Numerical integration over bounded intervals

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Padua, autumn, 2024
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"Numerical cubature and its applications"

Purpose

The purpose it to provide some basics on numerical integration over bounded intervals, that is

- description of symbolic software for computation of integrals;
- introduce the basic ideas on univariate polynomial interpolation;
- show some rules of interpolatory type, in the *Newton-Cotes* family, as midpoint, trapezoidal and Cavalieri-Simpson rules;
- disadvantages of Newton-Cotes rules and pros of composite rules;
- practical examples in Matlab/Octave.

The purpose is to compute

$$\int_a^b f(x)dx$$

where f is typically continuous in the interval $[a, b]$.

Remark

There are of course more general instances, e.g. f could have a finite number of jump discontinuities, but we will not take this case into account.

Remark (Pathological continuous functions)

Between the continuous functions there are many odd examples, as the Weierstrass function

$$f(x) = \sum_{k=1}^{+\infty} \frac{\sin(\pi k^2 x)}{\pi k^2}$$

that is continuous everywhere but differentiable only on a set of null measure (proof by Hardy).

Poincaré defined it as "an outrage against common sense"and Hermite as a "lamentable scourge".

- Analitically, a typical approach is to primitive) of the integrand.
- In view of the fundamental theorem of calculus (known as Torricelli-Barrow theorem), once a primitive is available then one can easily compute the definite integral.
- Unfortunately in many cases there are no known *primitives in terms of elementary functions* and one cannot adopt the technique described above.

Some examples of integrands with no primitive in terms of elementary functions, we have:

- the function $g : \mathbb{R} \to \mathbb{R}$ defined as

$$g(x) = \exp(x^2)$$

- the function $\texttt{sinc} : \mathbb{R} \to \mathbb{R}$ defined as

$$\texttt{sinc}(x) = \begin{cases} \frac{\sin(\pi t)}{\pi t}, & x \neq 0, \\ 1, & x = 0 \end{cases}$$

The proof concerning the absence of primitives in terms of elementary functions depend on the Liouville theorem.

There are several environments for symbolic calculus, that allow the computation of definite and indefinite integrals, e.g.

- Maple, offering also online services;
- Mathematica, offering also online services;
- Maxima, free, with interesting online demo.

We will propose some examples to understand their usage.

1. As first example we consider a rational function and its integration that can be done by decomposition in partial fractions.

$$\int \frac{1+x-x^2}{1+x^2}dx = \frac{\log(x^2+1)}{2} - x + 2\arctan(x) + C$$

   This technique has been discovered indipendently by Johann Bernoulli and Gottfried Leibniz (1702) and usually requires some boring computations.

2. As second example we consider $\exp(-x^2)$.

For each of them we will compute definite integrals in $[0,1]$.

We first use Maple via Matlab shell for the computation of indefinite integrals.

```
>> syms x;
>> int((1+x-x^2)/(1+x^2),x)

ans =

log(x^2 + 1)/2 - x + 2*atan(x)

>> int(exp(-x^2))

ans =

(pi^(1/2)*erf(x))/2

>> help erf
 erf Error function.
    Y = erf(X) is the error function for each element of X.  X must be
    real. The error function is defined as:

      erf(x) = 2/sqrt(pi) * integral from 0 to x of exp(-t^2) dt.

    See also erfc, erfcx, erfinv, erfcinv.

    Documentation for erf
    Other uses of erf

>>
```

Figure: Indefinite integrals in Maple (via Matlab shell).

Now we adopt Maple via Matlab shell for the computation of definite integrals.

```
>> int((1+x-x^2)/(1+x^2),0,1)

ans =

pi/2 + log(2)/2 - 1

>> format long
>> pi/2 + log(2)/2 - 1

ans =

   0.917369917074869

>> int(exp(-x^2),0,1)

ans =

(pi^(1/2)*erf(1))/2

>> (pi^(1/2)*erf(1))/2

ans =

   0.746824132812427

>>
```

Figure: Definite integrals in Maple (via Matlab shell).

We now make the computations of indefinite integrals via Mathematica, e.g. visiting the website Online Integral Calculator. To complete the computations, use the “equal ”symbol on the right.

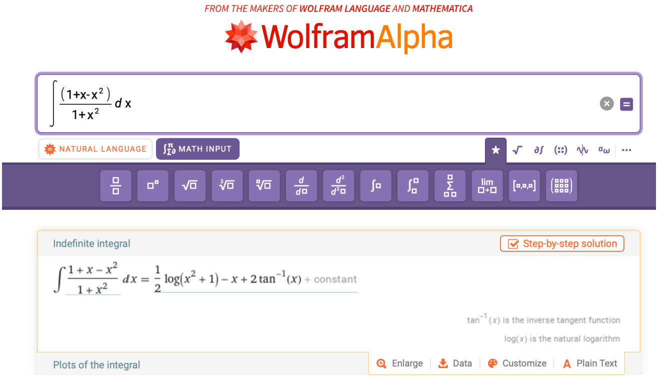


Figure: Indefinite integrals in Mathematica (via Online Integral Calculator).

About the second integral, we insert the new integrand and digit =on the right side of the box.

Figure: Indefinite integrals in Mathematica (via Online Integral Calculator).

Concerning definite integrals, writing the data in Math Input, clicking on *more digits*:

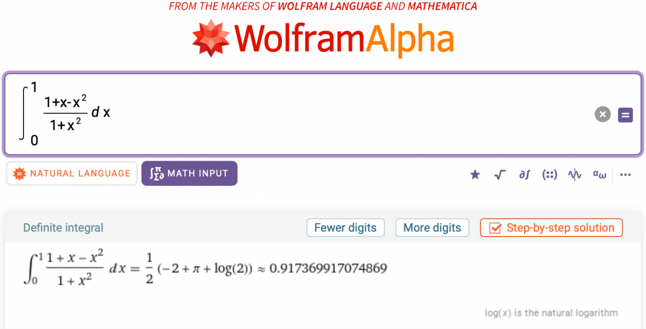


Figure: Definite integrals in Mathematica (via Online Integral Calculator).

We repeat the procedure for the second function.

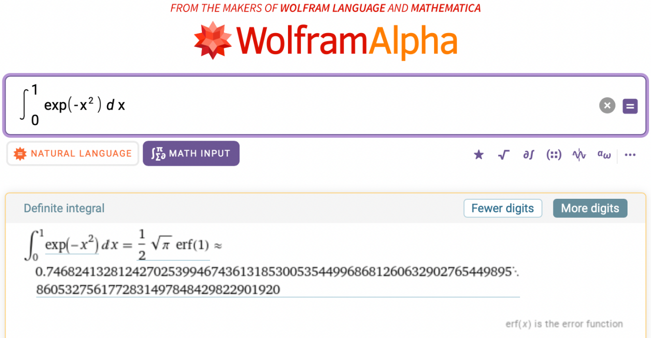


Figure: Definite integrals in Mathematica (via Online Integral Calculator).

We repeat the computation by means of Maxima demo homesite

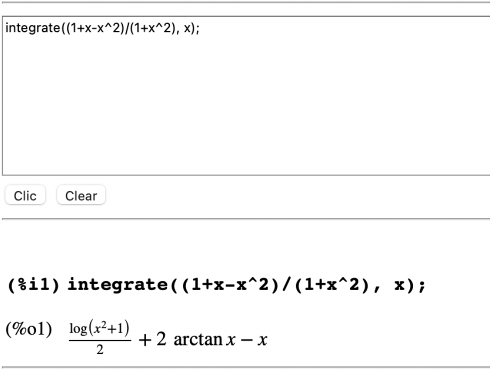


Figure: Indefinite integrals in Maxima (via online site).

Symbolic calculus of integrals

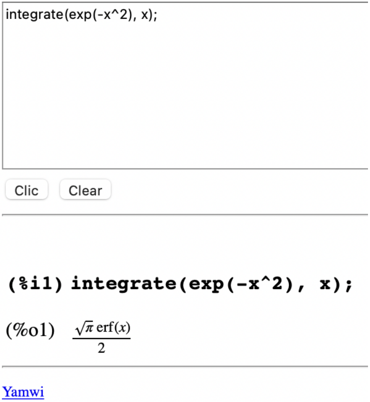


Figure: Indefinite integrals in Maxima (via online site).

```
integrate((1+x-x^2)/(1+x^2),x,0,1);
```

Clic Clear

```
(%i1) integrate((1+x-x^2)/(1+x^2),x,0,1);
```

(%o1) $\frac{\log 2+\pi-2}{2}$

Figure: Definite integrals in Maxima (via online site).

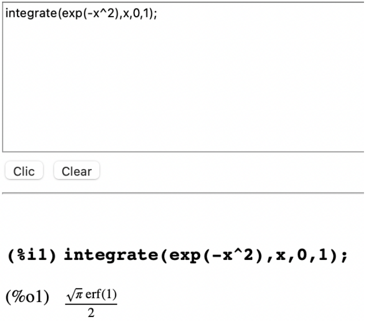


Figure: Definite integrals in Maxima (via online site).

Remark

The installation of Maxima may be not trivial (e.g. is not immediate on MacOS).

The previous definite integrals

$$\int_0^1 \frac{1+x-x^2}{1+x^2} dx \approx 0.917369917074869$$

$$\int_0^1 \exp(-x^2) dx \approx 0.746824132812427$$

may also be approximate by Matlab/Octave.

- We make the experiments in Octave, taking into account that the Matlab version is equal.
- To this purpose one can use OctaveOnline as well as Octave/Matlab.

We show how we can compute definite integrals in Matlab/Octave by adaptive algorithm implemented in `integral`, controlling the absolute and relative error, with tolerances decided by the users. This is guaranteed numerically but not mathematically (the code may fail in special and rare examples).

Consider that the function definition in Matlab/Octave requires the knowledge of pointwise operations, using "`.`".

```
>> f=@(x) (1+x-x.^2)./(1+x.^2);
>> Q = integral (f,0,1,'AbsTol',10^(-12),'RelTol',10^(-12));
>> Q
Q = 0.917369917074869
>> g=@(x) exp(-x.^2);
>> Q = integral (g,0,1,'AbsTol',10^(-12),'RelTol',10^(-12));
>> Q
Q = 0.746824132812427
>>
```

Figure: Definite integrals in Octave.

Problema. (Univariate polynomial interpolation)

Given

- $n+1$ *distinct points* $x_0, \ldots, x_n$,
- *the valuea* $y_0, \ldots, y_n$

the problem of polynomial interpolation consists in computing

$$p_n(x) = a_0 + \ldots + a_n x^n$$

such that

$$p_n(x_i) = y_i, \;\; i = 0, \ldots, n.$$

Example (Straight line for two given points)

Given 2 distinct points x_0, x_1 and the values y_0, y_1, determine $p_1 \in \mathbb{P}_1$ such that

$$p_1(x_0) = y_0, \quad p_1(x_1) = y_1 \tag{1}$$

that is the straight line that *passes* for the couples (x_0, y_0), (x_1, y_1).

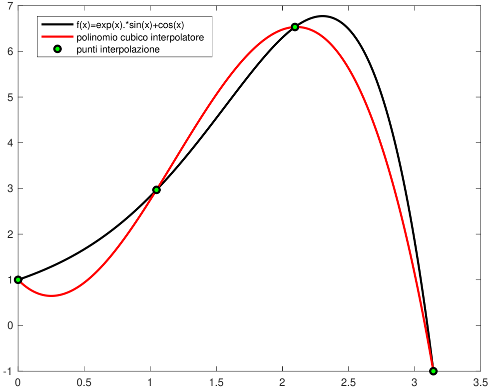


Figure: Example consisting in interpolating the function $f(x) = \exp(x)\sin(x) + \cos(x)$ (in black) by an algebraic polynomial of degree 3 (in red) in 4 equispaced points in $[0, \pi]$ (green circles). We plot the pertinent plots and the points $\{(x_k, y_k)\}_{k=0,\ldots,3}$, where $x_k = k\pi/3$ e $y_k = f(x_k)$, $k = 0, \ldots, 3$.

Theorem (Existence and uniqueness of polynomial interpolant)

GIven $n+1$ distinct points $x_0, x_1, \ldots, x_n$ and the values $vy_0, y_1, \ldots, y_n$, the polynomial $p_n \in \mathbb{P}_n$ such that

$$p_n(x_i) = y_i, \quad i = 0, \ldots, n.$$

exists and is unique.
Next

$$p_n(x) = \sum_{k=0}^{n} y_k L_k(x) = y_0 L_0(x) + y_1 L_1(x) + \ldots + y_n L_n(x)$$

where

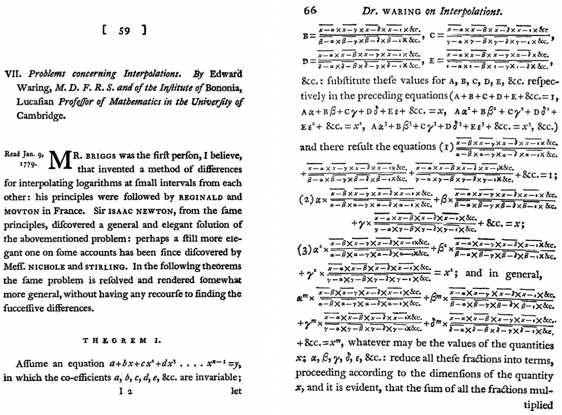
$$L_k(x) := \prod_{j=0, j\neq k}^{n} \frac{x - x_j}{x_k - x_j} = \frac{(x-x_0)\ldots(x-x_{k-1})(x-x_{k+1})\ldots(x-x_n)}{(x_k-x_0)\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots(x_k-x_n)}$$

is the k-th Lagrange polynomial w.r.t. the nodes $\{x_k\}_{k=0,\ldots,n}$.

The Lagrange polynomials where discovered by

- Waring in Problems concerning interpolations in 1779,
- rediscovered by Euler in 1783,
- published by Lagrange in Leçon Cinquième. Sur l'usage des courbes dans la solution des problèmes in 1795 (volume 7, p.286).

For an interesting note see A Chronology of Interpolation: From Ancient Astronomy to Modern Signal and Image Processing.



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VII. *Problems concerning Interpolations.* By Edward Waring, *M. D. F. R. S. and of the Inſtitute of* Bononia, Lucaſian *Profeſſor of Mathematics in the Univerſity of* Cambridge.

Read Jan. 9, 1779.

MR. BRIGGS was the firſt perſon, I believe, that invented a method of differences for interpolating logarithms at ſmall intervals from each other: his principles were followed by REGINALD and MOVTON in France. Sir ISAAC NEWTON, from the ſame principles, diſcovered a general and elegant ſolution of the abovementioned problem: perhaps a ſtill more elegant one on ſome accounts has been ſince diſcovered by Meſſ. NICHOLE and STIRLING. In the following theorems the ſame problem is reſolved and rendered ſomewhat more general, without having any recourſe to finding the ſucceſſive differences.

THEOREM I.

Aſſume an equation $a+bx+cx^2+dx^3 \dots x^{n-1}=y$, in which the co-efficients a, b, c, d, e, &c. are invariable;

I 2 let

66 *Dr.* WARING *on Interpolations.*

$$B=\frac{\overline{x-\alpha}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\epsilon}\times\text{\&c.}}{\overline{\beta-\alpha}\times\overline{\beta-\gamma}\times\overline{\beta-\delta}\times\overline{\beta-\epsilon}\times\text{\&c.}},\quad C=\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\delta}\times\overline{x-\epsilon}\times\text{\&c.}}{\overline{\gamma-\alpha}\times\overline{\gamma-\beta}\times\overline{\gamma-\delta}\times\overline{\gamma-\epsilon}\times\text{\&c.}},$$

$$D=\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\epsilon}\times\text{\&c.}}{\overline{\delta-\alpha}\times\overline{\delta-\beta}\times\overline{\delta-\gamma}\times\overline{\delta-\epsilon}\times\text{\&c.}},\quad E=\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\text{\&c.}}{\overline{\epsilon-\alpha}\times\overline{\epsilon-\beta}\times\overline{\epsilon-\gamma}\times\overline{\epsilon-\delta}\times\text{\&c.}},$$

&c.: ſubſtitute theſe values for A, B, C, D, E, &c. reſpectively in the preceding equations ($A+B+C+D+E+\text{\&c.}=1$, $A\alpha+B\beta+C\gamma+D\delta+E\epsilon+\text{\&c.}=x$, $A\alpha^2+B\beta^2+C\gamma^2+D\delta^2+E\epsilon^2+\text{\&c.}=x^2$, $A\alpha^3+B\beta^3+C\gamma^3+D\delta^3+E\epsilon^3+\text{\&c.}=x^3$, &c.) and there reſult the equations

$$(1)\ \frac{\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\epsilon}\times\text{\&c.}}{\overline{\alpha-\beta}\times\overline{\alpha-\gamma}\times\overline{\alpha-\delta}\times\overline{\alpha-\epsilon}\times\text{\&c.}}+\frac{\overline{x-\alpha}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\epsilon}\times\text{\&c.}}{\overline{\beta-\alpha}\times\overline{\beta-\gamma}\times\overline{\beta-\delta}\times\overline{\beta-\epsilon}\times\text{\&c.}}+\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\delta}\times\overline{x-\epsilon}\times\text{\&c.}}{\overline{\gamma-\alpha}\times\overline{\gamma-\beta}\times\overline{\gamma-\delta}\times\overline{\gamma-\epsilon}\times\text{\&c.}}+\text{\&c.}=1;$$

$$(2)\ \alpha\times\frac{\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\epsilon}\times\text{\&c.}}{\overline{\alpha-\beta}\times\overline{\alpha-\gamma}\times\overline{\alpha-\delta}\times\overline{\alpha-\epsilon}\times\text{\&c.}}+\beta\times\frac{\overline{x-\alpha}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\epsilon}\times\text{\&c.}}{\overline{\beta-\alpha}\times\overline{\beta-\gamma}\times\overline{\beta-\delta}\times\overline{\beta-\epsilon}\times\text{\&c.}}+\gamma\times\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\delta}\times\overline{x-\epsilon}\times\text{\&c.}}{\overline{\gamma-\alpha}\times\overline{\gamma-\beta}\times\overline{\gamma-\delta}\times\overline{\gamma-\epsilon}\times\text{\&c.}}+\text{\&c.}=x;$$

$$(3)\ \alpha^2\times\frac{\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\epsilon}\times\text{\&c.}}{\overline{\alpha-\beta}\times\overline{\alpha-\gamma}\times\overline{\alpha-\delta}\times\overline{\alpha-\epsilon}\times\text{\&c.}}+\beta^2\times\frac{\overline{x-\alpha}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\epsilon}\times\text{\&c.}}{\overline{\beta-\alpha}\times\overline{\beta-\gamma}\times\overline{\beta-\delta}\times\overline{\beta-\epsilon}\times\text{\&c.}}+\gamma^2\times\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\delta}\times\overline{x-\epsilon}\times\text{\&c.}}{\overline{\gamma-\alpha}\times\overline{\gamma-\beta}\times\overline{\gamma-\delta}\times\overline{\gamma-\epsilon}\times\text{\&c.}}=x^2;$$

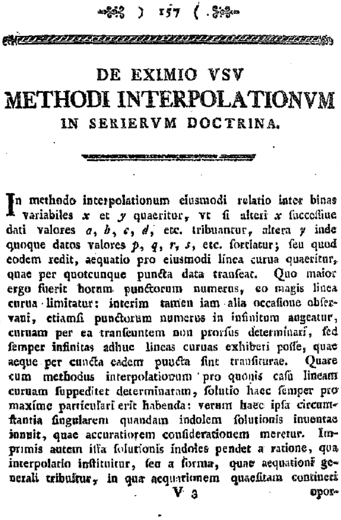
and in general,

$$\alpha^m\times\frac{\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\epsilon}\times\text{\&c.}}{\overline{\alpha-\beta}\times\overline{\alpha-\gamma}\times\overline{\alpha-\delta}\times\overline{\alpha-\epsilon}\times\text{\&c.}}+\beta^m\times\frac{\overline{x-\alpha}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\epsilon}\times\text{\&c.}}{\overline{\beta-\alpha}\times\overline{\beta-\gamma}\times\overline{\beta-\delta}\times\overline{\beta-\epsilon}\times\text{\&c.}}+\gamma^m\times\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\delta}\times\overline{x-\epsilon}\times\text{\&c.}}{\overline{\gamma-\alpha}\times\overline{\gamma-\beta}\times\overline{\gamma-\delta}\times\overline{\gamma-\epsilon}\times\text{\&c.}}+\delta^m\times\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\epsilon}\times\text{\&c.}}{\overline{\delta-\alpha}\times\overline{\delta-\beta}\times\overline{\delta-\gamma}\times\overline{\delta-\epsilon}\times\text{\&c.}}$$

$+\text{\&c.}=x^m$, whatever may be the values of the quantities x; $\alpha, \beta, \gamma, \delta, \epsilon$, &c.: reduce all theſe fractions into terms, proceeding according to the dimenſions of the quantity x, and it is evident, that the ſum of all the fractions multiplied

Figure: The paper by Waring in which the Lagrange polynomials were discovered.

Historical remark



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DE EXIMIO VSV
METHODI INTERPOLATIONVM
IN SERIERVM DOCTRINA.

In methodo interpolationum eiusmodi relatio inter binas variabiles x et y quaeritur, vt ſi alteri x ſucceſſiue dati valores a, b, c, d, etc. tribuantur, altera y inde quoque datos valores p, q, r, s, etc. ſortiatur; ſeu quod eodem redit, aequatio pro eiusmodi linea curua quaeritur, quae per quotcunque puncta data tranſeat. Quo maior ergo fuerit horum punctorum numerus, eo magis linea curua limitatur: interim tamen iam alia occaſione obſeruaui, etiamſi punctorum numerus in infinitum augeatur, curuam per ea tranſeuntem non prorſus determinari, ſed ſemper infinitas adhuc lineas curuas exhiberi poſſe, quae aeque per cuncta eadem puncta ſint tranſiturae. Quare cum methodus interpolationum pro quouis caſu lineam curuam ſuppeditet determinatam, ſolutio haec ſemper pro maxime particulari erit habenda: verum haec ipſa circumſtantia ſingularem quandam indolem ſolutionis inuentae innuit, quae accuratiorem conſiderationem meretur. Imprimis autem iſta ſolutionis indoles pendet a ratione, qua interpolatio inſtituitur, ſeu a forma, quae aequationi generali tribuitur, in qua aequationem quaeſitam contineri opor-

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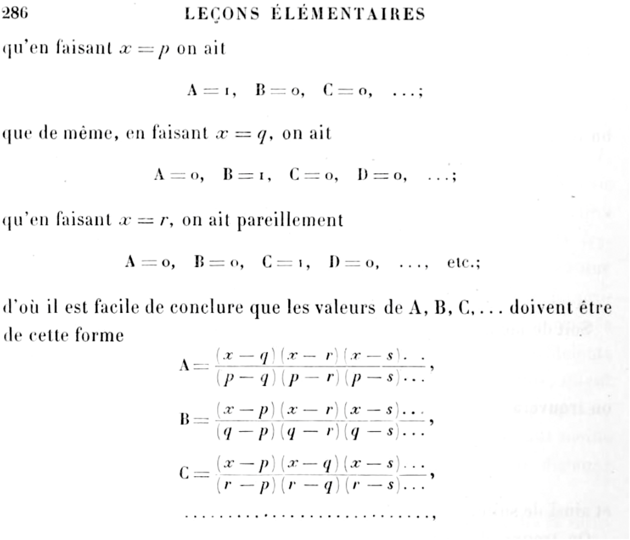
Figure: Paper by Eulero in which the Lagrange polynomials were rediscovered.

Historical remark



286 LEÇONS ÉLÉMENTAIRES

qu'en faisant $x = p$ on ait

$$A = 1, \quad B = 0, \quad C = 0, \quad \ldots;$$

que de même, en faisant $x = q$, on ait

$$A = 0, \quad B = 1, \quad C = 0, \quad D = 0, \quad \ldots;$$

qu'en faisant $x = r$, on ait pareillement

$$A = 0, \quad B = 0, \quad C = 1, \quad D = 0, \quad \ldots, \quad \text{etc.};$$

d'où il est facile de conclure que les valeurs de A, B, C, ... doivent être de cette forme

$$A = \frac{(x-q)(x-r)(x-s)\ldots}{(p-q)(p-r)(p-s)\ldots},$$

$$B = \frac{(x-p)(x-r)(x-s)\ldots}{(q-p)(q-r)(q-s)\ldots},$$

$$C = \frac{(x-p)(x-q)(x-s)\ldots}{(r-p)(r-q)(r-s)\ldots},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

Figure: Work by Lagrange on the relative polynomials.

One may believe that if we take more and more equispaced nodes in $[a, b]$, that is

$$x_k = a + k\,\frac{(b-a)}{n},\;\; k = 0, \dots, n$$

increasing n, the polynomial p_n will approximate better and better the function f, but it is not so.

Runge discovered a famous counterexample in Über die Darstellung willkrlicher Functionen und die Interpolation zwichen äquidistanten Ordinaten, p.243, (1901), that is $f \in C^\infty([-5, 5])$ defined by

$$f(x) = \frac{1}{1+x^2},\;\; x \in [-5, 5]$$

in which increasing n, $\max_{x\in[a,b]} |f(x) - p_n(x)|$ does not converge to 0.

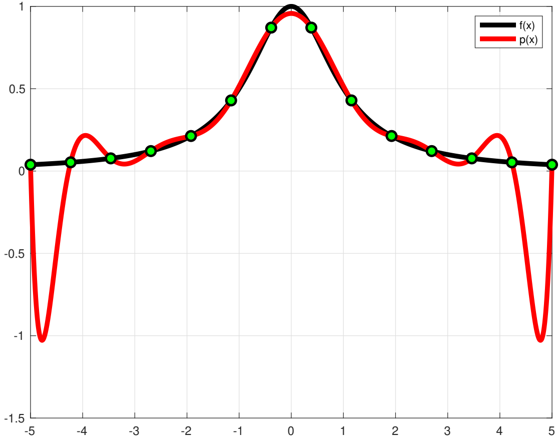
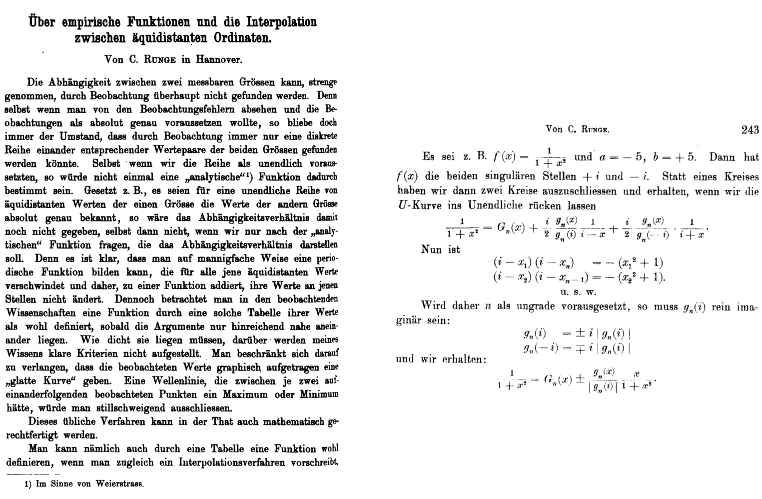


Figure: Plot illustrating the polynomial interpolant (of the Runge function) of degree 12 based on 13 equispaced nodes. The plot in red is the interpolant, while the plot in black is the Runge function. The green dots are the couples to be interpolated. Observe the wild obscillations at extrema. Increasing the degree things get worse!

Runge counterexample



Über empirische Funktionen und die Interpolation zwischen äquidistanten Ordinaten.

Von C. Runge in Hannover.

Die Abhängigkeit zwischen zwei messbaren Grössen kann, strenge genommen, durch Beobachtung überhaupt nicht gefunden werden. Denn selbst wenn man von den Beobachtungsfehlern absehen und die Beobachtungen als absolut genau voraussetzen wollte, so bliebe doch immer der Umstand, dass durch Beobachtung immer nur eine diskrete Reihe einander entsprechender Wertepaare der beiden Grössen gefunden werden könnte. Selbst wenn wir die Reihe als unendlich voraussetzten, so würde nicht einmal eine „analytische“[1]) Funktion dadurch bestimmt sein. Gesetzt z. B., es seien für eine unendliche Reihe von äquidistanten Werten der einen Grösse die Werte der andern Grösse absolut genau bekannt, so wäre das Abhängigkeitsverhältnis damit noch nicht gegeben, selbst dann nicht, wenn wir nur nach der „analytischen“ Funktion fragen, die das Abhängigkeitsverhältnis darstellen soll. Denn es ist klar, dass man auf mannigfache Weise eine periodische Funktion bilden kann, die für alle jene äquidistanten Werte verschwindet und daher, zu einer Funktion addiert, ihre Werte an jenen Stellen nicht ändert. Dennoch betrachtet man in den beobachtenden Wissenschaften eine Funktion durch eine solche Tabelle ihrer Werte als wohl definiert, sobald die Argumente nur hinreichend nahe aneinander liegen. Wie dicht sie liegen müssen, darüber werden meines Wissens klare Kriterien nicht aufgestellt. Man beschränkt sich darauf zu verlangen, dass die beobachteten Werte graphisch aufgetragen eine „glatte Kurve“ geben. Eine Wellenlinie, die zwischen je zwei aufeinanderfolgenden beobachteten Punkten ein Maximum oder Minimum hätte, würde man stillschweigend ausschliessen.

Dieses übliche Verfahren kann in der That auch mathematisch gerechtfertigt werden.

Man kann nämlich auch durch eine Tabelle eine Funktion wohl definieren, wenn man zugleich ein Interpolationsverfahren vorschreibt.

1) Im Sinne von Weierstrass.

Es sei z. B. $f(x) = \frac{1}{1+x^2}$ und $a = -5$, $b = +5$. Dann hat $f(x)$ die beiden singulären Stellen $+i$ und $-i$. Statt eines Kreises haben wir dann zwei Kreise auszuschliessen und erhalten, wenn wir die U-Kurve ins Unendliche rücken lassen

$$\frac{1}{1+x^2} = G_n(x) + \frac{i}{2}\frac{g_n(x)}{g_n(i)}\frac{1}{i-x} + \frac{i}{2}\frac{g_n(x)}{g_n(-i)} \cdot \frac{1}{i+x}.$$

Nun ist

$$(i-x_1)(i-x_n) = -(x_1^2+1)$$
$$(i-x_2)(i-x_{n-1}) = -(x_2^2+1).$$

u. s. w.

Wird daher n als ungrade vorausgesetzt, so muss $g_n(i)$ rein imaginär sein:

$$g_n(i) = \pm i\,|g_n(i)|$$
$$g_n(-i) = \mp i\,|g_n(i)|$$

und wir erhalten:

$$\frac{1}{1+x^2} = G_n(x) \pm \frac{g_n(x)}{|g_n(i)|}\frac{x}{1+x^2}.$$

Figure: Original paper on Runge counterexample.

The following theorem holds.

Theorem (Stability of integration)

If $\tilde{f}, f \in C([a,b])$, and $[a,b]$ is a bounded interval then

$$\left| \int_a^b f(x)dx - \int_a^b \tilde{f}(x)dx \right| \leq (b-a) \max_{x \in [a,b]} |f(x) - \tilde{f}(x)|.$$

Thus, if we assume that

- the points $\{x_k\}_{k=0,\ldots,n} \subset [a,b]$ are distinct,
- the polynomial $\tilde{f} = p_n$ that interpolates $(x_0, f(x_0)), \ldots, (x_n, f(x_n))$ is a *good* approximation of $f \in C([a,b])$.

then

$$I(p_n) = \int_a^b p_n(x)dx \approx I(f) = \int_a^b f(x)dx$$

with the advantage that the computation of $I(p_n)$ is usually easier than that of $I(f)$.

In virtue of what we have seen, let L_k be the Lagrange polynomials relatively to the set $\{x_k\}_{k=0,\dots,n}$, and the weights

$$w_k = \int_a^b L_k(x)dx$$

and get

$$\begin{aligned}\int_a^b f(x)dx &\approx \int_a^b p_n(x)dx = \int_a^b \sum_{k=0}^n f(x_k)L_k(x)dx \\ &= \sum_{k=0}^n \int_a^b f(x_k)L_k(x)dx = \sum_{k=0}^n f(x_k)\int_a^b L_k(x)dx \\ &= \sum_{k=0}^n w_k f(x_k). \qquad (2)\end{aligned}$$

that is

$$\int_a^b f(x)dx \approx \sum_{k=0}^n w_k f(x_k).$$

Observe that if $f \in \mathbb{P}_n$ then it it is exactly the polynomial p_n that interpolates the couples $(x_k, f(x_k))$, $k = 0, \ldots, n$ (by the uniqueness of the polynomial interpolant) and then

$$\int_a^b f(x)dx = \int_a^b p_n(x)dx = \int_a^b \sum_{k=0}^n f(x_k)L_k(x)dx = \ldots = \sum_{k=0}^n w_k f(x_k).$$

Cnosequently, if $f \in \mathbb{P}_n$ then $\sum_{k=0}^n w_k f(x_k)$ is equal to $\int_a^b f(x)dx$.

We will say that the degree of exactness of an interpolatory rule in $n+1$ nodes is at least n.

Thus, an interpolatory rule on

- 1 node, surely integrates exactly the constants;
- 2 nodes, surely integrates exactly the polynomials of degree 0, 1;
- 3 nodes, surely integrates exactly the polynomials of degree 0, 1, 2.

We will see that sometimes things can be even better.

The fact that

$$\int_a^b f(x)dx \approx \sum_{k=0}^{n} w_k f(x_k), \quad w_k = \int_a^b L_k(x)dx$$

says that to approximate the required integral it is not necessary to

1. compute the polynomial interpolant,
2. determine its primitive,
3. apply the fundamental theorem of integral calculus,

but

1. compute the weights $\{w_k\}_{k=0,\ldots,n}$ relatively to the nodes $\{x_k\}_{k=0,\ldots,n}$,
2. use the function evaluations $\{f(x_k)\}_{k=0,\ldots,n}$.

Remark

This fact is important, since if we change the integrand we minimize the computation, since the weights w_k, $k = 0, \ldots, n$ do not change, depending only on the nodes.

Definition (Rectangular rule)

Let

- $f \in C([a, b])$, $-\infty < a < b < +\infty$,
- $x_0 \in [a, b]$.

The rettangular rule is defined by

$$\int_a^b f(x)dx \approx w_0 f(x_0) = (b - a)f(x_0) := S_0^*(f). \quad (3)$$

If $x_0 = \frac{a+b}{2}$ we get the so called midpoint rule.

For the midpoint rule, the following error estimate holds

Theorem (Midpoint rule error estimate)

If $f \in C^{(2)}([a,b])$ then the midpoint rule error estimate is

$$E_0(f) := I(f) - S_0^*(f) = \frac{(b-a)^3}{24} f^{(2)}(\xi), \;\; \xi \in (a,b).$$

Observe that if $f \in \mathbb{P}_1$ allora

$$E_0(f) := I(f) - S_0^*(f) = 0$$

thus the rule is exact.

This is a little surprising since being based on just one function evaluation one may believe it integrates exactly just the constants and not polynomials of degree 1.

Rectangular rule

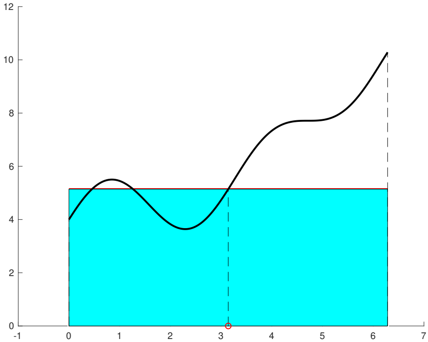


Figure: Rectangular rule with node $x_0 = (a+b)/2$, $a = 0$, $b = 2\pi$, for approximating $\int_0^{2\pi} (3 + \sin(2x) + \cos(x) + x)\, dx$ (midpoint rule computes the area in cyan).

Trapezoidal rule.

Definition (Trapezoidal rule)

The trapezoidal rule is defined by

$$\int_a^b f(x)dx \approx S_1(f) := \frac{b-a}{2} \cdot (f(a) + f(b)).$$

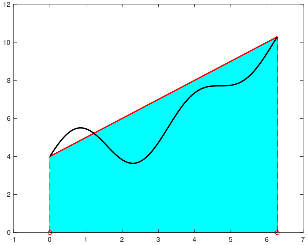


Figure: Trapezoidal rule for the approximation of $\int_0^{2\pi} 3 + \sin(2x) + \cos(x) + x\, dx$ (the rule computes the area in cyan).

Theorem (Error of the Trapezoidal rule)

If $f \in C^2([a,b])$ *then the error of the trapezoidal rule is*

$$E_1(f) := I(f) - S_1(f) = \frac{-(b-a)^3}{12} f^{(2)}(\xi), \quad \xi \in (a,b).$$

Theorem (Degree of exactness of the trapezoidal rule)

The degree of exactness of the trapezoidal rule is exactly 1.

Definition (Cavalieri-Simpson rule)

The Cavalieri-Simpson rule is defined as $\int_a^b f(x)dx \approx S_2(f)$ with

$$S_2(f) := \frac{b-a}{6} \cdot f(a) + \frac{2(b-a)}{3} \cdot f\left(\frac{a+b}{2}\right) + \frac{b-a}{6} \cdot f(b).$$

This interpolatory rule is equivalent to determine the definite integral for the polynomial with nodes the interval extrema a, b and their midpoint $(a+b)/2$.

Consequently, its degree of exactness is at least 2.

Cavalieri-Simpson rule

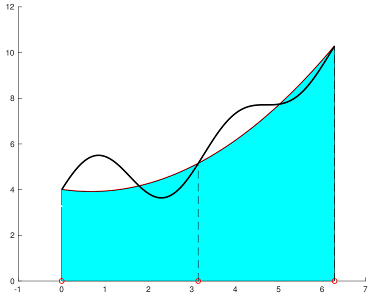


Figure: Cavalieri-Simpson rule for the computation of $\int_0^{2\pi} 3 + \sin(2x) + \cos(x) + x \, dx$, it determines the area in cyan).

Theorem (Cavalieri-Simpson rule error)

If $f \in C^4([a,b])$ the error made by Cavalieri-Simpson rule is

$$E_2(f) := I(f) - S_2(f) = \frac{-h^5}{90} f^{(4)}(\xi), \quad h = \frac{b-a}{2}$$

with $\xi \in (a,b)$.

Theorem (Degree of exactness of Cavalieri-Simpson rule)

The degree of exactness of Cavalieri-Simpson rule is exactly 3.

Remark

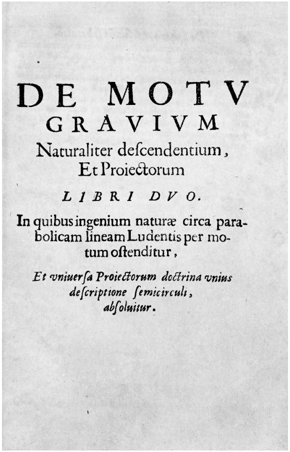
As in the case of midpoint rule, this result is remarkable, since it is expected to be 2, since the formula of interpolatory type is based on 3 nodes.

Observe that integration had a complicated history.

Cavalieri did not have the modern concept of integration since he was researcher between 1629 and 1647, while

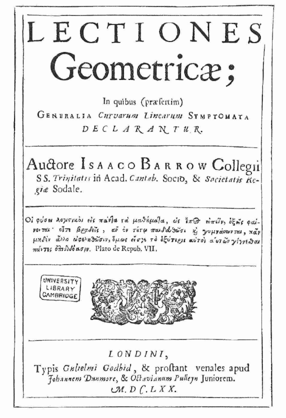
- the fundamental of integral calculus by (Torricelli-Barrow) appears in a primitive form in Lectiones geometricae by Barrow (1670) and in Geometriae pars universalis by Gregory (1668), see [1];
- Riemann integral was introduced in december 1853 (in his habilitation thesis);
- Lebesgue integral was discovered in 1904.

Cavalieri, by means of the so called method of indivisibles computed $\int_a^b x^n dx$ for $n = 1, \ldots, 9$.



DE MOTV
GRAVIVM
Naturaliter deſcendentium,
Et Proiectorum
LIBRI DVO.
In quibus ingenium naturæ circa parabolicam lineam Ludentis per motum oſtenditur,
Et vniuerſa Proiectorum doctrina vnius deſcriptione ſemicirculi, abſoluitur.

Figure: Work by Torricelli in which the fundamental theorem of integral calculus is introduced geometrically (link between displacement on a straight line and velocity).



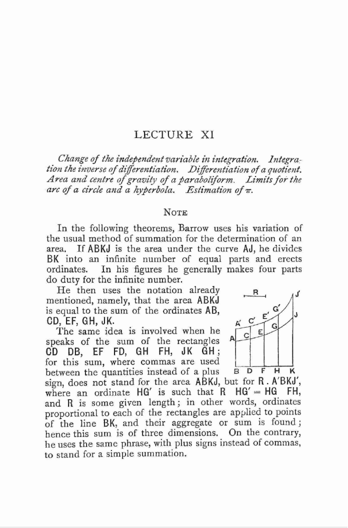
LECTIONES
Geometricæ;

In quibus (præfertim)
GENERALIA *Curvarum Linearum* SYMPTOMATA
DECLARANTUR.

Auctore ISAACO BARROW Collegii
SS. *Trinitatis* in Acad. *Cantab.* Socio, & *Societatis Regiæ* Sodale.

Οἱ φύσει λογιστικοὶ εἰς πάντα τὰ μαθήματα, ὡς ἔπος εἰπεῖν, ὀξεῖς φαίνονται· οἵ τε βραδεῖς, ἂν ἐν τούτῳ παιδευθῶσι καὶ γυμνάσωνται, κἂν μηδὲν ἄλλο ὠφεληθῶσιν, ὅμως εἴς γε τὸ ὀξύτεροι αὐτοὶ αὑτῶν γίγνεσθαι πάντες ἐπιδιδόασιν. Plato de Repub. VII.

LONDINI,
Typis *Gulielmi Godbid*, & proftant venales apud *Johannem Dunmore*, & *Octavianum Pulleyn* Juniorem.
M. D C. L X X.



LECTURE XI

Change of the independent variable in integration. Integration the inverse of differentiation. Differentiation of a quotient. Area and centre of gravity of a paraboliform. Limits for the arc of a circle and a hyperbola. Estimation of π.

NOTE

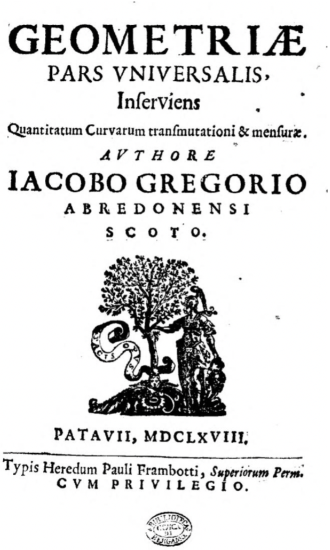
In the following theorems, Barrow uses his variation of the usual method of summation for the determination of an area. If ABKJ is the area under the curve AJ, he divides BK into an infinite number of equal parts and erects ordinates. In his figures he generally makes four parts do duty for the infinite number.

He then uses the notation already mentioned, namely, that the area ABKJ is equal to the sum of the ordinates AB, CD, EF, GH, JK.

The same idea is involved when he speaks of the sum of the rectangles CD DB, EF FD, GH FH, JK GH; for this sum, where commas are used between the quantities instead of a plus sign, does not stand for the area ABKJ, but for R . A'BKJ', where an ordinate HG' is such that R HG' = HG FH, and R is some given length; in other words, ordinates proportional to each of the rectangles are applied to points of the line BK, and their aggregate or sum is found; hence this sum is of three dimensions. On the contrary, he uses the same phrase, with plus signs instead of commas, to stand for a simple summation.

Figure: Work by Barrow in which a geometrical form of fundamental theorem of calculus. On this subject, see also the paper Barrow and Leibniz on the fundamental theorem of the calculus.

Historical notes


GEOMETRIÆ
PARS VNIVERSALIS,
Inſerviens
Quantitatum Curvarum tranſmutationi & menſuræ.
AVTHORE
IACOBO GREGORIO
ABREDONENSI
SCOTO.

PATAVII, MDCLXVIII.

Typis Heredum Pauli Frambotti, *Superiorum Perm.*
CVM PRIVILEGIO.

Figure: *Geometriae pars universalis*, published in Padua in 1668.

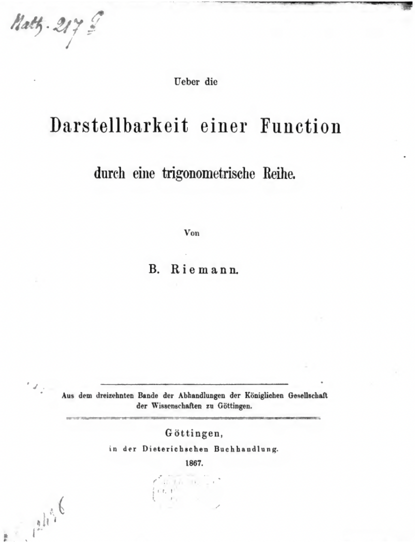
At the webpage Riemann integral we find this interesting remark.

The Riemann integral was introduced in Bernhard Riemann's paper "Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe" (On the representability of a function by a trigonometric series; i.e., when can a function be represented by a trigonometric series).

This paper was submitted to the University of Göttingen in 1854 as Riemann's Habilitationsschrift (qualification to become an instructor).

It was published in 1868 in Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen (Proceedings of the Royal Philosophical Society at Göttingen), vol. 13, pages 87-132.

For Riemann's definition of his integral, see section 4, "Über den Begriff eines bestimmten Integrals und den Umfang seiner Gültigkeit" (On the concept of a definite integral and the extent of its validity), pages 101–103.



Ueber die

Darstellbarkeit einer Function

durch eine trigonometrische Reihe.

Von

B. Riemann.

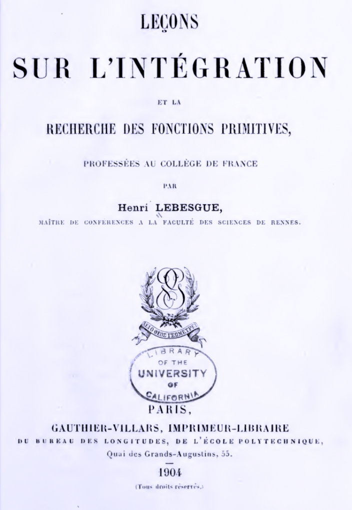
Aus dem dreizehnten Bande der Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen.

Göttingen,

in der Dieterichschen Buchhandlung.

1867.

Figure: Monograph in which the Riemann integral was introduced.



LEÇONS

SUR L'INTÉGRATION

ET LA

RECHERCHE DES FONCTIONS PRIMITIVES,

PROFESSÉES AU COLLÈGE DE FRANCE

PAR

Henri LEBESGUE,

MAÎTRE DE CONFÉRENCES A LA FACULTÉ DES SCIENCES DE RENNES.

PARIS,

GAUTHIER-VILLARS, IMPRIMEUR-LIBRAIRE

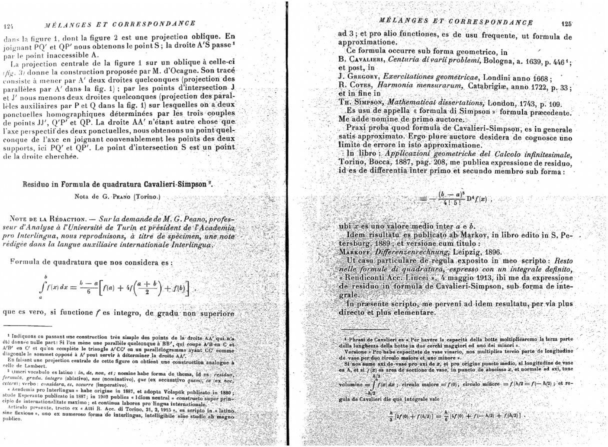
DU BUREAU DES LONGITUDES, DE L'ÉCOLE POLYTECHNIQUE,

Quai des Grands-Augustins, 55.

1904

Figure: Monograph in which the Lebesgue integral was introduced.

About the ascription to Cavalieri, Peano wrote in Residuo in Formula de quadratura Cavalieri-Simpson (1916), in a simplified latin language, that this formula was discovered by Cavalieri (1639), Gregory (1668), Cotes (1722) and Simpson (1743).



124 MÉLANGES ET CORRESPONDANCE

dans la figure 1, dont la figure 2 est une projection oblique. En joignant PQ′ et QP′ nous obtenons le point S ; la droite A′S passe[1] par le point inaccessible A.

La projection centrale de la figure 1 sur un oblique à celle-ci (*fig.* 3) donne la construction proposée par M. d'Ocagne. Son tracé consiste à mener par A′ deux droites quelconques (projection des parallèles par A′ dans la fig. 1) ; par les points d'intersection J et J′ nous menons deux droites quelconques (projection des parallèles auxiliaires par P et Q dans la fig. 1) sur lesquelles on a deux ponctuelles homographiques déterminées par les trois couples de points JJ′, Q′P′ et QP. La droite AA′ n'étant autre chose que l'axe perspectif des deux ponctuelles, nous obtenons un point quelconque de l'axe en joignant convenablement les points des deux supports, ici PQ′ et QP′. Le point d'intersection S est un point de la droite cherchée.

Residuo in Formula de quadratura Cavalieri-Simpson [2].

Nota de G. Peano (Torino.)

Note de la Rédaction. — *Sur la demande de M. G. Peano, professeur d'Analyse à l'Université de Turin et président de l'Academia pro Interlingua, nous reproduisons, à titre de spécimen, une note rédigée dans la langue auxiliaire internationale Interlingua.*

Formula de quadratura que nos considera es :

$$\int_a^b f(x)\,dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right],$$

que es vero, si functione f es integro, de gradu non superiore

[1] Indiquons en passant une construction très simple des points de la droite AA′ qui n'a été donnée nulle part: Si l'on mène une parallèle quelconque à BB′, qui coupe A′B en C et A′B′ en C′ et qu'on complète le triangle A′CC′ en un parallélogramme ayant CC′ comme diagonale le sommet opposé à A′ peut servir à déterminer la droite AA′.

En faisant une projection centrale de cette figure on obtient une construction analogue à celle de Lambert.

[2] Omni vocabulo es latino : *in, de, non, et ;* nomine habe forma de thema, id es : *residuo, formula, gradu, integro* (ablativo), *nos* (nominativo), *que* (ex accusativo *quem*), *ce* (ex *hoc, cetero*) ; verbo : *considera, es, occurre* (imperativo).

« Academia pro Interlingua » habe origine in 1887, et adopta Volapük publicato in 1880 ; stude Esperanto publicato in 1887 ; in 1902 publica « Idiom neutral » constructo super principio de internationalitate maximo ; et continua labores pro lingua internationale.

Articulo presente, tracto ex « Atti R. Acc. di Torino, 21, 2, 1915 », es scripto in « latino sine flexione », uno ex numeroso forma de interlingua, intelligibile sine studio ab magno publico.

MÉLANGES ET CORRESPONDANCE 125

ad 3 ; et pro alio functiones, es de usu frequente, ut formula de approximatione.

Ce formula occurre sub forma geometrico, in

B. Cavalieri, *Centuria di varii problemi*, Bologna, a. 1639, p. 446 [1] ;

et post, in

J. Gregory, *Exercitationes geometricae*, Londini anno 1668 ;

R. Cotes, *Harmonia mensurarum*, Catabrigiæ, anno 1722, p. 33 ;

et in fine in

Th. Simpson, *Mathematical dissertations*, London, 1743, p. 109.

Es usu de appella « formula di Simpson » formula præcedente. Me adde nomine de primo auctore.

Praxi proba quod formula de Cavalieri-Simpson, es in generale satis approximato. Ergo plure auctore desidera de cognosce uno limite de errore in isto approximatione.

In libro : *Applicazioni geometriche del Calcolo infinitesimale*, Torino, Bocca, 1887, pag. 208, me publica expressione de residuo, id es de differentia inter primo et secundo membro sub forma :

$$= -\frac{(b-a)^5}{4!\,5!} D^4 f(x),$$

ubi x es uno valore medio inter a e b.

Idem risultatu es publicato ab Markov, in libro edito in S. Petersburg, 1889 ; et versione cum titulo :

Markoff, *Differenzenrechnung*, Leipzig, 1896.

Ut casu particulare de regula exposito in meo scripto : *Resto nelle formule di quadratura, espresso con un integrale definito*, « Rendiconti Acc. Lincei », 4 maggio 1913, ibi me da expressione de residuo in formula de Cavalieri-Simpson, sub forma de integrale.

In præsente scripto, me perveni ad idem resultatu, per via plus directo et plus elementare.

[1] Phrasi de Cavalieri es « Per havere la capacità della botte moltiplicaremo la terza parte della lunghezza della botte in due cerchi maggiori ed uno dei minori ».

Versione « Pro habe capacitate de vase vinario, nos multiplica tercio parte de longitudine de vase per duo circulo maiore et uno minore ».

Si nos sume axi de vase pro axi de x, et pro origine puncto medio, si longitudine de vase es h, et si $f(x)$ es area de sectione de vase, in puncto de abscissa x, et normale ad axi, tunc volumine $= \int_{-h/2}^{h/2} f(x)\,dx$; circulo maiore $= f(0)$, circulo minore $= f(h/2 = f(-h/2)$; et regula de Cavalieri dic que integrale vale :

$$\frac{h}{3}[2f(0) + f(h/2)] = \frac{h}{6}[4f(0) + f(-h/2) + f(h/2)].$$

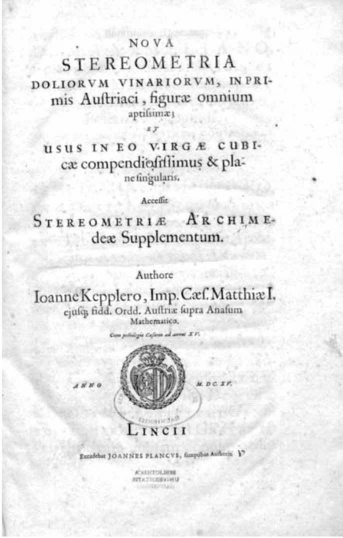
Figure: Peano paper and historical notes on the ascription of this rule.

It is surprising why Cavalieri for what purpose Cavalieri used this rule.

At that time there were many efforts on solids of rotation. For instance, Kepler in Nova stereometria doliorum vinariorum that is *New measure of the volume of barrels of wine* justified an empirical method used by austrian coopers, by means of a geometrical approach.

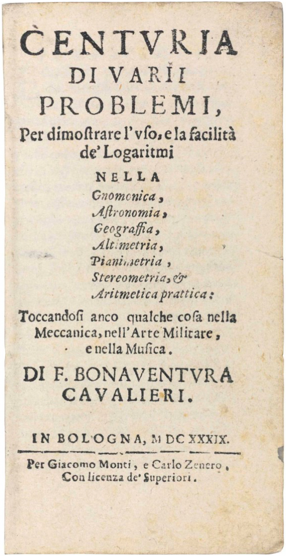
Also Cavalieri had some interest on the topic. In Note - Postille Matematiche, Gabrio Piola, discusses of the work *Centuria di varii problemi*, by Cavalieri. In particular at page 83, he writes

> Problem 80 concerns the measure of elliptical-circular barrels, giving a rule that is exactly the same that nowadays stems from Rossi-Amulis formula, demonstrated in 1806.



NOVA
STEREOMETRIA
DOLIORVM VINARIORVM, IN PRI-
mis Auſtriaci, figuræ omnium
aptiſsimæ;
ET
USUS IN EO VIRGÆ CUBI-
cæ compendioſiſſimus & pla-
ne ſingularis.
Acceſſit
STEREOMETRIÆ ARCHIME-
deæ Supplementum.
Authore
Ioanne Kepplero, Imp. Cæſ. Matthiæ I.
ejuſq; fidd. Ordd. Auſtriæ ſupra Anaſum
Mathematico.
Cum privilegio Cæſareo ad annos XV.
ANNO M. DC. XV.
LINCII
Excudebat JOANNES PLANCVS, ſumptibus Authoris.

Figure: Cover of Kepler work *Nova stereometria doliorum vinariorum* on the volume of barrels.



CENTVRIA
DI VARII
PROBLEMI,
Per dimoſtrare l'vſo, e la facilità
de' Logaritmi
NELLA
Gnomonica,
Aſtronomia,
Geograffia,
Altimetria,
Pianimetria,
Stereometria, &
Aritmetica prattica:
Toccandoſi anco qualche coſa nella
Meccanica, nell'Arte Militare,
e nella Muſica.
DI F. BONAVENTVRA
CAVALIERI.

IN BOLOGNA, M DC XXXIX.

Per Giacomo Monti, e Carlo Zenero,
Con licenza de' Superiori.

Figure: The cover of the work by Cavalieri, suggested by Peano.

[1] Phrasi de Cavalieri es « Per havere la capacità della botte moltiplicaremo la terza parte della lunghezza della botte in due cerchi maggiori ed uno dei minori ».

Versione « Pro habe capacitate de vase vinario, nos multiplica tercio parte de longitudine de vase per duo circulo maiore et uno minore ».

Si nos sume axi de vase pro axi de x, et pro origine puncto medio, si longitudine de vase es h, et si $f(x)$ es area de sectione de vase, in puncto de abscissa x, et normale ad axi, tunc

volumine $= \int_{-h/2}^{h/2} f(x)\,dx$; circulo maiore $= f(0)$, circulo minore $= f(h/2 = f(-h/2)$; et regula de Cavalieri dic que integrale vale :

$$\frac{h}{3}[2f(0) + f(h/2)] = \frac{h}{6}[4f(0) + f(-h/2) + f(h/2)] .$$

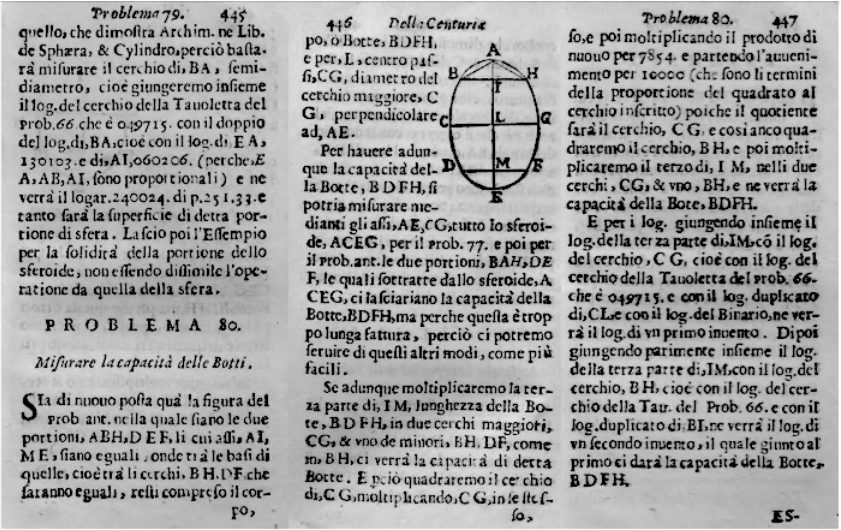
Figure: Remark by Peano on the volume of barrels.

1948. In Vino Veritas:—"Si adunque moltiplicaremo la terza parte di *IM* lunghezza della Botte *BDFH* in due cerchi maggiore *CG* e uno de minore *BH*, *DF* come in *BH*, ci verrà la capacità di detta Botte." F B. Cavalieri, *Una centuria di varii Problemi nella Prattica Astrologica*, (Bologna, 1639), Problema 80. [Per Dr. G. N Watson. It was probably this formula for the capacity of a cask of wine which suggested to James Gregory (who spent the years 1665–8 in Padua) the more general formula for approximate integration, often written in the form

$$\int_0^{2h} y\,dx = h(y_0 + 4y_1 + y_2)/3,$$

which, in its turn, led to the outstanding generalisations due to Newton and Cotes, and subsequently to the minor generalisation known as 'Simpson's rule'; about this rule (published in 1743) there is not much to be said except that it is more accurate than the 'trapezoidal rule'] [Per Dr. G N. Watson]

Figure: Remark by G.N. Watson about Cavalieri and the volume of barrels.



Problema 79. 445

quello, che dimostra Archim. ne Lib. de Sphæra, & Cylindro, perciò basta- rà misurare il cerchio di, BA, semi- diametro, cioè giungeremo insieme il log. del cerchio della Tauoletta del prob. *66* che è 049715. con il doppio del log. di, BA, cioè con il log. di EA, 130103. e di, AI, 060206. (perche, *E* A, AB, AI, sono proportionali) e ne verrà il logar. 240024. di p. 251.33. e tanto sarà la superficie di detta por- tione di sfera. Lascio poi l'Essempio per la solidità della portione dello sferoide, non essendo dissimile l'ope- ratione da quella della sfera.

P R O B L E M A 80.

Misurare la capacità delle Botti.

SIa di nuouo posta quà la figura del prob. ant. nella quale siano le due portioni, ABH, DEF, li cui assi, AI, ME, siano eguali, onde trà le basi di quelle, cioè trà li cerchi, BH, DF che saranno eguali, resti compreso il cor- po,

446 Della Centuria

po, ò Botte, BDFH, e per, L, centro pas- si, CG, diametro del cerchio maggiore, C G, perpendicolare ad, AE.

Per hauere adun- que la capacità del- la Botte, B D F H, si potria misurare me- dianti gli assi, AE, CG, tutto lo sferoi- de, ACEG, per il prob. 77. e poi per il prob. ant. le due portioni, BA*H*, *DE* F, le quali sottratte dallo sferoide, A CEG, ci lasciariano la capacità della Botte, BDFH, ma perche questa è trop po lunga fattura, perciò ci potremo seruire di questi altri modi, come più facili.

Se adunque moltiplicaremo la ter- za parte di, I M, lunghezza della Bo- te, B D F H, in due cerchi maggiori, CG, & vno de minori, BH. DF, come in, B H, ci verrà la capacità di detta Botte. E però quadraremo il cerchio di, C G, moltiplicando, C G, in se stes- so,

Problema 80. 447

so, e poi moltiplicando il prodotto di nuouo per 7854. e partendo l'auueni- mento per 10000 (che sono li termini della proportione del quadrato al cerchio inscritto) poiche il quotiente sarà il cerchio, C G, e così anco qua- draremo il cerchio, B H, e poi molti- plicaremo il terzo di, I M, nelli due cerchi, CG, & vno, BH, e ne verrà la capacità della Bote, BDFH.

E per i log. giungendo insieme il log. della terza parte di, IM, cō il log. del cerchio, C G, cioè con il log. del cerchio della Tauoletta del prob. *66*. che è 049715. e con il log. duplicato di, CL, e con il log. del Binario, ne ver- rà il log. di vn primo inuento. Dipoi giungendo parimente insieme il log. della terza parte di, IM, con il log. del cerchio, B H, cioè con il log. del cer- chio della Tau. del Prob. *66*. e con il log. duplicato di, BI, ne verrà il log. di vn secondo inuento, il quale giunto al primo ci darà la capacità della Botte, B D F H.

ES-

Figure: Problem 80, at p.445 of the *Centuria di varii problemi* by Cavalieri.

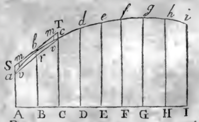
The same rule can be found in Mathematical dissertations On A Variety of Physical and Analytical subjects by Thomas Simpson. It is clear that T. Simpson is thinking to composite rules.

Of the Areas *of* Curves, &c. *by Approximation.*

PROPOSITION I.

Suppoſing a b c *to be a ſmall Portion of any Curve* a b f i, *and* A a, B b, C c, *three equidiſtant Ordinates; to find an Expreſſion in Terms of thoſe Ordinates, and the common Diſtance* A B, *that ſhall nearly exhibit the included Area* A C c b a A.

LET a common Parabola, having its Axis parallel to the given Ordinates, be deſcribed thro' the three Points *a*, *b*, *c*, of the propoſed Curve, or rather, to avoid confuſing the Figure, let that Curve itſelf repreſent a Portion of ſuch a Parabola; join A and C with a Right-Line, and make S *b* T parallel thereto, producing A *a*, and C *c* to meet S *b* T in S and T, and drawing *v m* from any Point *v*, in the Parabola, parallel to A S. Then *v m*, by the Property of the Parabola being to S *a*, as *b m*², to *b* S², or, in the duplicate Ratio of *b m*, the Space *b a* S *b*, included by the Parabola and the Right-Lines S *a*,



E e and

[110]

and S *b*, will be $\frac{1}{3}$ of the Parallelogram *bra* S *b*, for the ſame Reaſon that a Pyramid, whoſe Sections made by a Plane parallel to the Baſe, are in a duplicate Ratio of their Diſtances from the Vertex, is known to be $\frac{1}{3}$ of its circumſcribing Priſm. Wherefore, ſeeing B*b* × 2 A B is equal to the Area of A C T *b* S A, and $\overline{Aa+Cc} \times AB$ to that of A C *c r a* A, the former of theſe Quantities muſt exceed the Parabolic Area A C *c b a* A by juſt half what the latter wants of it; and therefore twice the former added to once the latter, will be juſt three times this Area, and conſequently the Area itſelf equal to $\frac{Aa+4Bb+Cc}{3} \times AB$; which Quantity, ſince a Parabola admits of infinite Variation of Curvature, ſo as to nearly coincide with any Curve for a ſmall Diſtance, muſt be equal alſo to the Area ſought very nearly.

Q. E. I.

Figure: Excerpt from *Mathematical dissertations On A Variety of Physical and Analytical subjects* by T. Simpson.

Considerazione.

Assume that $f \in C([a, b])$, where $[a, b]$ is bounded.

By Weierstrass theorem, since that function can be approximated arbitrary well and uniformly by polynomials, one may believe that that rules with higher degree may approximate arbitrarialy well the definite integral, but we shall see this is not true.

After the previous note on Runge function, one forsees that if the nodes are $\{x_k\}_{k=0,\ldots,n}$ some problems may happen.

We will see later the results.

Definition (Newton-Cotes rules)

Let $[a, b]$ be a bounded interval. A rule

$$S_n(f) = \sum_{i=0}^{n} w_i f(x_i) \approx \int_a^b f(x)dx$$

is of closed Newton-Cotes type if

- that the set of nodes is equispaced and contains the extrema, that is

$$x_i = a + \frac{i\,(b-a)}{n}, \;\; i = 0, \dots, n,$$

- the weights are

$$w_i = \int_a^b L_i(x)dx, \;\; i = 0, \dots, n, \;\; L_i(x) = \prod_{j=0,\, j\neq i}^{n} \frac{(x - x_i)}{x_j - x_i}.$$

Remark (Degree of exactness of closed Newton-Cotes formulae)

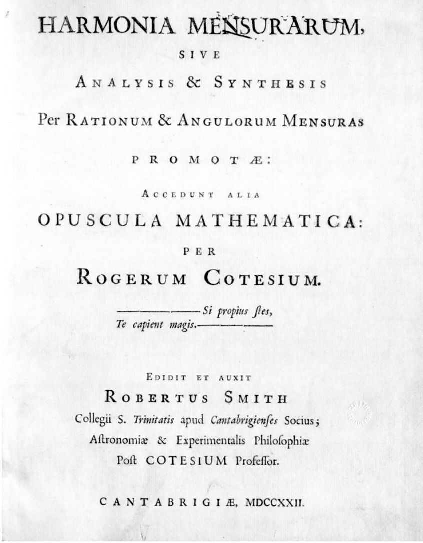
This rule is interpolatory and has $n+1$ nodes, thus it has degree of exactness at least n.

Remark (Facoltativa)

- *These rules were introduced by Newton in 1676 and described in Of Quadrature by Ordinates in 1695. The literature is not clear, it written they had 5 ordinates (so degree of exactness at least 4).*
- *Cotes, that was editor of the second version of the Principia, generalised the results by Newton, possibly in 1707, though they were published in 1722. In particular Cotes computed rules that had up to* 11 *nodes.*

  Considering Harmonia mensurarum, it is not easy to found where these rules are. An appendix is sometimes mentioned, but it is not easy to understand where it can be found.

  Historians say that Cotes was not cited and paid for his work in the second version of Newton's Principia. In spite of that, when Cotes died at 34 yo, Newton said If he had lived, we might have known something.



HARMONIA MENSURARUM,

SIVE

ANALYSIS & SYNTHESIS

Per RATIONUM & ANGULORUM MENSURAS

PROMOTÆ:

ACCEDUNT ALIA

OPUSCULA MATHEMATICA:

PER

ROGERUM COTESIUM.

——— *Si propius stes,*
Te capient magis. ———

EDIDIT ET AUXIT

ROBERTUS SMITH

Collegii S. *Trinitatis* apud *Cantabrigienses* Socius;
Astronomiæ & Experimentalis Philosophiæ
Post COTESIUM Professor.

CANTABRIGIÆ, MDCCXXII.

Figure: First page of *Harmonia mensurarum*

We show numerically the lack of convergence of the Newton-Cotes closed formula when applied to compute $\int_{-5}^{5} 1/(1+x^2)dx$ (i.e. we adopt as integrand the RUnge function). Since the interpolant is far from approximating the Runge function we believe that also the integrals will be very different.

n	*Integral Intp. Pol.*	*Absolute Error*
1	$3.846153846153846e-01$	$2.362e+00$
2	$6.794871794871796e+00$	$4.048e+00$
3	$2.081447963800905e+00$	$6.654e-01$
4	$2.374005305039788e+00$	$3.728e-01$
5	$2.307692307692308e+00$	$4.391e-01$
6	$3.870448673470800e+00$	$1.124e+00$
7	$2.898994409748379e+00$	$1.522e-01$
8	$1.500488907127907e+00$	$1.246e+00$
9	$2.398617897841837e+00$	$3.482e-01$
10	$4.673300555653490e+00$	$1.926e+00$
15	$4.155558992699889e+00$	$1.409e+00$
20	$-2.684955208653064e+01$	$2.960e+01$

Table: Newton-Cotes formulas failing to approximate $\int_{-5}^{5} 1/(1+x^2)dx \approx 2.746801533890032$.

composite rules are obtained integrating a piecewise polynomial interpolant of degree m.

As example,

- we subdivide $[a, b]$ in L equispaced intervals $[t_0, t_1], \ldots, [t_{L-1}, t_L]$;
- define in each $[t_{k-1}, t_k]$ a set of $m+1$ equispaced points $x_{m(k-1)} < \ldots < x_{mk}$ con $t_{k-1} = x_{m(k-1)}$ and $t_k = x_{mk}$;
- let "s_m"the piecewise polynomial interpolant of degree m, relatively to the subdivision $[t_k, t_{(k+1)}]$, with $k = 0, \ldots, L$, and couples $(x_j, f(x_j))$, $j = 0, \ldots, n = Lm$;
- suppose that $S_m(f, t_{k-1}, t_k)$ is a closed formula of Newton-Cotes close type, in the interval $[t_{k-1}, t_k]$, with $m+1$ nodes (so having at least degree of exactness m).

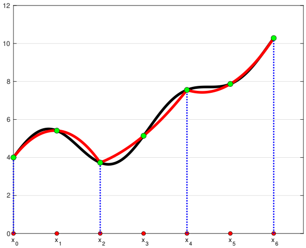


Figure: In black: the function $f(x) = 3 + \sin(2x) + \cos(x) + x$. In red: the piecewise interpolant s_m of degree $m = 2$, relatively to the subdivision $[t_k = x_{2k}, t_{k+1} = x_{2(k+1)}]$, with $k = 0, 1, 2$, and to the couples $(x_j, f(x_j))$, $j = 0, \ldots, n = 6$. In green: the couples $(x_j, f(x_j))$, $j = 0, \ldots, 6$.

- Being s_m a polynomial of degree m in each $[t_k, t_{k+1}]$,
- since $S_m(f, t_k, t_{(k+1))}$ has degree of exactness m,

by additivity of integration operator

$$\begin{aligned}\int_a^b f(x)dx &\approx \int_a^b s_m(x)dx \\ &= \int_{t_0=a}^{t_1} s_m(x)dx + \int_{t_1}^{t_2} s_m(x)dx + \ldots + \int_{t_{L-1}}^{x_L=b} s_m(x)dx \\ &= S_m(f, t_0, t_1) + S_m(f, t_1, t_2) + \ldots + S_m(f, t_{L-1}, t_L)\end{aligned}$$

Thus we obtained the approximation of the integral not by a Newton-Cotes rule with high degree of exactness "n", but applying the same rule with a low ADE "m"in each subinterval, summing up all the contributions.

Definition (Composite rule)

Let

1. $[a, b]$ a bounded interval,
2. $t_j = a + jh$ with $h = (b - a)/N$, $j = 0, \ldots, N$,
3. $S(f, \alpha, \beta))$ a rule in the bounded interval $[\alpha, \beta]$.

The quadrature formula

$$S^{(c)}(f, a, b, N) = \sum_{j=0}^{N-1} S(f, t_j, t_{j+1}) \tag{4}$$

è is a composite rule using S.

Remark

In this discussion, the points x_j, $j = 0, \ldots, N$, will be equispaced, but with some effort we can extend the analysis to a different distribution. In other terms:

- *if $t_0 = a < t_1 < \ldots < t_N = b$, one partitions $[a, b]$ as union of subintervals $[t_k, t_{k+1}]$, $k = 0, \ldots, N - 1$;*
- *in each subinterval we apply the same rule, summing up the contributions.*

Definition (Midpoint composite rule)

The midpoint composite rule is defined as

$$S_0^{(c)}(f, a, b, N) := \frac{b-a}{N} \sum_{k=0}^{N-1} f(x_k), \qquad (5)$$

where x_k is the midpoint of the $k+1$-th interval, that is

$$x_k = a + \frac{2k+1}{2} \cdot \frac{b-a}{N}, k = 0, \dots, N-1.$$

Subdividing $[a, b]$ in N equispaced interval $[t_k, t_{k+1}]$, $K = 0, \dots, N-1$ with $t_j = a + jh$, $j = 0, \dots, N$, $h = (b-a)/N$, if x_k is the midpoint of $[t_k, t_{k+1}]$ then

$$x_k = \frac{t_k + t_{k+1}}{2} = \frac{a + kh + a + (k+1)h}{2} = a + \frac{(2k+1)h}{2}.$$

and since $w_k = t_{k+1} - t_k = a + (k+1)h - (a + kh) = h = (b-a)/N$ we get

$$S_0^{(c)}(f, a, b, N) := \sum_{k=0}^{N-1} \frac{b-a}{N} f(x_k) = \frac{b-a}{N} \sum_{k=0}^{N-1} f(x_k).$$

Midpoint composite rule

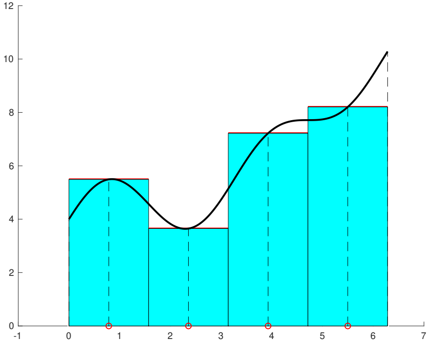


Figure: Midpoint composite rule and approximation of $\int_0^{2\pi} 3 + \sin(2x) + \cos(x) + x\,dx$ (the formula computes the area in cyan).

Theorem (Error of midpoint composite rule)

If $[a, b]$ is subdivided in N equispaced intervals of length $h = \frac{b-a}{N}$, then

$$E_0^{(c)}(f) := I(f) - S_0^{(c)}(f, a, b, N) = \frac{(b-a)}{24} h^2 f^{(2)}(\xi^*), \ \ \xi^* \in (a, b)$$

Remark (Degree of exactness)

From the formula above, it is immediate to get that ADE is exactly equal to 2.

Remark (Comparison with the midpoint rule)

The midpoint rule had an error

$$E_0(f) := I(f) - S_0(f) = \frac{(b-a)^3}{24} f^{(2)}(\xi), \quad \xi \in (a,b),$$

while for $N > 1$

$$\frac{(b-a)}{24} h^2 = \frac{(b-a)}{24} \left(\frac{b-a}{N}\right)^2 = \frac{(b-a)^3}{24N^2} < \frac{(b-a)^3}{24}.$$

Consequently, one may think that if f^2 *does not vary much, the composite rule tend to provide smaller errors (notice that in general* $\xi^* \neq \xi$*).*

Trapezoidal composite rule

Purpose. (How it is obtained)

To give some insight, suppose that $[a, b]$ is subdivided $N = 4$ equispaced subintervals $[t_k, t_{k+1}]$, $K = 0, \ldots, N-1 = 3$ con $t_j = a + jh$, $j = 0, \ldots, N = 4$, $h = (b-a)/N = (b-a)/4$.

Let $S_1(f, \alpha, \beta)$ the application of the trapezoidal rule relatively to f and to the interval $[\alpha, \beta]$,

- $S_1(f, t_0, t_1) = \frac{h}{2}(f(t_0) + f(t_1))$,
- $S_1(f, t_1, t_2) = \frac{h}{2}(f(t_1) + f(t_2))$,
- $S_1(f, t_2, t_3) = \frac{h}{2}(f(t_2) + f(t_3))$,
- $S_1(f, t_3, t_4) = \frac{h}{2}(f(t_3) + f(t_4))$,

thus being $N = 4$

$$
\begin{aligned}
S_1^{(c)}(f, a, b, 4) &= \frac{h}{2}(f(t_0) + f(t_1) + f(t_1) + f(t_2) + f(t_2) + f(t_3) + f(t_3) + f(t_4)) \\
&= \frac{h}{2}(f(t_0) + 2f(t_1) + 2f(t_2) + 2f(t_3) + f(t_4)) \\
&= \frac{b-a}{N}\left(\frac{1}{2}f(t_0) + f(t_1) + f(t_2) + f(t_3) + \frac{1}{2}f(t_4)\right)
\end{aligned}
$$

Definition (Trapezoidal composite rule)

Let $x_k = a + kh$, $k = 0, \ldots, N$, $h = (b-a)/N$, the trapezoidal composite rule is defined as

$$S_1^{(c)}(f, a, b, N) := \frac{b-a}{N}\left[\frac{f(x_0)}{2} + f(x_1) + \ldots + f(x_{N-1}) + \frac{f(x_N)}{2}\right],$$

In the previous assumptions,

$$E_1^{(c)}(f) := I(f) - S_1^{(c)}(f, a, b, N) = \frac{-(b-a)}{12} h^2 f^{(2)}(\xi), \;\; h = \frac{(b-a)}{N} \;\; \xi \in (a, b).$$

Remark (Grado di exactness)

Similarly to the basic rule, the degree of precision is exactly 1 *since*

$$|E_1^{(c)}(f)| := |I(f) - S_1^{(c)}(f, a, b, N)| = \frac{(b-a)}{12} h^2 \, |f^{(2)}(\xi)|, \;\; h = \frac{(b-a)}{N},$$

Remark (Comparison with the rule)

With regards to the rule, we had

$$|E_1(f)| := |I(f) - S_1(f)| = \frac{(b-a)^3}{12} \, |f^{(2)}(\xi)|, \;\; \xi \in (a, b).$$

but here, being $N > 1$

$$\frac{(b-a)}{12} h^2 = \frac{(b-a)}{12} \left(\frac{(b-a)}{N} \right)^2 = \frac{(b-a)^3}{12N^2} < \frac{(b-a)^3}{12},$$

and as in the case of the midpoint composite rule, we expect an inferior error increasing N *if* f^2 *does not vary much.*

Trapezoidal composite rule

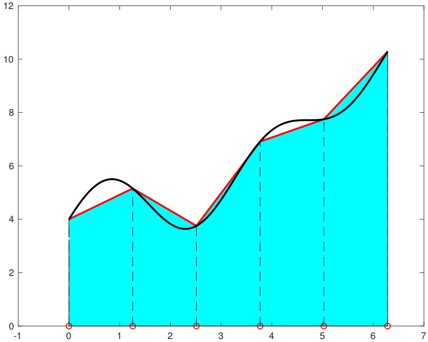


Figure: Trapezoidal composite rule for approximating $\int_0^{2\pi} 3 + \sin(2x) + \cos(x) + x\, dx$ (it computes the area in cyan, that is the area defines by the piecewise interpolant in red).

The trapezoidal composite rule has many interesting features., that are carefully described in [2]. In particular we can consider a note on a work by Poisson.

It appears to have been Poisson, in the 1820s, who first identified this effect [139]. The example Poisson chose has remained a favorite ever since: the perimeter of an ellipse, which he took to have axis lengths $1/\pi$ and $0.6/\pi$, giving the integral

$$I = \frac{1}{2\pi}\int_0^{2\pi} \sqrt{1 - 0.36\sin^2\theta}\, d\theta. \tag{1.1}$$

Poisson used this now-standard notation for definite integrals, but apparently it was not yet standard in the 1820s, for he comments that

> pour indiquer [les limites de l'intégrale] en même temps que l'intégrale, nous emploierons la notation très-commode que M. Fourrier a proposée.[1]

Perhaps in 1826 the spelling of Fourier's name wasn't yet standardized either!

The exact solution of (1.1) is

$$I = \frac{2}{\pi}E(0.36) = 0.90277992777219\ldots, \tag{1.2}$$

where E is the complete elliptic integral of the second kind. As trapezoidal rule approximations we can take

$$I_N = \frac{1}{N}\sum_{k=1}^{N} \sqrt{1 - 0.36\sin^2(2\pi k/N)}$$

for any positive integer N, or, equivalently, if N is divisible by 4, exploiting the four-fold symmetry as Poisson did,

$$I_N = \frac{4}{N}{\sum_{k=0}^{N/4}}' \sqrt{1 - 0.36\sin^2(2\pi k/N)}.$$

Consider the integrand

$$\frac{1}{2\pi}\sqrt{1-0.36\sin^2\theta}$$

and verify it is periodic with its derivatives in $[0, 2\pi]$. Both the geometry (meaning of the integral) and Maple confirm this.

```
>> syms f(x)
>> f(x)=(1/(2*pi))*sqrt(1-0.36*(sin(x))^2);
>> g=diff(f,x,1); g(2*pi)-g(0)
ans = 0
>> g=diff(f,x,2); g(2*pi)-g(0)
ans = 0
>> g=diff(f,x,3); g(2*pi)-g(0)
ans = 0
>> g=diff(f,x,4); g(2*pi)-g(0)
ans = 0
>> g=diff(f,x,25); g(2*pi)-g(0)
ans = 0
>> g=diff(f,x,30); g(2*pi)-g(0)
ans = 0
```

Euler-Maclaurin formula

Let h_N be the length of the generic subinterval of the equispaced subdivision of $[a, b]$.

Theorem (Euler-Mac Laurin formula, 1735)

If $f \in C^{2M+2}([a, b])$ then

$$\int_a^b f(x)dx = S_1^{(c)}(f, N) + \sum_{k=1}^{M} \frac{B_{2k}}{(2k)!} h_N^{2k} \left(f^{(2k-1)}(b) - f^{(2k-1)}(a)\right) - \frac{B_{2M+2}}{(2M+2)!} h_N^{(2M+2)}(b-a) f^{(2M+2)}(\xi), \ \ \xi \in (a, b)$$

where B_k are the Bernoulli numbers (Bernoulli, 1713).

If $f \in C^{2M+2}([a, b])$ e $f^{(2k-1)}(b) = f^{(2k-1)}(a)$, for $k = 1, \dots, M$

$$\int_a^b f(x)dx - S_1^{(c)}(f, N) = -\frac{B_{2M+2}}{(2M+2)!} h_N^{(2M+2)}(b-a) f^{(2M+2)}(\xi),$$

where $\xi \in (a, b)$ and from $h_N = (b-a)/N$, we get $\mathcal{E}_1^{(c)}(f) \approx \frac{C}{N^{2M+2}}$.

Euler-Maclaurin formula (Bernoulli numbers)

Bernoulli numbers might have been the first quantities computes numerically by a code on a machine, the so called Analytical Engine (1842), that is considered as the first modern calculator.

There is a controversy if the programmer was Babbage, the designer of the machine or Ada Lovelace, daughter of Lord Byron.

Figure: A component of the Analytical Engine.

From Euler-Mac Laurin formula, in the case of $f(\theta) := \frac{1}{2\pi}\sqrt{1-0.36\sin^2\theta}$ we have that

$$|E_1^{(c)}(f)| := |\int_a^b f(x)dx - S_1^{(c)}(f,N)|$$

is more rapid than N^{-M} for any M (here $N$is the number of subintervals).

Actually it is of *geometric* type, that is $|E_1^{(c)}(f)| \approx \alpha\gamma^{-N}$, for suitable α, γ (cf. [2], p.387). he

N	$\lvert E_1^{(c)}(f)\rvert$	N	$\lvert E_1^{(c)}(f)\rvert$
2	$9.7e-02$	32	0
4	$2.8e-03$	64	0
8	$1.1e-05$	128	0
16	$5.4e-10$	256	0
32	$1.1e-16$	512	0

Table: Error of Trapezoidal composite rule on Poisson example, subdividing $[a, b]$ in N equispaced intervals

Numerical integration of periodic functions is an important subject due to its applications.

It is a fundamental ingredient of FFT algorithm that requires the evaluations of these quantities for specific integrands.

The Fast Fourier Transform is used in

- elaboration of digital signals (fundamental for the mp3 compression),
- solution of PDEs;
- algorithms for multiplication of integers of large magnitude.

This algorithm was discovered by Cooley-Tukey in 1965 (but some sources say it was known is some form to Gauss!).

Remark

In IEEE Guest Editors' Introduction: The Top 10 Algorithms. it is written:

The FFT is perhaps the most ubiquitous algorithm in use today to analyze and manipulate digital or discrete data.

Cavalieri-Simpson composite rule

Purpose. (Composite Cavalieri-Simpson rule)

As illustration, let us subdivide $[a, b]$ *in* $N = 4$ *equispaced intervals* $[t_k, t_{k+1}]$, $k = 0, \dots, N-1 = 3$ *with* $t_j = a + jh$, $j = 0, \dots, N = 4$, $h = (b-a)/N$.

Defining with $S_2(f, t_k, t_{k+1})$ *the application of Cavalieri-Simpson rule relatively to* f *and to the interval* $[t_k, t_{k+1}]$, *letting* $c_k = \frac{t_k + t_{k+1}}{2} = a + \frac{2k+1}{2} \cdot \frac{b-a}{N}$ *be the midpoint of* $[t_k, t_{k+1}]$,

- $S_2(f, t_0, t_1) = \frac{h}{6}(f(t_0) + 4 \cdot f(c_0) + f(t_1))$,
- $S_2(f, t_1, t_2) = \frac{h}{6}(f(t_1) + 4 \cdot f(c_1) + f(t_2))$,
- $S_2(f, t_2, t_3) = \frac{h}{6}(f(t_2) + 4 \cdot f(c_2) + f(t_3))$,
- $S_2(f, t_3, t_4) = \frac{h}{6}(f(t_3) + 4 \cdot f(c_3) + f(t_4))$,

we obtain from $N = 4$

$$\begin{aligned} S_2^{(c)}(f, a, b, 4) &= \frac{h}{6}(f(t_0) + 4 \cdot f(c_0) + f(t_1)) + \frac{h}{6}(f(t_1) + 4 \cdot f(c_1) + f(t_2)) \\ &+ \frac{h}{6}(f(t_2) + 4 \cdot f(c_2) + f(t_3)) + \frac{h}{6}(f(t_3) + 4 \cdot f(c_3) + f(t_4)) = \dots \\ &= \frac{h}{6}f(t_0) + \frac{2h}{6}(f(t_1) + f(t_2) + f(t_3)) + \frac{h}{6}f(t_4) \\ &+ \frac{4h}{6}(f(c_0) + f(c_1) + f(c_2) + f(c_3)). \end{aligned} \tag{6}$$

Cavalieri-Simpson composite rule

From

$$S_2^{(c)}(f,a,b,4) = \frac{h}{6}f(t_0) + \frac{2h}{6}(f(t_1)+f(t_2)+f(t_3)) + \frac{h}{6}f(t_4) + \frac{4h}{6}(f(c_0)+f(c_1)+f(c_2)+f(c_3)). \quad (7)$$

setting $t_0 = x_0$, $c_0 = x_1$, $t_1 = x_2$, $c_1 = x_3$, $t_2 = x_4$, $c_2 = x_5$, $t_3 = x_6$, $c_3 = x_7$, $t_4 = x_8$, *we get for* $N = 4$ *(i.e. the number of subdivisions)*

$$S_2^{(c)}(f,a,b,4) = \frac{h}{6}\left(f(x_0) + 2\sum_{r=1}^{N-1} f(x_{2r}) + 4\sum_{s=0}^{N-1} f(x_{2s+1}) + f(x_{2N})\right) \quad (8)$$

This ideas can be easily generalized to any N.

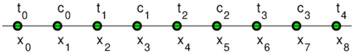


Figure: Relation between t_k, c_k e x_j.

Definition (Cavalieri-Simpson composite rule)

Set $x_k = a + kh/2$, $k = 0, \ldots, 2N$, $h = (b-a)/N$, Cavalieri-Simpson composite rule is defined as

$$S_2^{(c)}(f, a, b, N) = \frac{h}{6}\left[f(x_0) + 2\sum_{r=1}^{N-1} f(x_{2r}) + 4\sum_{s=0}^{N-1} f(x_{2s+1}) + f(x_{2N})\right] \quad (9)$$

Theorem (Error of Cavalieri-Simpson composite rule)

In the assumptions of subvisions via equispaced intervals of length h, the integration error is

$$E_2^{(c)}(f) := I(f) - S_2^{(c)}(f, a, b, N) = \frac{-(b-a)}{180}\left(\frac{h}{2}\right)^4 f^{(4)}(\xi), \quad \xi \in (a, b)$$

Remark (Degree of exactness of Cavalieri-Simpson composite rule)

The degree of exactness is exactly 3, as Cavalieri-Simpson rule, but if $N > 1$ the length h is inferior and consequently an inferior absolute integration error is expected.

Cavalieri-Simpson composite rule

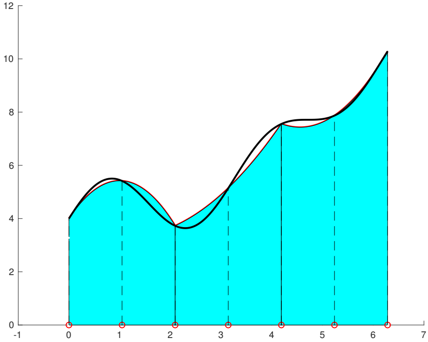


Figure: Cavalieri-Simpson composite rule for the computation of $\int_0^{2\pi} 3 + \sin(2x) + \cos(x) + x\, dx$ (the rule determines the area in cyan).

In this section we analyse some examples in which we apply composite rules to integrate some test integrands in $f \in C([a,b])$.

Example (1)

Approximate the definite integral

$$I = \int_0^{\pi} \exp(x)\cos(x)dx = -(\exp(\pi)+1)/2.$$

by means of composite rules $S_k^{(c)}(f,0,\pi,N)$, $N = 1,2,4,\ldots,512$, $k = 0,1,2$.

Remark

Note that the integrand belongs to $C^{\infty}([0,2\pi])$ (actually it is entire).

N	$\lvert E_0^{(c)}(f)\rvert$	$\lvert E_1^{(c)}(f)\rvert$	$\lvert E_2^{(c)}(f)\rvert$	$\#_N^{\mathrm{R}}$	$\#_N^{\mathrm{T}}$	$\#_N^{\mathrm{CS}}$
1	$1.2e+01$	$2.3e+01$	$4.8e-01$	1	2	3
2	$2.8e+00$	$5.3e+00$	$8.5e-02$	2	3	5
4	$6.4e-01$	$1.3e+00$	$6.1e-03$	4	5	9
8	$1.6e-01$	$3.1e-01$	$3.9e-04$	8	9	17
16	$3.9e-02$	$7.8e-02$	$2.5e-05$	16	17	33
32	$9.7e-03$	$1.9e-02$	$1.6e-06$	32	33	65
64	$2.4e-03$	$4.8e-03$	$9.7e-08$	64	65	129
128	$6.1e-04$	$1.2e-03$	$6.1e-09$	128	129	257
256	$1.5e-04$	$3.0e-04$	$3.8e-10$	256	257	513
512	$3.8e-05$	$7.6e-05$	$2.4e-11$	512	513	1025

Table: Comparison of midpoint, trapezoidal and Cavalieri-Simpson composite rule, for N equispaced intervals relatively to the computation of $I = \int_0^{\pi} f(x)dx$ con $f(x) = \exp(x)\cos(x)dx$,describing the absolute errors $|E_0^{(c)}(f)| = |I(f,a,b) - S_0^{(c)}(f,a,b,N)$, $|E_1^{(c)}(f)| = |I(f,a,b) - S_1^{(c)}(f,a,b,N)|$, $|E_2^{(c)}(f)| = |I(f,a,b) - S_2^{(c)}(f,a,b,N)|$, for each formula and the respective number of nodes $\#_N^{\mathrm{R}}$, $\#_N^{\mathrm{T}}$, $\#_N^{\mathrm{CS}}$.

Remark (optional)

In the second table we show the ratio between two successive errors for each formula. The value $(E_k^{(c)}(f))_N$, $k = 0, 1, 2$, is the absolute integration error by S_k, for computing $\int_a^b f(x)dx$, over N subdivisions, while

$$(r_k^{(c)}(f))_N = \frac{(E_k^{(c)}(f))_N}{(E_k^{(c)}(f))_{2N}}$$

is the ratio for $k = 0, 1, 2$ (in order, midpoint, trapezoidal and Cavalieri-Simpson composite rule).

N	$(r_0^{(c)}(f))_N$	$(r_1^{(c)}(f))_N$	$(r_2^{(c)}(f))_N$
1	4.33	4.27	5.59
2	4.34	4.20	13.92
4	4.10	4.06	15.54
8	4.03	4.02	15.89
16	4.01	4.00	15.97
32	4.00	4.00	15.99
64	4.00	4.00	16.00
128	4.00	4.00	16.00
256	4.00	4.00	16.00

Table: Ratios relatively to $I = \int_0^\pi f(x)dx$ with $f(x) = \exp(x)\cos(x)dx$, showing the ratios between two successive absolute errors of the formulas.

Example (2)

Approximate the definite integral

$$I = \int_{-5}^{5} \frac{1}{1+x^2}\,dx \approx 2.7468015338900322319659608$$

by composite rules $S_k^{(c)}(f, -5, 5, N)$, $N = 1, 2, 4, \ldots, 1024$, $k = 0, 1, 2$.

Remark

The integrand belongs to $C^\infty([-5, 5])$. Thus we can apply the error theorems for all the composite rules.

N	$\lvert E_0^{(c)}(f)\rvert$	$\lvert E_1^{(c)}(f)\rvert$	$\lvert E_2^{(c)}(f)\rvert$	$\#_N^{\mathrm{R}}$	$\#_N^{\mathrm{T}}$	$\#_N^{\mathrm{CS}}$
1	$7.3e+00$	$2.4e+00$	$4.0e+00$	1	2	3
2	$1.4e+00$	$2.4e+00$	$9.6e-02$	2	3	5
4	$4.6e-01$	$5.4e-01$	$1.3e-01$	4	5	9
8	$3.9e-02$	$3.8e-02$	$1.3e-02$	8	9	17
16	$2.1e-04$	$6.9e-04$	$9.1e-05$	16	17	33
32	$1.2e-04$	$2.4e-04$	$4.5e-08$	32	33	65
64	$3.0e-05$	$6.0e-05$	$2.6e-09$	64	65	129
128	$7.5e-06$	$1.5e-05$	$1.6e-10$	128	129	257
256	$1.9e-06$	$3.8e-06$	$1.0e-11$	256	257	513
512	$4.7e-07$	$9.4e-07$	$6.4e-13$	512	513	1025
1024	$1.2e-07$	$2.4e-07$	$4.0e-14$	1024	1025	2049

Table: Comparison of midpoint, trapezoidal and Cavalieri-Simpson composite rule, for N equispaced intervals relatively to the computation of $I = \int_{-5}^{5} f(x)dx$ con $f(x) = 1/(1+x^2)$, describing the absolute errors $|E_0^{(c)}(f)| = |I(f,a,b) - S_0^{(c)}(f,a,b,N)|$, $|E_1^{(c)}(f)| = |I(f,a,b) - S_1^{(c)}(f,a,b,N)|$, $|E_2^{(c)}(f)| = |I(f,a,b) - S_2^{(c)}(f,a,b,N)|$, for each formula and the respective number of nodes $\#_N^{\mathrm{R}}$, $\#_N^{\mathrm{T}}$, $\#_N^{\mathrm{CS}}$.

Some numerical comparisons: example 2 (optional)

N	$(r_0^{(c)}(f))_N$	$(r_1^{(c)}(f))_N$	$(r_2^{(c)}(f))_N$
1	3.61	3.89	14.78
2	3.96	4.00	15.33
4	3.96	3.91	15.00
8	4.04	4.18	15.38
16	4.07	3.93	15.85
32	3.89	3.94	15.77
64	4.04	3.94	15.29
128	4.05	4.00	17.00
256	3.93	4.09	15.15
512	4.00	3.93	16.10

Table: Ratios for the composite rules concerning the computation of $I = \int_{-5}^{5} f(x)dx$ for $f(x) = 1/(1+x^2)dx$

Example (3)

Approximate the definite integral

$$I = \int_0^1 x^3 \sqrt{x} dx = 2/9.$$

by composite rules $S_k^{(c)}(f, 0, 1, N)$, $N = 1, 2, 4, \ldots, 1024$, $k = 0, 1, 2$.

Remark

The integrand belongs to $C^3([0,1])$. Thus we can take into account all the error formulas but that of composite Cavalieri-Simpson rule that requires $f \in C^4([0,1])$

N	$\lvert E_0^{(c)}(f)\rvert$	$\lvert E_1^{(c)}(f)\rvert$	$\lvert E_2^{(c)}(f)\rvert$	$\#_N^{R}$	$\#_N^{T}$	$\#_N^{CS}$
1	$1.3e-01$	$2.8e-01$	$3.4e-03$	1	2	3
2	$3.6e-02$	$7.2e-02$	$2.3e-04$	2	3	5
4	$9.1e-03$	$1.8e-02$	$1.5e-05$	4	5	9
8	$2.3e-03$	$4.6e-03$	$1.0e-06$	8	9	17
16	$5.7e-04$	$1.1e-03$	$6.5e-08$	16	17	33
32	$1.4e-04$	$2.8e-04$	$4.1e-09$	32	33	65
64	$3.6e-05$	$7.1e-05$	$2.6e-10$	64	65	129
128	$8.9e-06$	$1.8e-05$	$1.7e-11$	128	129	257
256	$2.2e-06$	$4.5e-06$	$1.0e-12$	256	257	513
512	$5.6e-07$	$1.1e-06$	$6.6e-14$	512	513	1025
1024	$1.4e-07$	$2.8e-07$	$4.1e-15$	1024	1025	2049

Table: Comparison of midpoint, trapezoidal and Cavalieri-Simpson composite rule, for N equispaced intervals relatively to the computation of $I = \int_0^1 f(x)dx$ con $f(x) = x^3\sqrt{x}dx$,describing the absolute errors $|E_0^{(c)}(f)| = |I(f,a,b) - S_0^{(c)}(f,a,b,N)$, $|E_1^{(c)}(f)| = |I(f,a,b) - S_1^{(c)}(f,a,b,N)|$, $|E_2^{(c)}(f)| = |I(f,a,b) - S_2^{(c)}(f,a,b,N)|$, for each formula and the respective number of nodes $\#_N^{R}$, $\#_N^{T}$, $\#_N^{CS}$.

Remark (optional)

In the second table we show the ratio between two successive errors for each formula. The value $(E_k^{(c)}(f))_N$, $k = 0, 1, 2$, *is the absolute integration error by* S_k, *for computing* $\int_a^b f(x)dx$, *over N subdvisions, while*

$$(r_k^{(c)}(f))_N = \frac{(E_k^{(c)}(f))_N}{(E_k^{(c)}(f))_{2N}}$$

is the ratio for $k = 0, 1, 2$ *(in order, midpoint, trapezoidal and Cavalieri-Simpson composite rule).*

From the tables we see that the ratio for

- *composite midpoint and trapezoidal rule is approximatively 4,*
- *composite Cavalieri-Simpson rule is approximatively 16,*

and thus the errors are of the form C^*h^2 *and* C^*h^4.

N	$(r_0^{(c)}(f))_N$	$(r_1^{(c)}(f))_N$	$(r_2^{(c)}(f))_N$
1	3.76	3.86	14.56
2	3.93	3.96	15.00
4	3.98	3.99	15.31
8	4.00	4.00	15.53
16	4.00	4.00	15.67
32	4.00	4.00	15.77
64	4.00	4.00	15.84
128	4.00	4.00	15.89
256	4.00	4.00	15.92
512	4.00	4.00	16.06

Table: Ratios for the composite rules concerning the computation of $I = \int_0^1 f(x)dx$ for $f(x) = x^3\sqrt{x}dx$

Example (4)

Approximate the definite integral

$$I = \int_0^1 \sqrt{x} dx = 2/3.$$

by the composite rules $S_k^{(c)}(f, 0, 1, N)$, $N = 1, 2, 4, \ldots, 2048$, $k = 0, 1, 2$.

Remark

Notice that the integrand belongs to $C([0,1])$ but not to $C^1([0,1])$ (singularity in 0). Thus we cannot apply the error formulas since they require $f \in C^2([0,1])$ or even $f \in C^4([0,1])$.

N	$\lvert E_0^{(c)}(f)\rvert$	$\lvert E_1^{(c)}(f)\rvert$	$\lvert E_2^{(c)}(f)\rvert$	$\#_N^{R}$	$\#_N^{T}$	$\#_N^{CS}$
1	$4.0e-02$	$1.7e-01$	$2.9e-02$	1	2	3
2	$1.6e-02$	$6.3e-02$	$1.0e-02$	2	3	5
4	$6.3e-03$	$2.3e-02$	$3.6e-03$	4	5	9
8	$2.4e-03$	$8.5e-03$	$1.3e-03$	8	9	17
16	$8.7e-04$	$3.1e-03$	$4.5e-04$	16	17	33
32	$3.2e-04$	$1.1e-03$	$1.6e-04$	32	33	65
64	$1.1e-04$	$4.0e-04$	$5.6e-05$	64	65	129
128	$4.1e-05$	$1.4e-04$	$2.0e-05$	128	129	257
256	$1.5e-05$	$5.0e-05$	$7.0e-06$	256	257	513
512	$5.2e-06$	$1.8e-05$	$2.5e-06$	512	513	1025
1024	$1.8e-06$	$6.3e-06$	$8.8e-07$	1024	1025	2049
2048	$6.5e-07$	$2.2e-06$	$3.1e-07$	2048	2049	4097

Table: Comparison of midpoint, trapezoidal and Cavalieri-Simpson composite rule, for N equispaced intervals relatively to the computation of $I = \int_0^1 f(x)dx$ con $f(x) = \sqrt{x}dx$,describing the absolute errors and cardinalities.

An effect of the low regularity of the integrand is the slow convergence of the rule. THe fact that converge, independently of the regularity stems from the Polya-Steklov theorem.

Remark (optional)

In the second table we show the ratio between two successive errors for each formula. The value $(E_k^{(c)}(f))_N$, $k = 0, 1, 2$, *is the absolute integration error by* S_k, *for computing* $\int_a^b f(x)dx$, *over N subdvisions, while*

$$(r_k^{(c)}(f))_N = \frac{(E_k^{(c)}(f))_N}{(E_k^{(c)}(f))_{2N}}$$

is the ratio for $k = 0, 1, 2$ *(in order, midpoint, trapezoidal and Cavalieri-Simpson composite rule).*

The values of midpoint, trapezoidal and Cavalieri-Simpsons composite rule tend to $2.83 \approx 2^{1.5}$ *showing a convergence of the order* $C\,h^{1.5}$.

Some numerical comparisons: example 4 (optional)

N	$(r_0^{(c)}(f))_N$	$(r_1^{(c)}(f))_N$	$(r_2^{(c)}(f))_N$
1	2.47	2.64	2.82
2	2.59	2.70	2.83
4	2.67	2.74	2.83
8	2.72	2.77	2.83
16	2.75	2.79	2.83
32	2.78	2.80	2.83
64	2.79	2.81	2.83
128	2.80	2.81	2.83
256	2.81	2.82	2.83
512	2.82	2.82	2.83
1024	2.82	2.82	2.83

Table: Ratio for the composite rules concerning the computation of $I = \int_0^1 f(x)dx$ for $f(x) = \sqrt{x}dx$

Example (5)

Approximate the definite integral

$$I = \int_0^{100} \exp(-x^2)\, dx = \frac{\sqrt{\pi}}{2} \cdot \mathrm{erf}(100)$$

where $\mathrm{erf}(x)$ is the error function, by composite rules $S_k^{(c)}(f, 0, 1000, N)$, $N = 1, 2, 4, \ldots, 256$, $k = 0, 1, 2$.

Remark

Take into account that $\exp(-x^2)$ *has no primitive via elementary functions, hence* I *cannot be computed by fundamental theorem of calculus.*

In the first table we consider $N = 1, 2, 4, \ldots, 256$.

N	$\lvert E_0^{(c)}(f)\rvert$	$\lvert E_1^{(c)}(f)\rvert$	$\lvert E_2^{(c)}(f)\rvert$	$\#_N^{\mathrm{R}}$	$\#_N^{\mathrm{T}}$	$\#_N^{\mathrm{CS}}$
1	$8.9e-01$	$4.9e+01$	$1.6e+01$	1	2	3
2	$8.9e-01$	$2.4e+01$	$7.4e+00$	2	3	5
4	$8.9e-01$	$1.2e+01$	$3.3e+00$	4	5	9
8	$8.9e-01$	$5.4e+00$	$1.2e+00$	8	9	17
16	$8.9e-01$	$2.2e+00$	$1.6e-01$	16	17	33
32	$6.1e-01$	$6.8e-01$	$1.8e-01$	32	33	65
64	$3.1e-02$	$3.1e-02$	$1.0e-02$	64	65	129
128	$1.7e-07$	$1.7e-07$	$5.6e-08$	128	129	257
256	$1.1e-16$	$1.1e-16$	$2.2e-16$	256	257	513

Table: Comparison of midpoint, trapezoidal and Cavalieri-Simpson composite rule, for N equispaced intervals relatively to the computation of $I = \int_0^{100} f(x)dx$ con $f(x) = \exp(-x^2)\, dx$,describing the absolute errors $|E_0^{(c)}(f)| = |I(f,a,b) - S_0^{(c)}(f,a,b,N)$, $|E_1^{(c)}(f)| = |I(f,a,b) - S_1^{(c)}(f,a,b,N)|$, $|E_2^{(c)}(f)| = |I(f,a,b) - S_2^{(c)}(f,a,b,N)|$, for each formula and the respective number of nodes $\#_N^{\mathrm{R}}$, $\#_N^{\mathrm{T}}$, $\#_N^{\mathrm{CS}}$.

Commento. (optional)

- *For $n \leq 64$ the error slowly decreases, as consequence of the fact that the interval has length 100 and thus the number of function evaluations is too small and h too large.*
- *In the second table, we show the ratio between two successive integration errors $|(E_k^{(c)}(f))_N|$, $k = 0, 1, 2$, relatively to the formula S_k, and its approximation of $\int_a^b f(x)dx$, considering N equispaced intervals., We evaluate*

$$(r_k^{(c)}(f))_N = \frac{|(E_k^{(c)}(f))_N|}{|(E_k^{(c)}(f))_{2N}|}$$

for $k = 0, 1, 2$ (that is composite midpoint, trapezoidal and Cavalieri-Simpson rule).

Some numerical comparisons: example 5 (optional)

N	$(r_0^{(c)}(f))_N$	$(r_1^{(c)}(f))_N$	$(r_2^{(c)}(f))_N$
1	1.00	2.04	2.12
2	1.00	2.08	2.27
4	1.00	2.17	2.74
8	1.00	2.40	7.69
16	1.44	3.31	0.85
32	19.74	21.74	17.74
64	184935.89	184937.89	184933.89
128	1515188455.00	1515188459.00	252531408.50

Table: Ratio for the composite rules concerning the computation of $I = \int_0^{100} f(x)dx$ con $f(x) = \exp(-x^2)\, dx$, in cui si descrivono i rapporti tra 2 errori successivi per ogni formula.

Observe that the values of composite midpoint and trapezoidal rule do not tend to 4 and those of composite Cavalieri-Simpson rule do not tend to 16.

- David M. Bressoud, *Historical Reflections on Teaching the Fundamental Theorem of Integral Calculus*, The American Mathematical Monthly , Vol. 118, No. 2 (February 2011), pp. 99-115.
- Lloyd N. Trefethen, *The Exponentially Convergent Trapezoidal Rule*, SIAM review 2014, Vol. 56, No. 3, pp. 385–458.