Computing Fekete and Lebesgue points: simplex, square, disk

Alvise Sommariva

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Collaborators/Group

Joint work with

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- Francesca Rapetti (University of Nice)

Other collaborators:

- Stefano De Marchi (University of Padua)
- Len Bos (University of Verona)
- Norman Levenberg (University of Indiana)
- Jean Paul Calvi (University of Toulouse)
Purpose of our research

- Computation and analysis of good nodes for interpolation and approximation in 2D and 3D.
- Computation and analysis of good nodes for cubature in 2D and 3D.
- Providing Matlab/Octave codes to the users.

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Good interpolation sets

Let $\Omega \subset \mathbb{R}^d$ be a compact domain. We denote by $\text{vdm}(\mathcal{P})$ the Vandermonde matrix (w.r.t. some polynomial basis $\{\phi_k\}$) evaluated at the discrete set $\mathcal{P}$.

Classical good interpolation sets are:

- Fekete points (maximization of absolute value of Vandermonde determinant).
- Leja points (known $\xi_1, \ldots, \xi_k$, the point $\xi_{k+1}$ maximizes $\text{vdm}(\xi_1, \ldots, \xi_k, x)$, a generalization of the classical 1D Leja points).
- Lebesgue points (minimization of Lebesgue constant, i.e. the $\infty$-norm of the interpolation operator).

Note: Few of these sets are known explicitly.
Due to the theoretical difficulties, one tries to compute the Fekete, Leja or Lebesgue points $\{\xi_k\}$ numerically.

Approximate Fekete Points and Discrete Leja Points are obtained

- Generating an *admissible* of *weakly admissible mesh* on the compact domain $\Omega$ (i.e. sets satisfying a particular polynomial inequality).
Due to the theoretical difficulties, one tries to compute the Fekete, Leja or Lebesgue points \( \{ \xi_k \} \) numerically.

Approximate Fekete Points and Discrete Leja Points are obtained by:

- Generating an \textit{admissible} of \textit{weakly admissible mesh} on the compact domain \( \Omega \) (i.e. sets satisfying a particular polynomial inequality).
- Extracting by linear algebra routines sets that mimic the Fekete or Leja points (in the continuum).
Figure: Weakly admissible mesh on a non-convex polygon (degree 10) and Approximate Fekete Points (red circles)
Pros:

1. For mild degrees: fast computation of good points for interpolation on very general domains.
2. Many good meshes are known for simplex, squares, polygons, disk and some trivariate domains.
**Pros:**
1. For mild degrees: fast computation of good points for interpolation on very general domains.
2. Many good meshes are known for simplex, squares, polygons, disk and some trivariate domains.

**Cons:**
1. Possible difficulties in computing small admissible or weakly admissible meshes.
2. The points are usually not the actual Fekete/Leja points (though they have the same asymptotic behavior w.r.t.
equilibrium measure).
In this work, for a fixed degree $N$, we compute points $\mathcal{P} = \{\xi_k\}_{k=1,\ldots,M}$ on interval, simplex, square and unit disk that have low Lebesgue constant $\Lambda_N(\mathcal{P})$.

Remember that $\Lambda_N(\mathcal{P})$ is the norm of the interpolation operator in $\infty$-norm and correspond to

$$
\Lambda_N(\mathcal{P}) = \max_{x \in \Omega} \sum_{k=1}^{M} |L_k(x)|
$$

where

- $L_K$ are the Lagrange polynomials w.r.t $\mathcal{P}$,
- $M = \text{bynomial}(N + d, d)$ is the dimension of $\mathbb{P}_N$, the space of total polynomials of degree $N$ in $\mathbb{R}^d$. 
Lebesgue constant

Some facts:

- There are sets \( \{\xi_k\}_{k=1,\ldots,M} \) that minimize the Lebesgue constant (the so-called Lebesgue points).

At degree \( N \), the Fekete points \( \{\xi_k\}_{k=1,\ldots,M} \) possess Lebesgue constant less or equal than the cardinality \( M \) that in 2D is equal to \( \frac{(N+1)(N+2)}{2} \). In general, they do not minimize the Lebesgue constant.

- Let \( C(\Omega) \) be the space of continuous functions in \( \Omega \). If \( f \in C(\Omega) \), \( p_N \) the interpolant of \( f \) in \( P \) and \( p^* N \) \( \in P \) is the best approximant of \( f \) in \( P \) then it is easy to show that

\[
\|f - p_N\|_{\infty} \leq (1 + \Lambda_N(P)) \|f - p^* N\|_{\infty}.
\]

Small Lebesgue constant implies

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- Let $C(\Omega)$ be the space of continuous functions in $\Omega$. If $f \in C(\Omega)$, $p_N$ the interpolant of $f$ in $\mathcal{P}$ and $p_N^* \in \mathbb{P}_N$ is the best approximant of $f$ in $\mathbb{P}_N$ than it is easy to show that

$$\|f - p_N\|_\infty \leq (1 + \Lambda_N(\mathcal{P}))\|f - p_N^*\|_\infty.$$  

Small Lebesgue constant implies $\|f - p_N\|_\infty \approx \|f - p_N^*\|_\infty$. 

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Computing Fekete and Lebesgue points: simplex, square, disk
Provide (almost) Fekete and Lebesgue points on simplex, square and disk ($N \leq 18$). By affine mapping, these points can be used on any simplex, parallelogram and ellipse.
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Provide freeware Matlab codes for the computation of these sets.
The Lebesgue points $\mathcal{P}$ minimize the Lebesgue constant $\Lambda_N(\mathcal{P})$.

- We used \texttt{fmincon} (constrained minimization), \texttt{fminsearch}, \texttt{fminunc} (unconstrained minimization) and the Differential Evolution algorithm for computing the (almost-)optimal sets.
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- We use Matlab codes to compute the optimal points. Special settings/strategies to avoid erratic behaviors.
- We consider a fine mesh $\mathcal{T}$ of about 62500 test points. At degree $N$, the value target function $F_1$ on a unisolvent set $\mathcal{P}$ is

$$F_1(\mathcal{P}) := \max_{x \in \mathcal{T}} \sum_{k=1}^{M} |L_k(x)| \approx \max_{x \in \Omega} \sum_{k=1}^{M} |L_k(x)| := \Lambda_N(\mathcal{P}),$$

where $M = \text{bynomial}(N + d, d)$. 

The Fekete points $\mathcal{P}$ maximize the absolute value of the determinant of the Vandermonde matrix $V(\mathcal{P}) = [\phi_k(\xi_s)]$ with respect to a basis $\{\phi_k\}$ of $\mathbb{P}_N$. Reminder: Fekete points do not depend on the basis $\{\phi_k\}$.

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- We use Matlab codes to compute the (almost-)optimal points. Special settings/strategies to avoid erratic behaviors. Fekete points can be computed faster than Lebesgue points.
Computation of Fekete points

The Fekete points $\mathcal{P}$ maximize the absolute value of the determinant of the Vandermonde matrix $V(\mathcal{P}) = [\phi_k(\xi_s)]$ with respect to a basis $\{\phi_k\}$ of $\mathbb{P}_N$. Reminder: Fekete points do not depend on the basis $\{\phi_k\}$.

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- At degree $N$, the value target function $F_2$ on a set $\mathcal{P}$ of cardinality $M = \text{bynomial}(N + d, d)$ is

$$F_2(\mathcal{P}) := -|\det V(\mathcal{P})|$$
Many pointsets $\mathcal{P}$ on the simplex have been proposed in the literature. The best results were achieved by

- Heinrichs (2005) for the Lebesgue points.

We will compare our results with their ones.
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- General distribution with Gauss-Legendre-Lobatto points on the sides of the simplex (LEBGL).

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- General distribution with Gauss-Legendre-Lobatto points on the sides of the simplex (LEBGL).
- Symmetric distribution with Gauss-Legendre-Lobatto points on the sides of the simplex (LEBGLS).

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- Symmetric distribution with Gauss-Legendre-Lobatto points on the sides of the simplex (LEBGLS).
- Fekete points (FEK).

We will compare our results with their ones.
Table: \( \Lambda_N \) on the simplex. New sets: LEB, LEBGL, LEBGLS, FEK.

<table>
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<th>deg</th>
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<th>6</th>
<th>9</th>
<th>12</th>
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<td>-</td>
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<td>-</td>
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<td>33.07</td>
<td>83.18</td>
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Figure: (Almost-)Lebesgue points on the simplex (degree 10)
The Padua points (at degree $n$) are defined as follows. Let

\[ C_{n+1} = \{ z_j = \cos((j - 1)/n), j = 1, \ldots, n + 1 \} \]

and

\[ CE_{n+1} = \{ z_j \in C_{n+1}, j - 1 \text{ even} \} \]
\[ CO_{n+1} = \{ z_j \in C_{n+1}, j - 1 \text{ odd} \}. \]

Then

\[ Pad_n = (CE_{n+1} \times CO_{n+2}) \cup (CO_{n+1} \times CE_{n+2}) \subseteq C_{n+1} \times C_{n+2}. \]

The Lebesgue constant is $O(\log^2(n))$ (almost optimal). We can obtain new family of Padua points using Jacobi-Lobatto points instead of $C_{n+1}$. We will denote by $PdJ$ the Jacobi set with (almost-)lower Lebesgue constant.
Lebesgue/Fekete points on the square: results

Table: Lebesgue constants of some interpolation sets in the square.

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<tr>
<th>deg</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
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Figure: (Almost-)Lebesgue points on the square (degree 10).
Table: Lebesgue constants of (quasi)-Fekete and (quasi)-Lebesgue points in the unit disk.

<table>
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<tr>
<th>deg</th>
<th>1</th>
<th>2</th>
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(Almost-)Lebesgue points on the square: a figure

Figure: (Almost-)Lebesgue points on the square (degree 8): *polygonal distribution on concentric circles*
Lebesgue/Fekete points on the disk: distribution on concentric circles

**Table:** Cardinality distribution (number of vertices of the regular polygons) on concentric circles of (quasi-)Fekete and (quasi-)Lebesgue points in the unit disk.

<table>
<thead>
<tr>
<th>deg</th>
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| \(N\) | 3  | 6  | 10 | 15 | 21 | 28 | 36 | 45 | 55 | 66 | 78 |
See I. Yaman’s talk: *Radially orthogonal multivariate basis function* for a possible explanation of this particular distribution.

ALVANIA: S8.

Friday 14th October, 12-12.30.
References


Sets

Point sets are available at the homepage

http://www.math.unipd.it/~alvise/software.html