# Computing Fekete and Lebesgue points: simplex, square, disk

Alvise Sommariva

Università degli Studi di Padova
Dipartimento di Matematica Pura e Applicata

October 16, 2011

# Collaborators/Group

Joint work with

- Matteo Briani, Luca Mezzalira, Marco Vianello (University of Padua)
- Francesca Rapetti (University of Nice)

Other collaborators:

- Stefano De Marchi (University of Padua)
- Len Bos (University of Verona)
- Norman Levenberg (University of Indiana)
- Jean Paul Calvi (University of Toulouse)

# Purpose of our research

- Computation and analysis of good nodes for interpolation and approximation in 2D and 3D.

# Purpose of our research

- Computation and analysis of good nodes for interpolation and approximation in 2D and 3D.
- Computation and analysis of good nodes for cubature in 2D and 3D.

# Purpose of our research

- Computation and analysis of good nodes for interpolation and approximation in 2D and 3D.
- Computation and analysis of good nodes for cubature in 2D and 3D.
- Providing Matlab/Octave codes to the users.

# Good interpolation sets

Let $\Omega \subset \mathbb{R}^d$ be a compact domain. We denote by $\mathtt{vdm}(\mathcal{P})$ the Vandermonde matrix (w.r.t. some polynomial basis $\{\phi_k\}$) evaluated at the discrete set $\mathcal{P}$.

Classical good interpolation sets are:

- Fekete points (maximization of absolute value of Vandermonde determinant).
- Leja points (known $\xi_1, \ldots, \xi_k$, the point $\xi_{k+1}$ maximizes $\mathtt{vdm}(\xi_1, \ldots, \xi_k, x)$, a generalization of the classical 1D Leja points).
- Lebesgue points (minimization of Lebesgue constant, i.e. the $\infty$-norm of the interpolation operator).

<u>Note</u>: Few of these sets are known explicitly.

# Approximate Fekete Points and Discrete Leja Points

Due to the theoretical difficulties, one tries to compute the Fekete, Leja or Lebesgue points $\{\xi_k\}$ numerically.

Approximate Fekete Points and Discrete Leja Points are obtained

- Generating an *admissible* of *weakly admissible mesh* on the compact domain $\Omega$ (i.e. sets satisfying a particular polynomial inequality).

# Approximate Fekete Points and Discrete Leja Points

Due to the theoretical difficulties, one tries to compute the Fekete, Leja or Lebesgue points $\{\xi_k\}$ numerically.

Approximate Fekete Points and Discrete Leja Points are obtained

- Generating an *admissible* of *weakly admissible mesh* on the compact domain $\Omega$ (i.e. sets satisfying a particular polynomial inequality).
- Extracting by linear algebra routines sets that mimic the Fekete or Leja points (in the continuum).

# Approximate Fekete Points and Discrete Leja Points

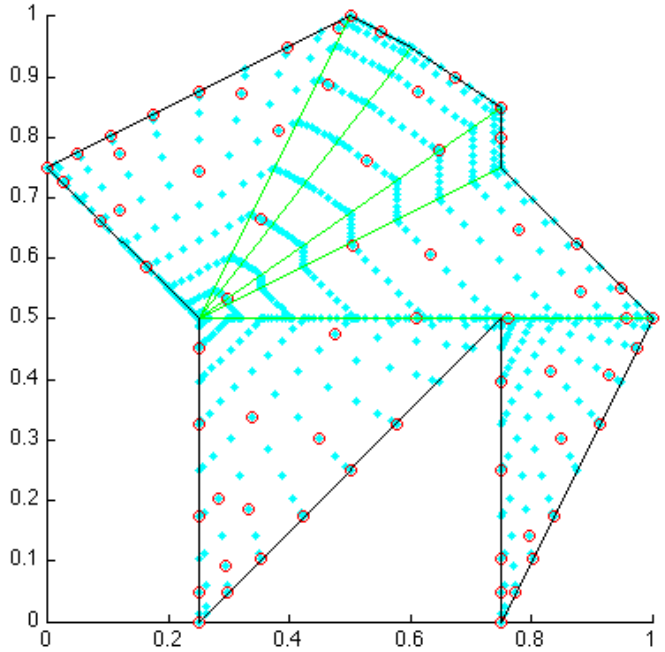


Figure: Weakly admissible mesh on a non-convex polygon (degree 10) and Approximate Fekete Points (red circles)

# Approximate Fekete Points and Discrete Leja Points

- Pros:

  1. For mild degrees: fast computation of good points for interpolation on very general domains.

  2. Many good meshes are known for simplex, squares, polygons, disk and some trivariate domains.

# Approximate Fekete Points and Discrete Leja Points

- Pros:

  1. For mild degrees: fast computation of good points for interpolation on very general domains.

  2. Many good meshes are known for simplex, squares, polygons, disk and some trivariate domains.

- Cons:

  1. Possible difficulties in computing *small* admissible or weakly admissible meshes.

  2. The points are usually not the actual Fekete/Leja points (though they have the same asymptotic behavior w.r.t. equilibrium measure).

# Lebesgue constant

In this work, for a fixed degree $N$, we compute points $\mathcal{P} = \{\xi_k\}_{k=1,\ldots,M}$ on interval, simplex, square and unit disk that have *low* Lebesgue constant $\Lambda_N(\mathcal{P})$.

Remember that $\Lambda_N(\mathcal{P})$ is the norm of the interpolation operator in $\infty$-norm and correspond to

$$\Lambda_N(\mathcal{P}) = \max_{x \in \Omega} \sum_{k=1}^{M} |L_k(x)|$$

where

- $L_K$ are the Lagrange polynomials w.r.t $\mathcal{P}$,
- $M = \texttt{bynomial}(N+d, d)$ is the dimension of $\mathbb{P}_N$, the space of total polynomials of degree $N$ in $\mathbb{R}^d$ .

# Lebesgue constant

Some facts:

- There are sets $\{\xi_k\}_{k=1,\ldots,M}$ that minimize the Lebesgue constant (the so called Lebesgue points).

# Lebesgue constant

Some facts:

- There are sets $\{\xi_k\}_{k=1,\ldots,M}$ that minimize the Lebesgue constant (the so called Lebesgue points).
- At degree $N$, the Fekete points $\{\xi_k\}_{k=1,\ldots,M}$ possess Lebesgue constant less or equal than the cardinality $M$ that in 2D is equal to $(N+1)(N+2)/2$. In general they do not minimize the Lebesgue constant.

# Lebesgue constant

Some facts:

- There are sets $\{\xi_k\}_{k=1,\ldots,M}$ that minimize the Lebesgue constant (the so called Lebesgue points).
- At degree $N$, the Fekete points $\{\xi_k\}_{k=1,\ldots,M}$ possess Lebesgue constant less or equal than the cardinality $M$ that in 2D is equal to $(N+1)(N+2)/2$. In general they do not minimize the Lebesgue constant.
- Let $C(\Omega)$ be the space of continuous functions in $\Omega$. If $f \in C(\Omega)$, $p_N$ the interpolant of $f$ in $\mathcal{P}$ and $p_N^* \in \mathbb{P}_N$ is the best approximant of $f$ in $\mathbb{P}_N$ than it is easy to show that

$$\|f - p_N\|_\infty \leq (1 + \Lambda_N(\mathcal{P}))\|f - p_N^*\|_\infty.$$

Small Lebesgue constant implies $\|f - p_N\|_\infty \approx \|f - p_N^*\|_\infty$.

# Computation of Lebesgue/Fekete points

- Provide (almost) Fekete and Lebesgue points on simplex, square and disk ($N \leq 18$). By affine mapping, these points can be used on any simplex, parallelogram and ellipse.

# Computation of Lebesgue/Fekete points

- Provide (almost) Fekete and Lebesgue points on simplex, square and disk ($N \leq 18$). By affine mapping, these points can be used on any simplex, parallelogram and ellipse.
- Provide freeware Matlab codes for the computation of these sets.

# Computation of Lebesgue points

The Lebesgue points $\mathcal{P}$ minimize the Lebesgue constant $\Lambda_N(\mathcal{P})$.

- We used `fmincon` (constrained minimization), `fminsearch`, `fminunc` (unconstrained minimization) and the Differential Evolution algorithm for computing the (almost-)optimal sets.

# Computation of Lebesgue points

The Lebesgue points $\mathcal{P}$ minimize the Lebesgue constant $\Lambda_N(\mathcal{P})$.

- We used `fmincon` (constrained minimization), `fminsearch`, `fminunc` (unconstrained minimization) and the Differential Evolution algorithm for computing the (almost-)optimal sets.
- We use Matlab codes to compute the optimal points. Special settings/strategies to avoid erratic behaviors.

# Computation of Lebesgue points

The Lebesgue points $\mathcal{P}$ minimize the Lebesgue constant $\Lambda_N(\mathcal{P})$.

- We used `fmincon` (constrained minimization), `fminsearch`, `fminunc` (unconstrained minimization) and the Differential Evolution algorithm for computing the (almost-)optimal sets.
- We use Matlab codes to compute the optimal points. Special settings/strategies to avoid erratic behaviors.
- We consider a fine mesh $\mathcal{T}$ of about 62500 test points. At degree $N$, the value target function $F_1$ on a unisolvent set $\mathcal{P}$ is

$$F_1(\mathcal{P}) := \max_{x \in \mathcal{T}} \sum_{k=1}^{M} |L_k(x)| \approx \max_{x \in \Omega} \sum_{k=1}^{M} |L_k(x)| := \Lambda_N(\mathcal{P}),$$

where $M = \texttt{bynomial}(N + d, d)$.

# Computation of Fekete points

The Fekete points $\mathcal{P}$ maximize the absolute value of the determinant of the Vandermonde matrix $V(\mathcal{P}) = [\phi_k(\xi_s)]$ with respect to a basis $\{\phi_k\}$ of $\mathbb{P}_N$. Reminder: Fekete points do not depend on the basis $\{\phi_k\}$.

- As in the case of Lebesgue points, we used `fmincon` (constrained minimization), `fminsearch`, `fminunc` (unconstrained minimization) and the Differential Evolution algorithm for computing the (almost-)optimal sets.

# Computation of Fekete points

The Fekete points $\mathcal{P}$ maximize the absolute value of the determinant of the Vandermonde matrix $V(\mathcal{P}) = [\phi_k(\xi_s)]$ with respect to a basis $\{\phi_k\}$ of $\mathbb{P}_N$. Reminder: Fekete points do not depend on the basis $\{\phi_k\}$.

- As in the case of Lebesgue points, we used `fmincon` (constrained minimization), `fminsearch`, `fminunc` (unconstrained minimization) and the Differential Evolution algorithm for computing the (almost-)optimal sets.
- We use Matlab codes to compute the (almost-)optimal points. Special settings/strategies to avoid erratic behaviors. Fekete points can be computed faster than Lebesgue points.

# Computation of Fekete points

The Fekete points $\mathcal{P}$ maximize the absolute value of the determinant of the Vandermonde matrix $V(\mathcal{P}) = [\phi_k(\xi_s)]$ with respect to a basis $\{\phi_k\}$ of $\mathbb{P}_N$. Reminder: Fekete points do not depend on the basis $\{\phi_k\}$.

- As in the case of Lebesgue points, we used `fmincon` (constrained minimization), `fminsearch`, `fminunc` (unconstrained minimization) and the Differential Evolution algorithm for computing the (almost-)optimal sets.
- We use Matlab codes to compute the (almost-)optimal points. Special settings/strategies to avoid erratic behaviors. Fekete points can be computed faster than Lebesgue points.
- At degree $N$, the value target function $F_2$ on a set $\mathcal{P}$ of cardinality $M = \texttt{bynomial}(N + d, d)$ is

$$F_2(\mathcal{P}) := -|\det V(\mathcal{P})|$$

# Lebesgue/Fekete points on the simplex: literature

Many pointsets $\mathcal{P}$ on the simplex have been proposed in the literature. The best results were achieved by

- Heinrichs (2005) for the Lebesgue points .

We will compare our results with their ones.

# Lebesgue/Fekete points on the simplex: literature

Many pointsets $\mathcal{P}$ on the simplex have been proposed in the literature. The best results were achieved by

- Heinrichs (2005) for the Lebesgue points .
- Taylor, Wingate, Vincent (2005) and Roth (2005) for the Fekete points.

We will compare our results with their ones.

# Lebesgue/Fekete points on the simplex: literature

Many pointsets $\mathcal{P}$ on the simplex have been proposed in the literature. The best results were achieved by

- Heinrichs (2005) for the Lebesgue points .
- Taylor, Wingate, Vincent (2005) and Roth (2005) for the Fekete points.
- Other works by Chen, Babuska (1995), Hesthaven (1998), Blyth, Pozrikidis (2006), Warburton (2006) introduce non extremal pointsets.

We will compare our results with their ones.

# Lebesgue/Fekete points on the simplex: new sets.

Many pointsets $\mathcal{P}$ on the simplex have been proposed in the literature. The best results were achieved by

- General distribution (LEB).

We will compare our results with their ones.

# Lebesgue/Fekete points on the simplex: new sets.

Many pointsets $\mathcal{P}$ on the simplex have been proposed in the literature. The best results were achieved by

- General distribution (LEB).
- General distribution with Gauss-Legendre-Lobatto points on the sides of the simplex (LEBGL).

We will compare our results with their ones.

# Lebesgue/Fekete points on the simplex: new sets.

Many pointsets $\mathcal{P}$ on the simplex have been proposed in the literature. The best results were achieved by

- General distribution (LEB).
- General distribution with Gauss-Legendre-Lobatto points on the sides of the simplex (LEBGL).
- Symmetric distribution with Gauss-Legendre-Lobatto points on the sides of the simplex (LEBGLS).

We will compare our results with their ones.

# Lebesgue/Fekete points on the simplex: new sets.

Many pointsets $\mathcal{P}$ on the simplex have been proposed in the literature. The best results were achieved by

- General distribution (LEB).
- General distribution with Gauss-Legendre-Lobatto points on the sides of the simplex (LEBGL).
- Symmetric distribution with Gauss-Legendre-Lobatto points on the sides of the simplex (LEBGLS).
- Fekete points (FEK).

We will compare our results with their ones.

# Lebesgue/Fekete points on the simplex: results

Table: $\Lambda_N$ on the simplex. New sets: LEB, LEBGL, LEBGLS, FEK.

| deg | 3 | 6 | 9 | 12 | 15 | 18 |
|---|---|---|---|---|---|---|
| LEB | 1.97 | 3.39 | 5.28 | 6.90 | 8.91 | 9.88 |
| LEBGL | 2.11 | 3.59 | 5.51 | 7.13 | 8.91 | 9.88 |
| LEBGLS | 2.11 | 3.59 | 5.49 | 7.26 | 8.91 | 12.69 |
| FEK | 2.11 | 4.17 | 6.97 | 8.57 | 11.42 | 14.43 |
| TWV | 2.11 | 4.17 | 6.80 | 9.68 | 10.01 | 14.73 |
| HEI | - | 3.67 | 5.58 | 7.12 | 8.41 | 10.08 |
| HEI2 | - | 3.87 | 5.59 | 7.51 | 9.25 | 11.86 |
| WAR | 2.11 | 3.70 | 5.73 | 9.36 | 17.64 | 36.76 |
| BP | 2.11 | 3.87 | 7.39 | 17.78 | 49.59 | 156.16 |
| CB | 2.11 | 3.79 | 5.91 | 10.08 | - | - |
| HES | 2.11 | 4.07 | 6.94 | 12.39 | 29.68 | - |
| AFP | 2.26 | 5.35 | 11.33 | 14.81 | 29.52 | 36.34 |
| DLP | 3.88 | 14.68 | 17.79 | 31.37 | 33.07 | 83.18 |

# (Almost-)Lebesgue points on the simplex: a figure

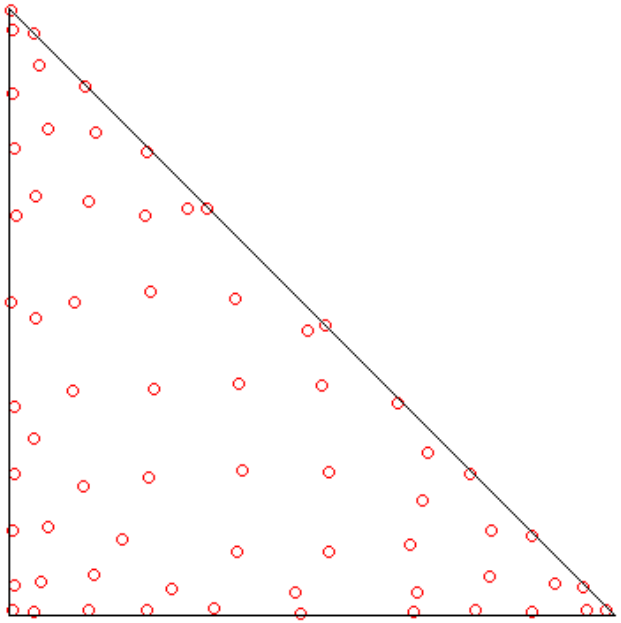

Figure: (Almost-)Lebesgue points on the simplex (degree 10)

# Lebesgue/Fekete points on the square: Padua Points

The Padua points (at degree $n$) are defined as follows. Let

$$C_{n+1} = \{z_j = \cos((j-1)/n), j = 1, \ldots, n+1\}$$

and

$$CE_{n+1} = \{z_j \in C_{n+1}, j-1 \text{ even}\}$$
$$CO_{n+1} = \{z_j \in C_{n+1}, j-1 \text{ odd}\}.$$

Then

$$Pad_n = (CE_{n+1} \times CO_{n+2}) \cup (CO_{n+1} \times CE_{n+2}) \subseteq C_{n+1} \times C_{n+2}.$$

The Lebesgue constant is $O(log^2(n))$ (almost optimal). We can obtain new family of Padua points using Jacobi-Lobatto points instead of $C_{n+1}$. We will denote by PdJ the Jacobi set with (almost-)lower Lebesgue constant.

# Lebesgue/Fekete points on the square: results

Table: Lebesgue constants of some interpolation sets in the square.

| deg | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| FEK | 2.00 | 2.93 | 3.72 | 4.00 | 4.73 | 6.07 | 5.48 | 5.96 | 6.21 | 6.65 |
| LEB | 2.00 | 2.38 | 2.73 | 3.24 | 3.59 | 3.98 | 4.33 | 4.85 | 5.12 | 5.23 |
| PdJ | 2.00 | 2.65 | 3.71 | 3.74 | 4.14 | 4.58 | 4.94 | 5.30 | 5.60 | 5.92 |
| Pd | 2.00 | 3.00 | 3.78 | 4.41 | 4.95 | 5.42 | 5.84 | 6.21 | 6.66 | 6.88 |

| deg | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|
| FEK | 6.88 | 7.37 | 7.55 | 7.95 | 8.05 | 8.33 | 8.43 | 8.68 | 8.84 | 8.99 |
| LEB | 5.97 | 6.26 | 6.50 | 6.82 | 7.00 | 7.14 | 7.41 | 7.61 | 7.86 | 8.05 |
| PdJ | 6.19 | 6.45 | 6.70 | 6.93 | 7.16 | 7.36 | 7.58 | 7.76 | 7.99 | 8.14 |
| Pd | 7.17 | 7.45 | 7.71 | 7.95 | 8.19 | 8.41 | 8.62 | 8.82 | 9.01 | 9.20 |

# (Almost-)Lebesgue points on the square: a figure

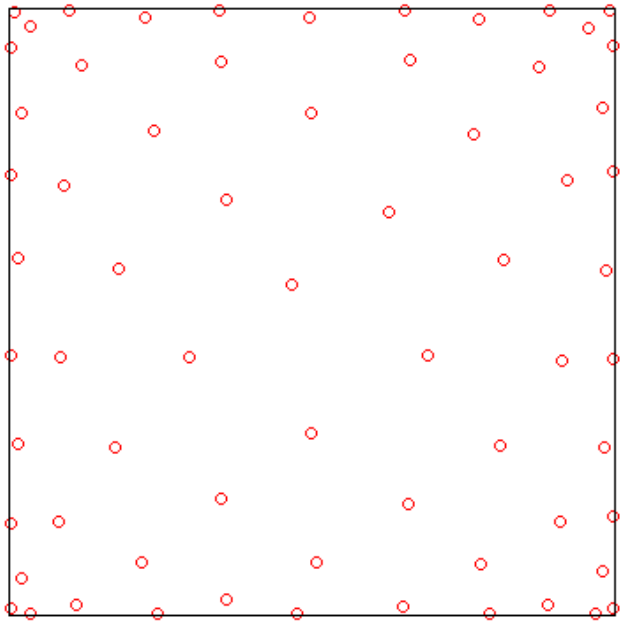

Figure: (Almost-)Lebesgue points on the square (degree 10).

# Lebesgue/Fekete points on the disk: results

Table: Lebesgue constants of (quasi)-Fekete and (quasi)-Lebesgue points in the unit disk.

| deg | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| FEK | 1.67 | 1.99 | 2.63 | 3.19 | 3.95 | 4.38 | 5.58 | 6.45 |
| LEB | 1.67 | 1.99 | 2.47 | 2.97 | 3.49 | 4.10 | 5.00 | 5.38 |

| deg | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|
| FEK | 7.02 | 7.84 | 8.92 | 14.85 | 17.02 | 18.51 | 21.44 | 28.91 |
| LEB | 6.54 | 7.14 | 7.84 | 8.97 | 10.14 | 11.14 | 13.08 | 14.72 |

# (Almost-)Lebesgue points on the square: a figure

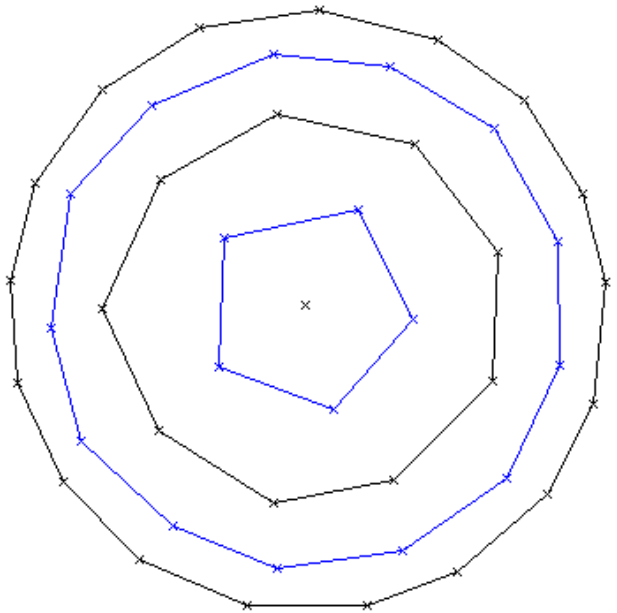

Figure: (Almost-)Lebesgue points on the square (degree 8): *polygonal distribution on concentric circles*

# Lebesgue/Fekete points on the disk: distribution on concentric circles

Table: Cardinality distribution (number of vertices of the regular polygons) on concentric circles of (quasi-)Fekete and (quasi-)Lebesgue points in the unit disk.

| deg | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 3 | 1 | 3 | 1 | 3 | 1 | 3 | 1 | 3 | 1 | 3 |
| | | 5 | 7 | 5 | 7 | 5 | 8 | 5 | 8 | 5 | 7 |
| | | | | 9 | 11 | 9 | 10 | 9 | 12 | 12 | 13 |
| | | | | | | 13 | 15 | 13 | 13 | 13 | 16 |
| | | | | | | | | 17 | 19 | 14 | 16 |
| | | | | | | | | | | 21 | 23 |
| $N$ | 3 | 6 | 10 | 15 | 21 | 28 | 36 | 45 | 55 | 66 | 78 |

# Lebesgue/Fekete points on the disk: some insight

See I. Yaman's talk: *Radially orthogonal multivariate basis function* for a possible explanation of this particular distribution.

ALVANIA: S8.

Friday 14th October, 12-12.30.

# References


- M. Briani, A. Sommariva, M. Vianello, *Computing Fekete and Lebesgue points : simplex, square, disk*, submitted.
- J.L. Calvi and N. Levenberg, *Uniform approximation by discrete least squares polynomials*, J. Approx. Theory, 152 (2008), pp.82-100.
- W. Heinrichs, *Improved Lebesgue constants on the triangle*, J. Comp. Phys., 207 (2005), pp.625-638.
- F. Rapetti, A. Sommariva, M. Vianello, *Computing symmetric Lebesgue points on the triangle*, submitted.
- A. Sommariva, M. Vianello, *Computing approximate Fekete points by QR factorizations of Vandermonde matrices*, Comput. Math. Appl., 57 (2009), pp.1324-1336.
- M.A. Taylor, B.A. Wingate, R.E. Vincent, *An algorithm for computing Fekete points in the triangle*, SIAM. J. Number. Anal., 38 (2000), pp.1707-1720.

# Sets

Point sets are available at the homepage

http://www.math.unipd.it/∼alvise/software.html