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CUBATURE FOR THE SPHERE AND THE DISCRETE SPHERICAL HARMONIC TRANSFORM*

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Abstract. Using a result of Bannai and Damerell, it is shown that a cubature formula with N points of degree $2s > 4$ for the surface of the n-dimensional sphere U_n cannot achieve the classical lower bound of $\dim \mathcal{P}^s$, where $\mathcal{P}^s$ is the space of all polynomials in n variables of at most degree s restricted to U_n. This implies that for $n > 2$ there does not exist a cubature-based discrete n-dimensional spherical harmonic transform for degree $s > 2$ with the same number of points as spectral coefficients.

Key words. cubature, sphere, spherical harmonic transform

AMS subject classifications. 65D32, 65M70

1. Introduction. Computing integrals over the surface of the sphere has many applications including the spectral transform algorithms used in most global climate models [3], [6]. For stability and accuracy, discrete transforms compute a least squares approximation of the spectral coefficients from a set of function values. To avoid solving the associated linear system, most transforms are based on exact cubature (or quadrature in one dimension) formulas. Here we denote a cubature formula which approximates the integral over U_n, the surface of the n-dimensional sphere, by

$$C_d(f) = \sum_{k=1}^{N} w_k f(x_k) \approx \int_{U_n} f(x)ds(x),$$

where the w_k are positive weights and the x_k are points on U_n, $1 \le k \le N$. A cubature formula is said to have degree d if

$$C_d(f) = \int_{U_n} f(x)ds(x) \qquad \forall f \in \mathcal{P}^d,$$

where $\mathcal{P}^d$ is the space of all polynomials of at most degree d in n variables restricted to U_n. The spherical harmonics Y_m^l are an orthonormal basis for $\mathcal{P}^d$ for $0 \le l \le d$ and $1 \le m \le M(l,n)$. The dimension of $\mathcal{P}^d$,

$$\dim \mathcal{P}^d = \sum_{l=0}^{d} M(l,n) = \frac{(n+2d-1)(n+d-2)!}{(n-1)!\,d!},$$

and the form of Y_m^l and $M(l,n)$ can be found in [13].

There has been a significant amount of work on the construction of cubature formulas for U_n which maximize d and minimize N, such as in [18], [9], [14], [7], [17], and [1]. Numerical methods for their construction have been given in [10], [12], and [8]. The classic lower bound for a degree $d = 2s$ cubature formula is $N \ge \dim \mathcal{P}^s$ ([18],

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[11]), but the above references show that for $n > 2$ there are only a few values of d where such minimal cubature formulas are known. In this paper, using the following result of [2], we show that for $d = 2s$, the lower bound for N of $\dim \mathcal{P}^s$ cannot be achieved for $n > 2$ and $s > 2$.

THEOREM 1 [2]. *A cubature formula of degree $2s$ with N equally weighted points does not exist for the surface of the n-dimensional sphere U_n, $n > 2$ with $N = \dim \mathcal{P}^s$ and $s > 2$.*

For domains such as the unit interval, the restriction that the quadrature points be equally weighted is quite severe and substantially increases the number of points needed to achieve a given degree; see [16, Chap. 6]. However, in §3 we show that for U_n, if a minimal ($N = \dim \mathcal{P}^s$) cubature formula exists then all the weights must be equal and thus the result of [2] applies.

2. Relation to the discrete spectral transform. A degree s discrete spherical harmonic transform (DST) must invert the synthesis operation

$$f(x) = \sum_{l=0}^{s} \sum_{m=1}^{M(l,n)} f_m^l Y_m^l(x) \tag{1}$$

by computing the integrals

$$f_m^l = \int_{U_n} f(x) Y_m^l(x)\, ds(x), \qquad 0 \le l \le s,\ 1 \le m \le M(l,n)$$

exactly for all $f \in \mathcal{P}^s$. If this is to be done by cubature, a necessary and sufficient condition is that the cubature formula be of degree $2s$ since the span of the product of all pairs of polynomials in $\mathcal{P}^s$ is $\mathcal{P}^{2s}$. The DST generated by such a cubature formula represents an $\mathbf{R}^N \to \mathbf{R}^P$ linear map from function values at the N cubature points to the $P = \dim \mathcal{P}^s$ spectral coefficients. That such a map, under a suitable norm, gives the least squares approximation of the spectral coefficients was shown by Swarztrauber [19]. This map inverts the $\mathbf{R}^P \to \mathbf{R}^N$ linear map given by the synthesis operation of evaluating a function given by P spectral coefficients at the N cubature points. Thus we have established that $N \ge P$. Furthermore, if $N = P$ then both maps must be inverses of each other.

It will be shown in the next section that there is no cubature formula with $N = P$ for $s > 2$, $n > 2$, and thus there can be no cubature-based invertible DST for $s > 2$, $n > 2$. But first we will discuss the two- and three-dimensional cases.

For $n = 2$, the restriction of polynomials in two variables to the boundary of the unit disk U_2 can be thought of as trigonometric polynomials on the interval $[0, 2\pi]$. For this case, we have the well-known discrete Fourier transform (DFT), which uses the trapezoidal rule with $2s + 1$ points to approximate the integrals which define the Fourier coefficients up to degree s. This approximation is exact for the dimension $2s + 1$ space of trigonometric polynomial up to degree s. The fast Fourier transform (FFT) uses the trapezoidal rule with one less point, but does not resolve all of the degree s trigonometric polynomials. Both of these quadrature-based transforms have $N = P$ and are invertible.

For $n = 3$, the DST most commonly used by spherical spectral methods relies on the spherical product Gauss cubature formula ([5], [18, §2.7]). For a function representable as a sum of spherical harmonics up to degree s, this degree $2s$ cubature formula requires function values at $(2s+1)(s+1)$ points to compute the $(s+1)^2$ spherical harmonic coefficients. Unlike the DFT, this DST does not provide an invertible mapping from the function values at the grid points to the spectral coefficients since $N > P$.

3. Nonexistence of minimal cubature. We can now easily show the following theorem.

THEOREM 2. *A cubature formula of degree* $2s$ *does not exist for the surface of the n-dimensional sphere* U_n, $n > 2$ *with* $N = \dim \mathcal{P}^s$ *and* $s > 2$.

Proof. Assume such a minimal cubature formula exists with weights w_k and points x_k, $1 \leq k \leq N$. Since the points x_k lie on the surface of the unit sphere, they may be treated as vectors with inner product $x_k \cdot x_k = 1$. From §2, we have that the associated DST applied to an arbitrary set of function values $\{f(x_k),\ 1 \leq k \leq N\}$ must be inverted by the synthesis operation (1) and thus

$$f(x_j) = \sum_{l=0}^{s} \sum_{m=1}^{M(l,n)} \left(\sum_{k=1}^{N} w_k Y_m^l(x_k) f(x_k) \right) Y_m^l(x_j), \qquad 1 \leq j \leq N.$$

Interchanging the order of summation, we have

$$f(x_j) = \sum_{k=1}^{N} w_k K(x_j \cdot x_k) f(x_k) \tag{2}$$

where

$$K(x_j \cdot x_k) = \sum_{l=0}^{s} \sum_{m=1}^{M(l,n)} Y_m^l(x_k) Y_m^l(x_j)$$

depends only on the inner product of x_j and x_k by the addition theorem for spherical harmonics [13]. Since equation (2) holds for arbitrary $\{f(x_k),\ 1 \leq k \leq N\}$, we have

$$w_k K(x_k \cdot x_j) = \begin{cases} 1, & k = j, \\ 0, & k \neq j, \end{cases}$$

and $w_k K(x_k \cdot x_k) = w_k K(1) = 1$ for all k. Thus w_k must be independent of k and we can apply Theorem 1 to finish the proof. We note that the reproducing kernel K appears in the literature ([15], [11], [4]), but here the use of spherical harmonics allows us to calculate the form of K.

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