \in \{1, 2, \ldots, Q_k(1)\},$$

is called the kth *characteristic matrix*. Thus, H_0 is the all-one vector of size n.

DEFINITION 3.5. For any $X \subset \Omega_d$ of size n, and for any $\alpha \in \mathbb{R}$, $-1 \leqslant \alpha \leqslant 1$, the $n \times n$ *distance matrix* D_α is defined by its elements $D_\alpha(\xi, \eta) = 1$ for $\langle \xi, \eta \rangle = \alpha$, and $D_\alpha(\xi, \eta) = 0$ otherwise, for $\xi, \eta \in X$. The sum of the elements of D_α is denoted by d_α.

THEOREM 3.6. *Let $X \subset \Omega_d$, and let A' be a finite set containing all inner products of the vectors of X. Then*

$$H_k H_k^T = \sum_{\alpha \in A'} Q_k(\alpha) D_\alpha,$$

where the $Q_k(x)$ are the Gegenbauer polynomials, H_k the characteristic matrices, and D_α the distance matrices.

Proof. The addition formula 3.3 and Definition 3.4 yield

$$H_k H_k^T = [Q_k(\langle \xi, \eta \rangle)]_{\xi, \eta \in X}.$$

Now apply Definition 3.5.

COROLLARY 3.7.

$$\|H_k^T H_0\|^2 = \sum_{\alpha \in A'} Q_k(\alpha) d_\alpha.$$

Proof. In the formula of Theorem 3.6, take the sum of the elements of the matrices.

COROLLARY 3.8. *For any polynomial* $F(x)$, *with Gegenbauer coefficients* $f_0, f_1, \ldots$, *the following holds:*

$$f_0 n^2 + \sum_{k=1}^{\infty} f_k \|H_k^T H_0\|^2 = \sum_{\alpha \in A'} F(\alpha) d_\alpha.$$

Proof. Use $F(x) = \sum f_k Q_k(x)$ with Theorem 3.6 so as to obtain

$$\sum_{k=0}^{\infty} f_k H_k H_k^T = \sum_{\alpha \in A'} F(\alpha) D_\alpha,$$

and take the sum of the elements of the matrices.

LEMMA 3.9.

$$\|H_i^T H_j - n\Delta_{i,j}\|^2 = \sum_{k=1}^{i+j} q_k(i,j) \|H_k^T H_0\|^2,$$

where $q_k(i,j)$ *is as in Lemma* 2.5, *and* $\Delta_{i,j}$ *denotes the appropriate zero matrix for* $i \neq j$ *and unit matrix for* $i = j$.

Proof. We refer to [11], Lemma 4.5.

4. Spherical codes

DEFINITION 4.1. Let A be a subset of the interval $[-1, 1[$. A *spherical A-code*, for short an *A-code*, is a non-empty subset X of the unit sphere in $\mathbb{R}^d$, satisfying $\langle \xi, \eta \rangle \in A$, for all $\xi \neq \eta \in X$.

Thus, an A-code is a set of unit vectors with angles from the prescribed set arc cos A, or a set of points on Ω_d with distances from the prescribed set $(2(1 - A))^{1/2}$. We shall use the notation $A' := A \cup \{1\}$.

DEFINITION 4.2. A polynomial $F(x) \in \mathbb{R}[x]$ is *compatible* with the set A if $\forall_{\alpha \in A}(F(\alpha) \leqslant 0)$.

THEOREM 4.3. *Let* $F(x)$, *with Gegenbauer coefficients* $f_0 > 0$ *and* $f_k \geqslant 0$, *for all k, be compatible with the set A. Then the cardinality n of any A-code X satisfies*

$$n \leqslant F(1)/f_0.$$

Equality holds if and only if, for all $\xi \neq \eta \in X$, *and for all* $k \geqslant 1$,

$$F(\langle \xi, \eta \rangle) = 0, \qquad f_k H_k^T H_0 = 0.$$

Proof. This is an immediate consequence of Corollary 3.8, since $d_1 = n$.

EXAMPLE 4.4. For given β, with $-1 \leqslant \beta < 0$, let A be any subset of the interval $[-1, \beta]$. The polynomial $F(x) = x - \beta$ is compatible with A, and $f_0 = -\beta > 0, f_1 = 1/d > 0$. Hence Theorem 4.3 applies, yielding $n \leqslant 1 - 1/\beta$. An A-code of given dimension $r \leqslant d$ achieves this bound if and only if it is an r-dimensional regular simplex, with $\beta = -1/r$.

EXAMPLE 4.5. For given α and β, with

$$-1 \leqslant \alpha \leqslant \beta < 1, \qquad \alpha + \beta \leqslant 0, \qquad \alpha\beta > -1/d,$$

let A be any subset of $[\alpha, \beta]$. Then the polynomial $F(x) = (x - \alpha)(x - \beta)$ has the Gegenbauer coefficients

$$f_0 = \alpha\beta + 1/d, \qquad f_1 = -(\alpha + \beta)/d, \qquad f_2 = 2/d(d+2),$$

which are non-negative. Application of Theorem 4.3 yields

$$n \leqslant d(1 - \alpha)(1 - \beta)/(1 + d\alpha\beta)$$

for any A-code X. In addition, equality is only possible if A contains α and β, and if X is an $\{\alpha, \beta\}$-code.

In the special case $\beta = -\alpha$, the lines spanned by the vectors of an $\{\alpha, \beta\}$-code constitute a set of equiangular lines in $\mathbb{R}^d$. The sets achieving the above bound are equivalent to the regular 2-graphs, cf. [24], [26]. Also, the case of $\{\alpha, \beta\}$-codes with $\alpha + \beta < 0$ leads to equiangular lines, now in $\mathbb{R}^{d+1}$. Indeed, cf. [19], since $\alpha + \beta < 0$ we may determine R and φ such that

$$1 - \alpha = R^2(1 - \cos\varphi), \qquad 1 - \beta = R^2(1 + \cos\varphi), \qquad R > 1, \qquad 0 < \varphi < \pi.$$

Now define $Y \subset \mathbb{R}^{d+1}$ as follows

$$Y = \{R^{-1}((R^2 - 1)^{1/2}, \xi); \xi \in X\}.$$

Clearly, Y is a $\{\pm\cos\varphi\}$-code on Ω_{d+1}. Hence it carries a set of equiangular lines with the angle φ.

EXAMPLE 4.6. For a given β, with $0 \leqslant \beta < d^{-1/2}$, let A be any subset of $[-1, \beta]$. Define

$$\alpha := -(1 + \beta)/(1 + d\beta),$$

so $-1 \leqslant \alpha < 0$. The polynomial

$$F(x) := (x - \alpha)^2(x - \beta)$$

is compatible with A, and has non-negative Gegenbauer coefficients with $f_0 > 0$. Application of Theorem 4.3 yields

$$n \leqslant d(1 - \beta)(2 + (d + 1)\beta)/(1 - d\beta^2)$$

for any A-code X, and equality is only possible if X is an $\{\alpha, \beta\}$-code. It is interesting to observe that the bounds of Examples 4.5 and 4.6 coincide for this particular α.

EXAMPLE 4.7. For given α, β, γ, with $-1 \leqslant \alpha \leqslant \beta \leqslant \gamma < 1$, let A be any subset of $[-1, \alpha] \cup [\beta, \gamma]$. The polynomial $F(x) = (x - \alpha)(x - \beta)(x - \gamma)$ is compatible with A. It has non-negative Gegenbauer coefficients, with $f_0 > 0$, if

$$\alpha + \beta + \gamma \leqslant 0, \qquad \alpha\beta + \beta\gamma + \gamma\alpha \geqslant -3/(d + 2),$$
$$\alpha + \beta + \gamma < -d\alpha\beta\gamma.$$

Then Theorem 4.3 yields

$$n \leqslant -d(1 - \alpha)(1 - \beta)(1 - \gamma)/(\alpha + \beta + \gamma + d\alpha\beta\gamma)$$

for any A-code X, and equality is only possible if X is an $\{\alpha, \beta, \gamma\}$-code. In Example 9.3 we shall give a construction with

$$d = 23, \qquad n = 2048, \qquad \alpha = -9/23, \qquad \beta = -1/23, \qquad \gamma = 7/23.$$

We conclude this section by giving yet another bound for the cardinality of an A-code X. This so-called absolute bound only depends on the cardinality of A, not on its specific elements.

THEOREM 4.8. *For given $s = |A| < \infty$, the cardinality n of any A-code X satisfies $n \leqslant R_s(1)$.*

Proof. Cf. [16]. For A we define the annihilator polynomial

$$F(x) := \prod_{\alpha \in A} (x - \alpha)/(1 - \alpha).$$

For any $\eta \in X$ we define the function $F_\eta \colon \Omega_d \to \mathbb{R}$ by

$$F_\eta(\xi) := F(\langle \xi, \eta \rangle), \qquad \xi \in \Omega_d.$$

Thus F_η belongs to the linear space $\mathrm{Hom}(s) \oplus \mathrm{Hom}(s - 1)$, which has the dimension $R_s(1)$. By definition we have

$$F_\eta(\xi) = \delta_{\xi,\eta}, \qquad \text{for all} \quad \xi \in X,$$

so that the functions F_η are linearly independent. Hence their number $n = |X|$ cannot exceed the dimension of the linear space, which proves the theorem.

EXAMPLE 4.9. For $s = 1$ we have $n \leqslant d + 1$, with equality if and only if X is a regular d-simplex, as in Example 4.4.

EXAMPLE 4.10. For $s = 2$ we have

$$n \leqslant \tfrac{1}{2}d(d + 3).$$

Examples meeting the bound exist for

$$d = 2, \quad n = 5; \qquad d = 6, \quad n = 27; \qquad d = 22, \quad n = 275.$$

Indeed, the following numbers of equiangular lines exist [17]:

$$6 \text{ in } \mathbb{R}^3; \qquad 28 \text{ in } \mathbb{R}^7; \qquad 276 \text{ in } \mathbb{R}^{23}.$$

In each case we consider, with respect to a unit vector along any one line, the unit vectors at an obtuse angle along the other lines. These vectors determine a space of one dimension less, and provide the announced examples, cf. Example 4.5.

Theorem 4.8 has an application to graph theory.

THEOREM 4.11. *A regular graph on* n *vertices, whose* $(0, 1)$-*adjacency matrix* L *has the smallest eigenvalue* < -1 *of multiplicity* $n - d$, *satisfies*

$$n \leqslant \tfrac{1}{2}d(d+1) - 1.$$

Proof. Let k be the valency, and λ the smallest eigenvalue of L. Then

$$G := L - \lambda I - \frac{k - \lambda}{n} J$$

is positive semi-definite of rank $\leqslant d - 1$. Hence G is the Gram matrix of n vectors in $\mathbb{R}^{d-1}$ of equal length with three distinct inner products. Theorem 4.8 applies with $s = 2$, hence

$$n \leqslant \binom{d}{d-2} + \binom{d-1}{d-2} = \tfrac{1}{2}d(d+1) - 1.$$

EXAMPLE 4.12. There are no strongly regular graphs with

$$n = 28, \quad k = 18, \quad \lambda_1 = 4, \quad \lambda_2 = -2;$$
$$n = 276, \quad k = 165, \quad \lambda_1 = 27, \quad \lambda_2 = -3,$$

where, in both cases, k, λ_1 and λ_2 are the eigenvalues of L. This is well known for the 28-graph, but new for the 276-graph (Scott [23] showed the non-existence of rank 3 graphs with these parameters).

5. Spherical designs

DEFINITION 5.1. A finite non-empty set $X \subset \Omega_d$ is a *spherical t-design*, for short a *t-design*, for some $t \in \mathbb{N}$ if the following holds for $k = 0, 1, \ldots, t$:

$$\forall_{V \in \mathrm{Hom}(k)} \forall_{T \in O(d)} \Big(\sum_{\xi \in X} V(T\xi) = \sum_{\xi \in X} V(\xi) \Big) \cdot$$

Here $T\xi$ denotes the image of $\xi \in \Omega_d$ under the element T of the orthogonal group $O(d)$. Since Hom(k) is spanned by the monomials

$$\xi_1^{k_1}\xi_2^{k_2}\cdots\xi_d^{k_d}, \quad k_i \in \mathbb{N}, \qquad \sum_{i=1}^{d} k_i = k,$$

Definition 5.1 amounts to requiring that the kth moments of X are constants with respect to orthogonal transformations, for $k = 0, 1, \ldots, t$. Thus, a 1-design is a set $X \subset \Omega_d$ whose centre of mass is the centre of Ω_d, and for a 2-design, in addition, the inertia ellipsoid is a sphere. Another way to express the t-design property requires that, for $k = 0, 1, \ldots, t$,

$$\forall_{V\in \mathrm{Hom}(k)}\forall_{T\in O(d)}\left(n^{-1}\sum_{\xi\in X} V(T\xi) = \omega_d^{-1}\int_{\Omega_d} V(\xi)d\omega(\xi)\right),$$

that is, that the kth moments of TX equal the corresponding kth moments of Ω_d, for all $T \in O(d)$. Since for any $V \in \mathrm{Harm}(k)$, with $k \geqslant 1$, the above integral vanishes, Theorem 3.1 implies the following criterion for t-designs.

THEOREM 5.2. *A finite set $X \subset \Omega_d$ is a t-design if and only if*

$$\sum_{\xi\in X} W(\xi) = 0 \qquad \textit{for all} \quad W \in \sum_{k=1}^{t} \mathrm{Harm}(k).$$

THEOREM 5.3. *A finite set $X \subset \Omega_d$ is a t-design if and only if its characteristic matrices satisfy any one of the following conditions:*

(i) $H_k^T H_0 = 0$ for $k = 1, 2, \ldots, t$, or

(ii) $H_k^T H_l = n\Delta_{k,l}$, for $0 \leqslant k + l \leqslant t$.

Proof. The equivalence of Definition 5.1 and (i) follows from Theorem 5.2 and Definition 3.4. The equivalence of (i) and (ii) follows from Lemma 3.9.

Remark 5.4. For $t \geqslant 2$, let $e := \lfloor t/2 \rfloor$ and $r := e - (-1)^t$. Then

$$H_e^T H_e = nI \qquad \text{and} \qquad H_e^T H_r = 0$$

are necessary and sufficient conditions for a t-design. This is a consequence of Theorem 5.3 and Lemmas 2.5 and 3.9.

THEOREM 5.5. *For any A-code X, let $A' = A \cup \{1\}$ and let d_α denote the sum of the elements of the distance matrix D_α. Then*

$$\sum_{\alpha\in A'} d_\alpha Q_k(\alpha) \geqslant 0,$$

and equality holds for $k = 1, 2, \ldots, t$ if and only if X is a t-design.

Proof. Apply Corollary 3.7 and Theorem 5.3.

EXAMPLE 5.6. Remark 5.4 says that 2-designs X are characterized by

$$H_0^T H_1 = 0 \qquad \text{and} \qquad H_1^T H_1 = nI.$$

Now observe that the $n \times d$ characteristic matrix H_1 satisfies

$$H_1 H_1^T = d \sum_{\alpha \in A'} \alpha D_\alpha = d\, \mathrm{Gram}(X).$$

Hence X is a 1-design if and only if the Gram matrix Gram(X) of its inner products has vanishing row sums. The second condition for X to be a 2-design amounts to the following: Gram(X) has two eigenvalues, namely n/d and 0, the vectors of X span $\mathbb{R}^d$ and may be viewed as the orthogonal projections into $\mathbb{R}^d$ of n orthogonal vectors of length $\sqrt{n/d}$ in n-space, cf. [6], [12]. Examples for such sets X are abundant; for instance, any spanning set of unit vectors along equiangular lines which correspond to a regular 2-graph. However, such a set only yields a 2-design if it is a 1-design; we refer to [24], [25] for many examples of this situation.

EXAMPLE 5.7. A set X is *antipodal* whenever

$$\forall_{\xi \in X}(-\xi \in X).$$

Obviously, antipodal A-codes provide 1-designs, since

$$A'(X) = -A'(X), \qquad d_\alpha = d_{-\alpha}, \qquad \sum_{\alpha \in A'} d_\alpha Q_k(\alpha) = 0 \quad \text{for odd } k.$$

The antipodal codes on Ω_d are in 1–1 correspondence with the sets of lines through the origin of $\mathbb{R}^d$, the subject of [11]. An antipodal code on Ω_d is a 3-design if and only if the Gram matrix of a spanning set of vectors of the corresponding set of lines has two eigenvalues. This yields many examples for 3-designs; for instance, the antipodal codes corresponding to regular 2-graphs, cf. [24], [26]. For $d = 3$ the six vertices of the octahedron, and also the eight vertices of the cube, provide a 3-design.

EXAMPLE 5.8. For a 5-design Theorem 5.5 requires

$$\sum_{\alpha \in A'} \alpha^i d_\alpha = 0 \quad \text{for} \quad i = 1, 3, 5,$$

and

$$\sum_{\alpha \in A'} \alpha^2 d_\alpha = \frac{n^2}{d}, \qquad \sum_{\alpha \in A'} \alpha^4 d_\alpha = \frac{3n^2}{d(d+2)}.$$

For $d = 3$, the 12 vertices of the icosahedron, and also the 20 vertices of the dodecahedron, provide a 5-design. Further examples are given in subsequent sections.

Remark 5.9. The analogy with the classical t-designs, cf. [14], [5], [27], is explained as follows. For integers d, v with $1 \leqslant d \leqslant v/2$ we define the 'discrete d-sphere' in $\mathbb{R}^v$ by

$$\Omega := \left\{x = (x_1, \ldots, x_v) \in \mathbb{R}^v;\ x_i \in \{0, 1\},\ \sum_{i=1}^{v} x_i = d\right\},$$

whence $|\Omega| = \binom{v}{d}$. We define Hom(t) to be the set of all functions $f\colon \Omega \to \mathbb{R}$, which are represented by homogeneous polynomials $f(x)$ of degree $\leqslant 1$ in each coordinate x_i, and of total degree t. It turns out that the monomials

$$x_{i_1} x_{i_2} \cdots x_{i_t}$$

form a basis for Hom(t), hence dim Hom(t) $= \binom{v}{t}$. Now a classical t-design $t - (v, d, \lambda)$ is a collection X of d-subsets of a v-set, such that each t-subset is contained in a constant number λ of elements of X. In the setting above, this corresponds to a subset X of Ω subject to the condition

$$\forall_{f \in \mathrm{Hom}(t)} \forall_{T \in \mathrm{Sym}(v)} \left(\sum_{\xi \in X} f(T\xi) = \sum_{\xi \in X} f(\xi)\right).$$

This is equivalent to requiring that the sum over X of any monomial $\in$ Hom(t) is a constant with respect to Sym(v), that is, that the monomial takes the value 1 for a constant number, λ say, of elements of X. Thus Ω, Sym(v), and the classical t-designs, correspond to Ω_d, $O(d)$, and the spherical t-designs, respectively. This correspondence may be pushed still further; for details we refer to [9].

There is no upper bound to the number of points of a t-design, since the union of disjoint t-designs again is a t-design. The following theorem, which in some sense is dual to Theorem 4.3, provides a lower bound.

THEOREM 5.10. *Let $F(x)$, with Gegenbauer coefficients $f_0 > 0$ and $f_k \leqslant 0$ for all $k > t$, satisfy $F(1) > 0$ and $F(\alpha) \geqslant 0$ for all $\alpha \in [-1, 1]$. Then the cardinality of any t-design X satisfies*

$$n \geqslant F(1)/f_0.$$

Equality holds if and only if, for all $\xi \neq \eta \in X$, and for all $k > t$,

$$F(\langle \xi, \eta \rangle) = 0, \qquad f_k H_k^T H_0 = 0.$$

Proof. This is a consequence of Theorem 5.3 and Corollary 3.8.

THEOREM 5.11. *Let X be a (2e)-design. Then*

$$n = |X| \geqslant R_e(1).$$

Equality holds if and only if $A(X)$ consists of the zeros of $R_e(x)$.

Proof. Apply Theorem 5.10 for $F(x) = (R_e(x))^2$. It is easily verified, by use of the orthogonality relations for Gegenbauer polynomials and Theorem 2.4, that the bound specializes to

$$n \geqslant F(1)/f_0 = (R_e(1))^2/R_e(1) = R_e(1),$$

with equality if and only if all elements of $A(X)$ are zeros of $F(x)$. Now it readily follows from Theorem 4.8 that $n = R_e(1)$ implies $|A(X)| \geqslant e$, so that $A(X)$ must consist of the e zeros of $F(x)$.

THEOREM 5.12. *Let X be a $(2e + 1)$-design, Then*

$$n = |X| \geqslant 2C_e(1).$$

Equality holds if and only if $A(X)$ consist of -1 and the zeros of $C_e(X)$. Moreover, in the case of equality X is antipodal.

Proof. Apply Theorem 5.10 for $F(x) = (x + 1)(C_e(x))^2$, then the lower bound specializes to $2C_e(1)$, and the desired result about $A(X)$ is proved by an argument similar to that of Theorem 5.11. In order to prove the last statement, we define Y to be the set of the lines carried by the vectors of X. Clearly $|X| \leqslant 2|Y|$, with equality if and only if X is antipodal. If $|X| = 2C_e(1)$, then from $C_e(-x) = (-1)^e C_e(x)$ it follows that $(A(X))^2$ has cardinality $[e/2] + 1$ and contains 0 whenever e is odd. Therefore, the absolute bound for systems of lines [11] yields $|Y| \leqslant C_e(1)$, whence $|X| = 2\,|Y| = 2C_e(1)$, so that X is antipodal (and Y meets the absolute bound).

DEFINITION 5.13. A t-design is called *tight* if any of the bounds mentioned in Theorems 5.11 and 5.12 is attained.

Clearly, a tight t-design cannot be a $(t + 1)$-design. We conclude this section by some preliminary examples of tight t-designs.

EXAMPLE 5.14. For $d = 2$ and any t, a tight t-design is nothing but a regular $(t + 1)$-gon.

EXAMPLE 5.15. For any d, the $d + 1$ vertices of a regular simplex in $\mathbb{R}^d$ provide a tight 2-design. The $2d$ vertices of the cross polytope (the generalization of the octahedron) provide a tight 3-design. Notice that the 2^d vertices of the cube also provide a 3-design (not a 4-design), but not a tight 3-design for $d \geqslant 3$.

EXAMPLE 5.16. For $d = 3$ the icosahedron is the only tight 5-design.

6. Spherical (d, n, s, t)-configurations

DEFINITION 6.1. A (spherical) (d, n, s, t)-*configuration* is a set $X \subset \Omega_d$ of cardinality n, which is a t-design and an A-code with $|A| = s$.

Given $X \subset \Omega^d$, $|X| = n$, we denote by $s(X)$ and $t(X)$ the minimum s and the maximum t for which X is a (d, n, s, t)-configuration. Theorem 6.5 will provide a criterion for an A-code to be a t-design, in terms of the Gegenbauer coefficients $f_0, f_1, \ldots, f_s$ of an annihilator $F(x)$ of degree s for the set A.

DEFINITION 6.2. $F(x) \in \mathbb{R}[x]$ is an *annihilator* polynomial for a finite set $A \neq \varnothing$ with $1 \notin A$ if

$$F(1) = 1, \qquad \forall_{\alpha \in A}(F(\alpha) = 0).$$

LEMMA 6.3. *Let X be an A-code, and let $G(x)$ be an annihilator for A with Gegenbauer coefficients $g_0, g_1, \ldots$. Then*

$$n(1 - ng_0) = \sum_{k=1}^{\infty} g_k \|H_0^T H_k\|^2.$$

Proof. Apply Corollary 3.8.

THEOREM 6.4. *Let X be an A-code, $|X| = n$, $|A| = s$. The Gegenbauer coefficients of an annihilator $F(x)$ of degree s for A satisfy*

$$(\forall_{0 \leqslant i \leqslant s}(f_i \geqslant 0)) \Rightarrow (\forall_{0 \leqslant j \leqslant s}(f_j \leqslant 1/n)).$$

If, in addition, $f_j = 1/n$ for some $j \leqslant s$, then X is an A-code of maximum cardinality.

Proof. For any fixed $j \in \{0, 1, \ldots, s\}$, define

$$G(x) := F(x)Q_j(x)/Q_j(1).$$

Clearly, $G(x)$ is an annihilator for A. Lemma 2.6 implies that $g_0 = f_j$ and $g_k \geqslant 0$ for all k. Hence Lemma 6.3 yields $1 - nf_j = 1 - ng_0 \geqslant 0$. If equality holds, then the bound of Theorem 4.3 is attained, and X is an A-code of maximum cardinality.

THEOREM 6.5. *Let X be an A-code, with $|X| = n$, $|A| = s$, and let $F(x)$ be an annihilator of degree s for A with Gegenbauer coefficients $f_0, f_1, \ldots, f_s$. If X is a t-design with $t \geqslant s$, then $f_0 = f_1 = \cdots = f_{t-s} = 1/n$. Conversely, if $f_0 = f_1 = \cdots = f_r = 1/n$, and $f_{r+1} > 0, \ldots, f_s > 0$ for some $r \leqslant s$, then X is an $(r + s)$-design.*

Proof. First, suppose X is a t-design with $t \geqslant s$. For any fixed $j = 0, 1, \ldots, t - s$ the polynomial

$$G(x) := F(x)Q_j(x)/Q_j(1)$$

is an annihilator for A of degree $j + s \leqslant t$. Hence Theorem 5.3 and Lemma 6.3 yield $0 = 1 - ng_0 = 1 - nf_j$, by use of Lemma 2.6. Conversely, let us consider the annihilator

$$G(x) := x^r F(x)$$

for A of degree $r + s$. Assuming $f_0 = \cdots = f_r = 1/n$ and all $f_i > 0$, we conclude from Lemma 2.7 that $g_0 = 1/n$, $g_k > 0$ for $0 \leqslant k \leqslant r + s$. Lemma 6.3 implies $H_k^T H_0 = 0$ for $1 \leqslant k \leqslant r + s$, whence X is an $(r + s)$-design.

THEOREM 6.6. *Any* (d, n, s, t)*-configuration* X *satisfies*

$$t \leqslant 2s \qquad \text{and} \qquad n \leqslant R_s(1).$$

If $t = 2s$, *or if* $n = R_s(1)$, *then* X *is a tight* $(2s)$*-design.*

Proof. Let $F(x)$ be the annihilator of degree s for A. We first apply Theorem 6.5. If $t \geqslant s$, then $f_{t-s} \neq 0$, hence $t - s \leqslant s$. This proves $t \leqslant 2s$. In the case of equality Theorem 5.11 implies $n \geqslant R_s(1)$, whence $n = R_s(1)$ by Theorem 4.8, and X is a tight $(2s)$-design. For the second part of the theorem we observe that Theorem 3.6 implies

$$\sum_{k=0}^{s} f_k H_k H_k^T = I.$$

Hence the $n \times R_s(1)$-matrix

$$H := [H_0 \quad H_1 \quad \cdots \quad H_s]$$

has rank n, proving once again $n \leqslant R_s(1)$, cf. Theorem 4.8. Now suppose $n = R_s(1)$, then H is non-singular, and all f_k are positive. Therefore Theorem 6.4 implies that all $f_k \leqslant 1/n$. Hence

$$n \sum_{k=0}^{s} f_k Q_k(1) = n = R_s(1) = \sum_{k=0}^{s} Q_k(1)$$

implies $f_0 = f_1 = \cdots = f_s = 1/n$, $nF(x) = R_s(x)$, and it follows from Theorem 6.5 that X is a $(2s)$-design. Now the theorem is proved. It is interesting to observe that in the case of equality we have

$$HH^T = H^T H = nI.$$

EXAMPLE 6.7. Tight 4-designs have

$$s(X) = 2, \qquad t(X) = 4, \qquad n = R_2(1) = \tfrac{1}{2}d(d + 3).$$

Example 4.10 applies, and as a consequence of Theorem 6.6 we have three tight 4-designs, with

$$d = 2, \quad n = 5; \qquad d = 6, \quad n = 27; \qquad d = 22, \quad n = 275.$$

THEOREM 6.8. *Any (d, n, s, t)-configuration X, which is an A-code with $A' = -A'$, $|A| = s$, satisfies*

$$t \leqslant 2s - 1 \quad \text{and} \quad n \leqslant 2C_{s-1}(1).$$

If $t = 2s - 1$, or if $n = 2C_{s-1}(1)$, then X is an antipodal tight $(2s - 1)$-design.

Proof. Applying the absolute bound [11] to the set Y of the lines carried by the vectors of X we obtain

$$n = |X| \leqslant 2|Y| \leqslant 2C_{s-1}(1).$$

This also proves $t \leqslant 2s - 1$, since $t = 2s$ is excluded by Theorem 6.6. If $t = 2s - 1$, then Theorem 5.12 implies $n \geqslant 2C_{s-1}(1)$, which yields $n = 2C_{s-1}(1)$, and X is antipodal. Now suppose $n = 2C_{s-1}(1)$, then by the above inequality X is antipodal and Y attains the absolute bound for sets of lines. It follows from [11], theorem 6.1, that the annihilator $F(x)$ of degree s for A is given by

$$nF(x) = (1 + x)C_{s-1}(x) = \sum_{k=0}^{s-1} Q_k(x) + \lambda_s Q_s(x),$$

cf. Definitions 2.1 and 2.3. Therefore, Theorem 6.5 implies that X is a $(2s - 1)$-design. Now the theorem is proved. Sections 8 and 9 contain examples of $(2s - 1)$-designs which are not antipodal, hence not tight.

7. Distance invariance and association schemes

For any A-code X, the *valencies* $v_\alpha(\xi)$ and the *intersection numbers* $p_{\alpha,\beta}(\xi, \eta)$ are defined as follows.

DEFINITION 7.1.

$$\forall_{\alpha\in A'}\forall_{\xi\in X}(v_\alpha(\xi) := |\{\zeta \in X : \langle\xi, \zeta\rangle = \alpha\}|),$$
$$\forall_{\alpha,\beta\in A'}\forall_{\xi,\eta\in X}(p_{\alpha,\beta}(\xi, \eta) := |\{\zeta \in X : \langle\xi, \zeta\rangle = \alpha, \langle\eta, \zeta\rangle = \beta\}|).$$

DEFINITION 7.2. X is *distance invariant* if, for all $\alpha \in A'$, the valency $v_\alpha(\xi)$ is independent of $\xi \in X$. X carries an $s(X)$-class *association scheme* if, for all $\alpha, \beta \in A'$, the intersection number $p_{\alpha,\beta}(\xi, \eta)$ depends only on $\langle\xi, \eta\rangle$.

Thus, the association schemes of Bose and Mesner, cf. [4], [7], [13], are specialized to the present situation. It is interesting to point out that any abstract association scheme can be represented by means of an A-code of a suitable dimension, cf. Section 9. We observe that the triangle inequality on the sphere imposes restrictions on the intersection numbers, namely

$$(p_{\alpha,\beta}(\xi, \eta) \neq 0) \Rightarrow (2(1 - \alpha)(1 - \beta)(1 + \langle\xi, \eta\rangle) \geqslant (1 - \alpha - \beta + \langle\xi, \eta\rangle)^2).$$

For any integer $i \geqslant 0$, let x^i have the Gegenbauer expansion

$$x^i = \sum_{k=0}^{i} f_{i,k} Q_k(x).$$

The 'convolution' of x^i and x^j is defined to be the polynomial

$$F_{i,j}(x) := \sum_{k=0}^{\min(i,j)} f_{i,k} f_{j,k} Q_k(x).$$

LEMMA 7.3. *For $0 \leqslant i + j \leqslant t$, and for fixed $\gamma := \langle \xi, \eta \rangle$, the intersection numbers $p_{\alpha,\beta}(\xi, \eta)$ of a (d, n, s, t)-configuration satisfy the linear equation*

$$\sum_{\alpha\beta,\in A} \alpha^i \beta^j p_{\alpha,\beta}(\xi, \eta) = nF_{i,j}(\gamma) - \gamma^j - \gamma^i + \delta_{1,\gamma}.$$

Proof. By use of Theorem 5.3, part (ii), the t-design property implies

$$\left(\sum_{k=0}^{i} f_{i,k} H_k H_k^T\right)\left(\sum_{k=0}^{j} f_{j,k} H_k H_k^T\right) = n \sum_{k=0}^{\min(i,j)} f_{i,k} f_{j,k} H_k H_k^T.$$

We rewrite this by use of the addition formula which by Theorem 3.6 reads

$$H_k H_k^T = [Q_k(\langle \xi, \eta \rangle)]_{\xi,\eta \in X}.$$

Equate the (ξ, η)-entries on both sides of the formula above, and use the definition of $f_{i,k}$, then

$$\sum_{\alpha,\beta \in A'} \alpha^i \beta^j p_{\alpha,\beta}(\xi, \eta) = nF_{i,j}(\langle \xi, \eta \rangle).$$

This leads to the desired formula since, for $\langle \xi, \eta \rangle = \gamma$,

$$p_{\alpha,1}(\xi, \eta) = p_{1,\alpha}(\xi, \eta) = \delta_{\alpha,\gamma}.$$

THEOREM 7.4. *Let X be a (d, n, s, t)-configuration. If $t \geqslant s - 1$, then X is distance invariant. If $t \geqslant 2s - 2$, then X carries an $s(X)$-class association scheme. If $t \geqslant 2s - 3$, then, for any fixed $\langle \xi, \eta \rangle = \gamma$, the intersection numbers $p_{\alpha,\beta}(\xi, \eta)$ are uniquely determined by $p_{\gamma,\gamma}(\xi, \eta)$.*

Proof. Suppose $t \geqslant s - 1$, and apply Lemma 7.3 for $j = 0$, $\xi = \eta$, $\gamma = 1$:

$$\sum_{\alpha \in A} \alpha^i v_\alpha(\xi) = nF_{i,0}(1) - 1; \qquad 0 \leqslant i \leqslant s - 1.$$

This linear system of s equations with s unknowns $v_\alpha(\xi)$ has a Vandermonde, hence non-singular, matrix. Therefore, the valencies are uniquely determined, and are independent of ξ.

Next suppose $t \geqslant 2s - 2$. Now Lemma 7.3 yields a linear system of s^2 equations for $0 \leqslant i, j \leqslant s - 1$, with s^2 unknowns $p_{\alpha,\beta}(\xi, \eta)$. The matrix of this system is the direct product of two Vandermonde matrices, hence is non-singular. Therefore, for fixed $\gamma = \langle \xi, \eta \rangle$, the intersection numbers are uniquely determined. The third part of the theorem is proved analogously.

THEOREM 7.5. *Any tight t-design carries an s-class association scheme, with* $s = [t/2]$.

Proof. Apply Theorems 5.11, 5.12 and 7.4.

Remark 7.6. For $t(X) \geqslant 2s(X) - 2$, the Bose–Mesner algebra of the association scheme is easily described [7]. It is generated by I and the pairwise orthogonal idempotent matrices

$$J_k := n^{-1} H_k H_k^T, \qquad k = 0, 1, \ldots, s(X) - 1.$$

This allows us to give explicit formulae for the eigenvalues of the association scheme, cf. Example 8.4 below.

We conclude this section with the following non-existence theorem.

THEOREM 7.7. *The only tight* 6*-design is the regular heptagon in* $\mathbb{R}^2$.

Proof. For $d = 2$, the only tight t-designs are the regular $(t + 1)$-gons. So we have to show that the existence of a 6-design $X \subset \Omega_d$, with $d \geqslant 3$ and $n = R_3(1)$, leads to a contradiction. By Theorem 7.5 the set X is an A-code of degree $s(X) = |A(X)| = 3$ which carries a 3-class association scheme. For $d \geqslant 3$, the eigenvalues of the association scheme are integers, since for $d \geqslant 3$ the multiplicities $Q_k(1)$ are distinct. This implies that the elements α, β, γ of $A(X)$ are rational. On the other hand, by Theorem 5.11, these α, β, γ are the zeros of the polynomial

$$R_3(x) = \tfrac{1}{6} d((d + 2)(d + 4)x^3 + 3(d + 2)x^2 - 3(d + 2)x - 3),$$

and it is not difficult to show that any rational zero of $R_3(x)$ is the inverse of an integer. We now use an argument devised by van Lint [18] in the theory of perfect codes. By straightforward verification it follows that

$$R_3(-1/(d + 2)) > 0, \qquad R_3(-1/(d + 3)) < 0.$$

Hence $R_3(x)$ has a zero between $-1/(d + 2)$ and $-1/(d + 3)$, which obviously cannot be the inverse of an integer. This contradiction proves the theorem.

The present theorem, and Examples 4.9, 4.10 and 6.7, suggest the following:

CONJECTURE 7.8. *There exist no tight* $(2e)$*-designs in* Ω_d *for* $d \geqslant 3$ *and* $e \geqslant 3$.

8. Examples from sets of lines and derived configurations

The unit sphere $\Omega_{d+1} \subset \mathbb{R}^{d+1}$, with vectors $\zeta = (\varepsilon, \eta_1, \eta_2, \ldots, \eta_d) =: (\varepsilon; \eta)$ for short, is partitioned into spheres in parallel spaces of dimension d as follows.

$$\Omega_{d+1} = \bigcup_{-1 \leqslant \varepsilon \leqslant 1} \{(\varepsilon, \xi\sqrt{1 - \varepsilon^2}); \xi \in \Omega_d\}.$$

Let $Z \subset \Omega_{d+1}$ be any B-code containing $e := (1, 0, 0, \ldots, 0) = (1; 0)$. Define $B^* := B \backslash \{-1\}$ and $s^* := |B^*|$.

DEFINITION 8.1. The *derived code* Z_ε of Z, with respect to e and to any $\varepsilon \in B^*$, is the set

$$Z_\varepsilon := \{\xi \in \Omega_d : (\varepsilon, \xi\sqrt{1 - \varepsilon^2}) \in Z\}.$$

Clearly, Z_ε is an A-code on Ω_d with

$$A = \{(\beta - \varepsilon^2)/(1 - \varepsilon^2) : \beta \in B^*\}.$$

The following theorem is the spherical analogue of the Assmus–Mattson theorem on designs in codes [3], cf. also [8], theorem 5.3.

THEOREM 8.2. *Let $Z \subset \Omega_{d+1}$, containing e, be a t-design and a B-code, with $1 \leqslant s^* \leqslant t + 1$. Then any non-empty derived $Z_\varepsilon \subset \Omega_d$, with respect to e and to $\varepsilon \in B^*$, is a $(t + 1 - s^*)$-design.*

Proof. Define δ by $\delta = 1$ if $(-e) \in Z$ and $\delta = 0$ if $(-e) \notin Z$. For any r with $0 \leqslant r \leqslant t + 1 - s^*$, any $F_r \in \mathrm{Hom}_d(r)$, and any k with $r \leqslant k \leqslant t$, define $G_{r,k} \in \mathrm{Hom}_{d+1}(k)$ by

$$G_{r,k}(\zeta) = G_{r,k}(\varepsilon; \eta) := \varepsilon^{k-r}F_r(\eta) = \varepsilon^{k-r}(1 - \varepsilon^2)^{r/2}F_r(\xi),$$

for $\xi \in \Omega_d$. Then

$$\sum_{\zeta \in Z} G_{r,k}(\zeta) - G_{r,k}(e) - \delta G_{r,k}(-e) = \sum_{\varepsilon \in B^*} \varepsilon^{k-r}(1 - \varepsilon^2)^{r/2} \sum_{\xi \in Z_\varepsilon} F_r(\xi).$$

Any $T \in O(d)$ induces an orthogonal transformation of Ω_{d+1} fixing e which, applied to Z, leaves the left-hand side invariant, since Z is a t-design. Therefore, the right-hand side

$$\sum_{\varepsilon \in B^*} \varepsilon^{k-r}(1 - \varepsilon^2)^{r/2} \sum_{\xi \in TZ_\varepsilon} F_r(\xi)$$

is independent of $T \in O(d)$. For $k = r, r + 1, \ldots, r - 1 + s^*$, this yields s^* equations for the s^* unknowns

$$\sum_{\xi \in TZ_\varepsilon} F_r(\xi), \qquad \varepsilon \in B^*.$$

Since the (essentially Vandermonde) determinant

$$\det[\varepsilon^{k-r}(1 - \varepsilon^2)^{r/2}], \qquad \varepsilon \in B^*, k \in \{r, r + 1, \ldots, r - 1 + s^*\},$$

is non-zero, the unknowns are determined, that is, they are independent of $T \in O(d)$. This holds for any $r = 0, 1, \ldots, t + 1 - s^*$, and for any $F_r \in \mathrm{Hom}_d(r)$. Therefore, any $Z_\varepsilon \neq \varnothing$ is a $(t + 1 - s^*)$-design, and the theorem is proved.

Let Y be any finite non-empty set of lines through the origin of $\mathbb{R}^{d+1}$, and let Z be the set of the intersections of the lines with the unit sphere Ω_{d+1}. Then Z is an antipodal B-code for a set B satisfying $B' = -B'$, thus yielding an $(d' = d + 1, n', s', t')$-configuration. Such Z, and their derived Z_ε, are exposed in the following examples.

EXAMPLE 8.3. Let Y be equiangular in $\mathbb{R}^{d+1}$, so that Z is an antipodal B-code with $B = \{-1, \varepsilon, -\varepsilon\}$, $0 < \varepsilon < 1$. Clearly, Z is a 1-design. Now apply Example 4.7 and Theorem 6.5. If $\varepsilon^2 < 1/(d + 1)$, then Z is a 2-design, whence a 3-design, if and only if the bound

$$n' \leqslant \frac{2(d+1)(1-\varepsilon^2)}{1-\varepsilon^2(d+1)}$$

is attained. This corresponds to the special bound for equiangular lines. Theorem 6.8 implies that Z is a 4-design, whence a tight 5-design, if and only if $n' = (d + 1)(d + 2)$. This corresponds to the absolute bound for equiangular lines. In this case $\varepsilon = (d + 3)^{-1/2}$, and $1/\varepsilon$ is an integer if $d > 3$. Thus, the icosahedron, and the regular 2 graphs on 28 and on 276 vertices provide tight 5-designs with the parameters

$$(d', n', s', t') = (3, 12, 3, 5), (7, 56, 3, 5), (23, 552, 3, 5).$$

The known regular 2-graphs provide many $(d', n', 3, 3)$-configurations; there are several infinite series, cf. [24], [26].

The derived configuration Z_ε of Z is an A-code, with

$$A = \left\{-\frac{\varepsilon}{1-\varepsilon}, \frac{\varepsilon}{1+\varepsilon}\right\}.$$

By Theorem 8.2 we find a 2-design

$$(d, n, s, t) = \left(d, \frac{d}{1-\varepsilon^2(d+1)}, 2, 2\right)$$

corresponding to each regular 2-graph. This is a tight 4-design if and only if $\varepsilon = (d + 3)^{-1/2}$. We only know the existence of the following cases:

$$(d, n, s, t) = (2, 5, 2, 4), (6, 27, 2, 4), (22, 275, 2, 4).$$

Each tight 4-design provides a maximal solution to the problem of $[-1, \beta]$-codes, with $\beta = \varepsilon/(1 + \varepsilon)$; indeed, the bound of Example 4.6 is achieved. We point out that any tight 4-design necessarily is the derived of a tight 5-design.

EXAMPLE 8.4. Let Y be a set of lines in $\mathbb{R}^{d+1}$, each pair of which is either perpendicular or has a given angle arc cos ε. Then $Z \subset \Omega_{d+1}$ is an antipodal B-code with

$$B = \{-1, 0, \varepsilon, -\varepsilon\}, \qquad 0 < \varepsilon < 1.$$

By application of Theorems 6.4 and 6.5 it can be shown that, in the case $\varepsilon^2 < 3/(d+3)$, Z forms a 4-design, whence a 5-design, if and only if

$$n' = 2\,\frac{(d+1)(d+3)(1-\varepsilon^2)}{3-(d+3)\varepsilon^2},$$

that is, if the set of lines meets the special bound. Moreover, Z is a 6-design, whence a tight 7-design, if and only if the set of lines meets the absolute bound, that is,

$$n' = \tfrac{1}{3}(d+1)(d+2)(d+3).$$

Then $\varepsilon^2(d+5) = 3$ and $1/\varepsilon \in \mathbb{N}\backslash\{0, 1\}$. Thus, the known sets of lines provide two tight 7-designs with the parameters

$$(d', n', s', t') = (8, 240, 4, 7) \qquad \text{and} \qquad (23, 4600, 4, 7).$$

Indeed, the first configuration corresponds to the root system E_8 with $\varepsilon = \frac{1}{2}$ (the Gosset polytope 4_{21} in $\mathbb{R}^8$, cf. [6], p. 204); the second configuration corresponds to a subset of the Leech lattice with $\varepsilon = \frac{1}{3}$. Their construction is briefly indicated in Example 8.5.

By Theorem 8.2 the 5-designs Z yield derived 3-designs Z_ε, and the 7-designs Z yield derived 5-designs Z_ε. Restricting to the case $\varepsilon^2(d+5) = 3$, $1/\varepsilon \in \mathbb{N}\backslash\{0, 1\}$, of the absolute bound, we find for Z_ε:

$$A = \left\{\frac{\varepsilon}{1+\varepsilon}, \frac{-\varepsilon}{1-\varepsilon}, \frac{-\varepsilon^2}{1-\varepsilon^2}\right\},$$

$$n \prod_{\alpha\in A'} \frac{x-\alpha}{1-\alpha} = Q_0(x) + Q_1(x) + Q_2(x) + \frac{1-\varepsilon^2}{3-\varepsilon^2}\, Q_3(x),$$

$$(d, n, s, t) = \left(\frac{3-5\varepsilon^2}{\varepsilon^2}, \frac{3-5\varepsilon^2}{2\varepsilon^6}, 3, 5\right).$$

By Theorem 7.4 these configurations carry 3-class association schemes. For $\varepsilon = \frac{1}{2}$ we have the tight 5-design with parameters (7, 56, 3, 5) which was met in Example 8.3. For $\varepsilon = \frac{1}{3}$ we have the 5-design (22, 891, 3, 5), a first example of a non-tight $(2s-1)$-design. The eigenmatrix P and the multiplicity vector μ of its association scheme are as follows:

$$A' = \{-\tfrac{1}{2},\ -\tfrac{1}{8},\ \tfrac{1}{4},\ 1\}$$

$$P = \begin{bmatrix} 42 & 512 & 336 & 1 \\ -21 & -64 & 84 & 1 \\ 9 & -16 & 6 & 1 \\ -3 & 8 & -6 & 1 \end{bmatrix}, \qquad \mu = \begin{bmatrix} 1 \\ 22 \\ 252 \\ 616 \end{bmatrix}.$$

EXAMPLE 8.5. Following McKay [20] we consider the lattice generated by the integral linear combinations of the columns of

$$\frac{1}{2}\begin{bmatrix} 4I_k & C_k - I_k \\ O_k & I_k \end{bmatrix},$$

where C_k is a skew Hadamard matrix of order k with the constant diagonal $-I_k$. For $k = 4$ this is the Gosset lattice 5_{21} in $\mathbb{R}^8$; its 240 vertices of length $\sqrt{2}$ provide the first configuration of Example 8.4. For $k = 12$ this is the Leech lattice in $\mathbb{R}^{24}$. Its $2 * \binom{28}{5}$ vectors of length 2 provide an antipodal B-code $Z \subset \mathbb{R}^{24}$ with

$$B = \{-1, 0, \pm\tfrac{1}{2}, \pm\tfrac{1}{4}\},$$

and the corresponding set Y of lines meets the absolute bound [11]. From Theorem 6.8 it follows that Z is a tight 11-design with

$$(d', n', s', t') = (24, 196560, 6, 11).$$

The derived configuration of Z with respect to $\varepsilon = \frac{1}{2}$ is the (23, 4600, 4, 7)-configuration of Example 8.4. The derived configuration X of Z with respect to $\varepsilon = \frac{1}{4}$ has

$$A(X) = \{-\tfrac{3}{5}, -\tfrac{1}{3}, -\tfrac{1}{15}, \tfrac{1}{5}, \tfrac{7}{15}\}, \qquad t(X) = 7,$$
$$(d, n, s, t) = (23, 23 * 2^{11}, 5, 7).$$

Theorem 7.4 implies that X is distance invariant, but it does not guarantee that X carries a 5-class association scheme. However, the third part applies. For instance, for all $\xi, \eta \in X$ with $\langle \xi, \eta \rangle = -\frac{3}{5}$, the intersection numbers $p_{\alpha,\beta}(\xi, \eta)$ only depend on α and β, since the triangle property implies $p_{\gamma,\gamma}(\xi, \eta) = 0$ for these ξ, η.

EXAMPLE 8.6. The regular polytope {3, 3, 5} in $\mathbb{R}^4$, cf. [6], p. 153, and [12], is a configuration X with

$$A(X) = \left\{-1, 0, \pm\tfrac{1}{2}, \frac{\pm 1 \pm \sqrt{5}}{4}\right\}, \qquad (d, n, s, t) = (4, 120, 8, 11).$$

It suffices to observe that the annihilator for $A(X)$ of degree 8 has the expansion

$$120F(x) = Q_0(x) + Q_1(x) + Q_2(x) + Q_3(x) + \tfrac{4}{5}Q_4(x) + \tfrac{2}{3}Q_5(x) + \tfrac{3}{7}Q_6(x) + \tfrac{1}{4}Q_7(x) + \tfrac{1}{9}Q_8(x).$$

Hence Theorem 6.5 implies that X is a design of strength $t(X) = 11$.

9. Examples from association schemes

In order to obtain a further method of construction for (d, n, s, t)-configurations, we consider Bose–Mesner algebras [4], [7]. For any fixed $s \in \mathbb{N}$, let

$$D_0 = I, D_1, \ldots, D_s; \qquad \sum_{i=0}^{s} D_i = J,$$

be real non-zero symmetric matrices of size n, with entries $\in \{0, 1\}$, which generate an $(s + 1)$-dimensional linear algebra over $\mathbb{R}$. This algebra is called the *Bose–Mesner algebra*, for short BM algebra, of the s-class association scheme with adjacency matrices D_i. It is well known that any BM algebra $\mathscr{A}$ is commutative, and admits a unique basis of mutually orthogonal idempotents

$$J_0 = n^{-1}J, J_1, \ldots, J_s,$$

cf. [7]. Clearly, any $D \in \mathscr{A}$ is positive semi-definite whenever it has non-negative components with respect to this basis. Thus for any such D, with unit diagonal, of rank $d \geqslant 2$, there exists a $d \times n$ matrix C such that

$$C^T C = D = I + \alpha_1 D_1 + \cdots + \alpha_s D_s,$$

for well-defined $\alpha_i \in \mathbb{R}$. The columns of C represent unit vectors in $\mathbb{R}^d$ with mutual inner products α_i. If $\alpha_i \neq 1$, for all $i \geqslant 1$, then these columns are distinct, and constitute an A-code $X \subset \Omega_d$ with $A = \{\alpha_1, \ldots, \alpha_s\}$ and $|X| = n$.

EXAMPLE 9.1. Let $\mathscr{A}$ be a 3-dimensional BM algebra, that is, the adjacency algebra of a strongly regular graph [7]. Let J_1 be one of the non-trivial minimal idempotents of $\mathscr{A}$, and let $d = \operatorname{rank}(J_1)$ be the corresponding multiplicity. If the given graph is not a ladder graph or its complement, the matrix $D = d^{-1}J_1$ satisfies the above requirements, that is,

$$D = I + \alpha D_1 + \beta(J - I - D_1)$$

for some $\alpha, \beta < 1$, only depending on the spectrum of the graph. Any set $X \subset \Omega_d$ which has D as its Gram matrix is an $\{\alpha, \beta\}$-code of cardinality n. Using well-known identities concerning the spectrum of a strongly regular graph, one can easily verify, as a consequence of Theorem 4.3, that X is a 2-design with

$$n = d(1 - \alpha)(1 - \beta)/(1 + d\alpha\beta).$$

Therefore, if $\alpha + \beta \leqslant 0$ holds, X provides a maximal solution to the problem of $\{\alpha, \beta\}$-codes, cf. Example 4.5. Conversely, let there be given an $\{\alpha, \beta\}$-code $X \subset \Omega_d$, with $\alpha < \beta < -\alpha$, whose cardinality n achieves the bound of Example 4.5. It turns out that the annihilator of degree $s = 2$ for A satisfies $f_0 = 1/n, f_1 > 0, f_2 > 0$. Therefore, Theorems 6.5 and 7.4 imply that X is a 2-design, and carries a strongly regular graph.

EXAMPLE 9.2. Which of the 2-designs of Example 9.1 are 3-designs? This question has an interesting relation to the *Krein condition*. Let $J_0, J_1, \ldots, J_s$ be the basis of the mutually orthogonal idempotents of a BM algebra $\mathscr{A}$. The Hadamard product $J_i \circ J_j$, being a principal submatrix of the Kronecker product $J_i \otimes J_j$, has all its eigenvalues in the interval [0, 1], and belongs to $\mathscr{A}$. Therefore, the coefficients in

$$J_i \circ J_j = \sum_{k=0}^{s} q_{ij}^k J_k$$

satisfy $q_{ij}^k \geqslant 0$. This is the Krein condition for $\mathscr{A}$, cf.* [13], [22], [23] and also [7]. Now it turns out that in Example 9.1 the following conditions are equivalent:

$$f_1 \leqslant \frac{1}{n}, \qquad d\alpha\beta + \alpha + \beta + 1 \geqslant 0, \qquad q_{11}^1 \geqslant 0.$$

Since $f_1 = 1/n$ is a criterion for X to be a 3-design, we have the following elaboration of Example 9.1. A strongly regular graph with $\alpha + \beta < 0$ provides a 3-design if and only if $q_{11}^1 = 0$, in other words, cf. [7], if and only if its 'pseudo-dual' has no triangles. The first and second of the following examples are provided by the Clebsch graph and the Higman–Sims graph, which are 'dual' to their complements. The remaining examples are derived from the McLaughlin graph, cf. Example 6.7.

$$\begin{aligned}
A &= \{-\tfrac{3}{5}, \tfrac{1}{5}\}, &\quad (d, n, s, t) &= (5, 16, 2, 3),\\
A &= \{-\tfrac{4}{11}, \tfrac{1}{11}\}, &\quad (d, n, s, t) &= (22, 100, 2, 3),\\
A &= \{-\tfrac{1}{3}, \tfrac{1}{9}\}, &\quad (d, n, s, t) &= (21, 112, 2, 3),\\
A &= \{-\tfrac{2}{7}, \tfrac{1}{7}\}, &\quad (d, n, s, t) &= (21, 162, 2, 3).
\end{aligned}$$

Notice that such examples yield 3-designs with $s(X) = 2$ which are not tight (not antipodal).

EXAMPLE 9.3. Let Γ be the orthogonal complement of the binary Golay code, that is, the unique binary code of length $d = 23$, size $n = 2048$, with Hamming distances 8, 12, 16, cf. [10]. Mapping the Hamming cube into the unit sphere in the usual way, we obtain from Γ an A-code $X \subset \Omega_{23}$ with

$$A = \{-\tfrac{9}{23}, -\tfrac{1}{23}, \tfrac{7}{23}\}.$$

The Gegenbauer coefficients of the annihilator of degree 3 for A are easily checked to satisfy

$$0 < f_1 < f_3 < f_2 < f_0 = \frac{1}{n}.$$

* The present simple proof of the Krein condition also occurs in N. Biggs, 'Automorphic Graphs and the Krein Condition', *Geom. Dedic.* 5, 117–127 (1976).

Hence Theorem 6.5 implies that X is a 3-design of strength $t(X) = 3$, and a maximal code, cf. Example 4.7. Although we know that X carries a 3-class association scheme [7], we cannot deduce this property from Theorem 7.4. In fact, there might exist a maximal A-code (with necessarily the same parameters d_α as the Golay code) which does not carry an association scheme. This example shows the difference between the cases $s \geqslant 3$ and $s = 2$ (cf. Example 9.1).

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