MATHEMATICAL DISSERTATIONS

On a VARIETY of

PHYSICAL and ANALYTICAL SUBJECTS.

Containing, among other Particulars,

A Demonſtration of the true Figure which the Earth, or any Planet muſt acquire from its Rotation about an Axis.

A general Inveſtigation of the Attraction at the Surfaces of Bodies nearly ſpherical.

A Determination of the meridional Parts, and the Lengths of the ſeveral Degrees of the Meridian, according to the true Figure of the Earth.

An Inveſtigation of the Height of the Tides in the Ocean.

A new Theory of Aſtronomical-Refractions, with exact Tables deduced therefrom.

A new and very exact Method for approximating the Roots of Equations in Numbers; that quintuples the Number of Places at each Operation.

Several new Methods for the Summation of Series.

Some new and very uſeful Improvements in the inverſe Method of Fluxions.

THE WHOLE

In a general and perſpicuous Manner.

By THOMAS SIMPSON.

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TO

Martin Folkes, Esquire,

PRESIDENT

OF THE

ROYAL-SOCIETY.

S I R,

I Could not have wish'd for a greater Honour than your condescending to receive these Sheets under your Protection: As every Man is in justice answerable, both to his Patron and the Publick, for what he presumes to print, I hope I have taken care that they may not be wholly

wholly unworthy your peruſal. If they ſhall have the good Fortune to meet with your Approbation, I need not be anxious about their Reception elſewhere. In the mean time, Sir, I moſt earneſtly deſire, that this Addreſs may be underſtood as an humble Acknowledgment of the Favours which I a Stranger, however undeſerving, have received at your Hands; and which I ſhall always remember with the ſincereſt Gratitude.

I am,

Sir,

Your moſt obliged

Humble Servant,

THOMAS SIMPSON.

PREFACE.

IT is so natural when a Work of this kind appears in the World, to ask What there is new in it? *and the greater Part of those who set up for Judges, are so extreamly bent to depreciate every Thing to which they can frame the least Pretence of an Objection, that an Author, without any Imputation of Vanity, may sometimes be allow'd to set forth the Merits of his own Performance, in order to give his less discerning Readers a true Representation thereof: And this I hope will be thought a reasonable Apology for what I have to offer in behalf of the several Particulars that compose this Miscellany.*

The First, which is one of the most considerable Papers in the whole Work, is concerned in determining the Figure which a Planet, or an homogenous Fluid, must acquire from its Rotation about an Axis; wherein the true Figure, under such a Rotation, is not only universally demonstrated, but the particular Species thereof, according to any assigned Time of Revolution; in which it is proved that the Gravitation at any Point in the Surface, is accurately as a Perpendicular to the Surface at that Point, produced from thence to the Axis of Revolution; and that it is impossible for the Parts of the Fluid ever to come to an Equilibrium among themselves, when the Motion about the Axis is so great as to exceed a certain assignable Quantity; with several other Particulars never before touch'd upon by Any. I must own that, since my first drawing up this Paper, the World has been obliged with something very curious on this Head, by that celebrated Mathematician Mr. Mac-Laurin, *in which many of the* *ſame Things, are demonſtrated. But what I here offer was read before the Royal Society *, and the greater Part of this Work printed off, many Months before the Publication of that Gentleman's Book; for which Reaſon I ſhall think myſelf ſecure from any Imputations of Plagiariſm, eſpecially as there is not the leaſt Likeneſs between our two Methods.*

The ſecond Paper, contains a general Inveſtigation of the Attraction at the Surfaces of Bodies nearly ſpherical.

The Third, conſiders the Heights of the Tides in the Ocean.

The Fourth, exhibits a very eaſy Method for finding the Length of a Degree of the Meridian, and the meridional Parts anſwering to any given Latitude, according to the true ſpheroidal Figure of the Earth.

The Fifth, includes the Inveſtigation of the Curve deſcribed by a Ray of Light in paſſing thro' an elaſtic Medium, whoſe Denſity either reſpects a plane, or ſpherical Surface, and varies according to any given Law: Whence are derived ſome practical, and very uſeful Concluſions, relating to the Refraction which the Light of the Heavenly-Bodies ſuffers in its Paſſage thro' the Earth's Atmoſphere; with exact Tables thereof, laid down by the help of very accurate Obſervations.

The Sixth, treats of the Summation of Series; which, beſides containing ſeveral Matters intirely new, is much more general and extenſive than any Thing I have hitherto met with, for the ſame Purpoſe.

The Seventh, exhibits a new Method for finding the Values of Series by Approximation.

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* It was read before the Royal-Society in *March* and *April*, 1741, and had been printed in the Philoſophical Tranſactions, had not I deſired the contrary.

The

The Eighth, comprehends the Investigation of some very useful Theorems for approximating the Roots of Equations in Numbers, much more exact than any Thing hitherto publish'd; whereby the Number of Places is tripled, quadrupled, or even quintupled, at each Operation; to which are added, some easy and proper Applications, in illustration thereof.

The Ninth, relates to mechanic Quadratures, or the Method of approximating the Areas of Curves, by Means of equidistant Ordinates. This Method was originally an Invention of Sir Isaac Newton's, *since prosecuted by Mr.* De Moivre, *Mr.* Stirling, *and Others: However, as I here assume nothing to myself, but a Liberty of putting the Matter in such a Light, as I judge will be most plain and satisfactory to the Reader, I see no Reason why I may not be allow'd the same Privilege as Others.*

The Tenth, is concerned in finding and comparing of Fluents, and contains a great Variety of new and useful Improvements, being one of the most considerable Papers in the whole Work.

The Eleventh, is an easy Investigation of the Paths of Shadows, on the Plane of the Horizon.

The Twelfth, contains a Determination of the Time of the Year when Days lengthen the fastest, according to any assigned Excentricity of the Earth's Orbit.

The Thirteenth, shews how much the Descent of Bodies, is affected by the Earth's Rotation.

The Fourteenth, is a Demonstration of the Law of Motion, that a Body deflected by two Forces, tending to two fix'd Points, describes equal Solids in equal Times, about the Right-Line joining those Points.

The

The Fifteenth, ſhews in what Caſes a Body, acted on by a centripetal Force, may continually deſcend towards the Centre, yet never ſo far as to come within a certain Diſtance; and in what other Caſes it may continually aſcend, yet never riſe to a certain finite Altitude.

The Sixteenth and laſt, comprehends an eaſy and general Inveſtigation of all the principal Theorems relating to Compound-Intereſt and Annuities, without being obliged to ſum up the Terms of a geometrical Progreſſion.

Theſe ſix laſt Papers, tho' more ſimple in themſelves, and of leſs general Uſe than ſome of the preceding, may nevertheleſs be look'd upon as entertaining Speculations, and therefore not prove unacceptable.

ERRATA.

PAGE 2. l. laſt, and p. 3. l. 2. for $\overline{a-z^2}$ read $\overline{a-z}|^2$; p. 4. l. 10. for $\overline{a^n-x^n}|^m \times \overline{a^n-x^n}|^m$, r. $\overline{a^n-x^n}|^m \times \overline{a^n-x^n}$; p. 5. l. 3. dele the Semi-colon; p. 20. l. 21. for *whereof, &c.* read *whereof the Time of Revolution can be ſo ſhort, as of that, whoſe equatoreal Diameter is to its Axis, as* 2.7198 *to Unity*; p. 38. l. 12. for *paſſing*, r. *paſſing thro'*; p. 41. l. 4. for *Spherodial*, r. *Spheroidal*; p. 42. l. 9. for *Part*, r. *Parts*; p. 47. l. 26. for *by*, r. *be*; p. 61. l. 21. for *could*, r. *could have been*; p. 64. l. 17. for R, r. *k*; p. 70 l. 14. for *in*, r. *to*; p. 73. l. 16. for z, r. $-z$; p. 78. l. 8. after *Numbers* put a Semi-colon; p. 81. l. 15. for $+z^n \pm z^{2n}$, r. $\pm z^n + z^{2n}$; p. 86. l 13. for $k+2m$, r. $q+2m$; p. 99. l. laſt, for a^2, r. a^2x^m; p. 105. l. 9. for 613, r. 61.3; p. 122. l. 20. for *independant*, r. *independent*; p. 128. l. 8. for $1-\frac{b+1}{f+2}$, &c. r. $-1-\frac{b+1}{f+1}$, &c. p. 136. l. 5. for *Negative*, r. *negative Number*; p. 145. l. 2. for $p+3r$, r. $p+2r$; p 149. l. 12. dele the upper Vinculum; p. 151. l. 5. for $=z$, r. $=x$; p. 161. l. 19. for *gravitate*, r. *to gravitate*.

A MATHEMATICAL

DISSERTATION

ON THE

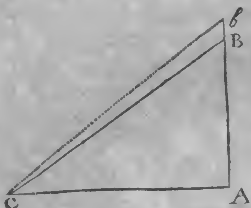
FIGURE

OF THE

EARTH.

LEMMA I.

SUPPOSING AC perpendicular to AB, and a Corpuscle at C to be attracted towards every Point or Particle in the Line AB, by Forces in the reciprocal duplicate Ratio of the Distances: To find the Ratio of the whole compounded Force, whereby the Corpuscle is urged in the Direction AC.



Let $AC = d$, $AB = x$, and $Bb = \dot{x}$: Therefore $BC^2 = d^2 + x^2$; consequently $\frac{1}{d^2 + x^2}$ will be as the Force of a Particle at B, in the Direction B C; but $\overline{d^2 + x^2}|^{\frac{1}{2}} : d :: \frac{1}{d^2 + x^2} : \frac{d}{\overline{d^2 + x^2}|^{\frac{3}{2}}}$ the Efficacy of that Particle in the proposed Direction A C; wherefore $\frac{d\dot{x}}{\overline{d^2 + x^2}|^{\frac{3}{2}}}$ is the Fluxion of the whole Force; whose Fluent $\frac{x}{d \times \overline{d^2 + x^2}|^{\frac{1}{2}}} = \frac{AB}{CA \times BC}$ is, therefore, the Force itself. Q. E. I.

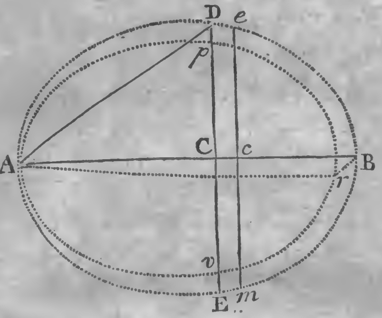
LEMMA II.

Supposing a Cuneus of uniformly dense Matter, compriz'd between two equal and similar elliptical Planes A D B E A, A p r v A, inclined to each other, at the common vertex A, of their first or second Axes, in an indefinitely small angle r A B; to find the attraction of the said Cuneus, whereby a Corpuscle at A is urged in the Direction of the Axis A B or A r.

Let D E and $m e$ be two Ordinates to the Axis A B, indefinitely near to each other, and let $AB = a$, $BC = z$, $Cc = \dot{z}$, $CD = y$, and the Sine of the Angle formed by the two Planes, to the Radius $1, = d$; and let the Equation of the Ellipsis be $y^2 = fz - z^2 - gz^2$ (which will answer either to the transverse, or conjugate, Axis, according as the Value of g, is taken negative or positive)

Now, it will be, as $1 : d :: a - z : d \times \overline{a - z}$, the Distance of the two Planes at the Ordinate D C, or the Depth of the proposed Matter at that Ordinate; which therefore drawn, into $\frac{\dot{z}\sqrt{fz - z^2 - gz^2}}{\overline{a - z} \times \sqrt{\overline{a - z}|^2 + fz - z^2 - gz^2}} \left\{ = \frac{DC \times Cc}{AC \times AD} \right\}$ the

the Attraction of the Particles in the surface DC*ce*D (*by the preceding Lemma*) gives $\frac{d\dot{z}\sqrt{fz - z^2 - gz^2}}{\sqrt{\overline{a-z}^2 + fz - z^2 - gz^2}}$



for the Fluxion of half the Force required. But when $fz - z^2 - gz^2$ becomes $= o$, z will be $= \frac{f}{1+g} = AB = a$; therefore by writing $\overline{1+g} \times a$ instead of f in the said Fluxion, it will be $\frac{dz^{\frac{1}{2}}\dot{z} \times \overline{1+g}|^{\frac{1}{2}}}{\overline{a+gz}|^{\frac{1}{2}}} = \frac{\overline{1+g}|^{\frac{1}{2}} \times dz^{\frac{1}{2}}\dot{z}}{a^{\frac{1}{2}}} \times \overline{1 - \frac{gz}{2a} + \frac{3g^2z^2}{2.4a^2} - \frac{3.5g^3z^3}{2.4.6a^3}}$ &c, the Fluent whereof, when z becomes $= a$, will be $ad \times \overline{1+g}|^{\frac{1}{2}} \times \overline{\frac{2}{3} - \frac{g}{5} + \frac{3g^2}{4.7}}$ &c. which, because $a \times \overline{1+g}|^{\frac{1}{2}}$ is $= f \times \overline{1+g}|^{-\frac{1}{2}} = f \times \overline{1 - \frac{g}{2} + \frac{3g^2}{2.4}}$ &c. will be $= df \times \frac{2}{3} - \frac{2.4g}{3.5} + \frac{2.4.6g^2}{3.5.7} - \frac{2.4.6.8g^3}{3.5.7.9}$ &c.

Q. E. I.

LEMMA

LEMMA III.

THE Fluent of $\overline{a^n - x^n}|^m \times x^{rn-1}\dot{x}$ *being given; tis proposed to find the Fluent of* $\overline{a^n - x^n}|^{m+p} \times x^{rn+vn-1}\dot{x}$ *when* $\overline{a^n - x^n}|^{m+1}$ *becomes* $=0$; *supposing p and v to be whole positive Numbers.*

Let $Q = \overline{a^n - x^n}|^{m+1} \times x^{rn}$, and let E, F, G, H, &c. denote the Fluents of $\overline{a^n - x^n}|^n \times x^{rn-1}\dot{x}$, $\overline{a^n - x^n}|^n \times x^{rn+n-1}\dot{x}$, $\overline{a^n - x^n}|^m \times x^{rn+2n-1}\dot{x}$ respectively: Then $\dot{Q}$ being $(= rn\dot{x} \times x^{rn-1} \times \overline{a^n - x^n}|^{m+1} - \overline{m+1} \times n\dot{x}x^{n-1} \times \overline{a^n - x^n}|^n \times x^{rn} = rn\dot{x}x^{rn-1} \times \overline{a^n - x^n}|^n \times \overline{a^n - x^n}|^m - \overline{m+1} \times n\dot{x}\, x^{rn-1} \times \overline{a^n - x^n}|^m \times x^{rn}) = rna^n \times \overline{a^n - x^n}|^m \times x^{rn-1}\dot{x} - \overline{r+m+1} \times n \times \overline{a^n - x^n}|^m \times x^{rn+n-1}\dot{x}$, if, instead of $\overline{a^n - x^n}|^m \times x^{rn-1}\dot{x}$, and $\overline{a^n - x^n}|^m \times x^{rn+n-1}\dot{x}$, their Equals $\dot{E}$ and $\dot{F}$ be here substituted, we shall get $\dot{Q} = rna^n\dot{E} - \overline{r+m+1 \times n} \times \dot{F}$; whence $Q = rn \cdot a^n E - \overline{r+m+1 \times n} \times F$, and consequently $F = \frac{rna^n E - Q}{n \times \overline{r+m+1}}$, which therefore when $\overline{a^n - x^n}|^{m+1}$, or Q becomes $=0$, will be $= \frac{ra^n E}{r+m+1}$. And it is manifest, by Inspection, from the very same Reasons that G is $= \frac{\overline{r+1} \times a^n F}{r+m+2}$, $H = \frac{\overline{r+2} \times a^n G}{r+m+3}$ &c. or $G = \frac{r \times \overline{r+1} \times a^{2n} E}{\overline{r+m+1} \times \overline{r+m+2}}$, $H = \frac{r \times \overline{r+1} \times \overline{r+2} \times a^{3n} E}{\overline{r+m+1} \times \overline{r+m+2} \times \overline{r+m+3}}$; and therefore the Fluent of $\overline{a^n - x^n}|^m \times x^{rn+vn-1}\dot{x}$, (in this Cirumstance) putting $r+m+1 = q$, will be $\frac{r \times \overline{r+1} \times \overline{r+2} \ldots \overline{r+v-1} \times a^{vn}E}{q \times \overline{q+1} \times \overline{q+2} \ldots \overline{q+v-1}}$. Now if this Fluent be denoted by P, and $x^{rn+vn-1}$ by K, the Fluent

ent of $\overline{a^n - x^n}|^m \times K x^n \dot{x}$ will, it is evident, be $= \frac{r+v}{q+v} \times a^n P$, which taken from $a^n P$, that of $\overline{a^n - x^n}|^m \times K a^n \dot{x}$, leaves $\frac{\overline{q-r} \times a^n P}{q+v} = \frac{\overline{m+1} \times a^n P}{q+v}$ for the Fluent of $\overline{a^n - x^n}|^m \times K a^n \dot{x}$; $-\overline{a^n - x^n}|^m \times K x^n \dot{x}$ or its Equal, $\overline{a^n - x^n}|^{m+1} \times K \dot{x}$; therefore the Fluent of $\overline{a^n - x^n}|^{m+1} \times K \dot{x}$ is to the Fluent of $\overline{a^n - x^n}|^m \times K \dot{x}$, as $\frac{\overline{m+1} \times a^n}{q+v}$, to 1; and, for the very same reasons, will the Fluent of $\overline{a^n - x^n}|^{m+2} \times K \dot{x}$ be to that of $\overline{a^n - x^n}|^{m+1} \times K \dot{x}$, as $\frac{\overline{m+2} \times a^n}{q+v+1}$: to 1, &c. whence it manifestly appears that the Fluent of $\overline{a^n - x^n}|^{m+p} \times K \dot{x}$, or of $\overline{a^n - x^n}|^{m+p} \times x^{rn+vn-1} \dot{x}$ will be expressed by $a^{pn} P \times \frac{m+1}{q+v} \times \frac{m+2}{q+v+1} \times \frac{m+3}{q+v+2} \ldots \times \frac{m+p}{q+v+p-1}$, or by $E a^{pn+vn}$ into $\frac{r.\,r+1.\,r+2.\,r+3 \ldots\ldots r+v-1}{q.\,q+1.\,q+2.\,q+3.\ \ q+4.\,q+5} \frac{\times m+1.\ m+2.\ m+3 \ldots\ldots m+p}{.\,q+6.\ q+7 \ldots\ldots r+m+v+p}$. Q. E. I.

COROLLARY. I.

If E be taken equal to the Fluent of $\overline{1-s^2}|^{-\frac{1}{2}} \times \dot{s}$, or $\frac{1}{4}$ Part of the Periphery of the Circle whose Radius is Unity, and $\overline{1-s^2}|^{\frac{1}{2}} = v$, and the Fluent of $\overline{1-ss}|^{-\frac{1}{2}}\ \dot{s} \times dv^{2t} s^{2w}$ or of $\frac{d\dot{s}}{\overline{1-ss}|^{\frac{1}{2}}} \times v^{2t} s^{2w}$, when $1-ss$ is $=0$ and t and w, whole Numbers be required: Then, by writing 1, for a; $\frac{1}{2}$, for r; $-\frac{1}{2}$, for m; 1, for q; t, for p; and w, for v in the general Expression foregoing, we shall have $d E \times \frac{1.\,3.\,5.\,7 \ldots\ldots \overline{2w-1}}{2.\ 4.\ 6.\ 8.\ \ \ 10.\ 12} \frac{\times 1.\,3.\,5.\,7 \ldots\ldots \overline{2t-1}.}{.14 \ldots\ldots\ 2w+2t.}$ for the Fluent in this Case.

 COROL-

COROLLARY. II.

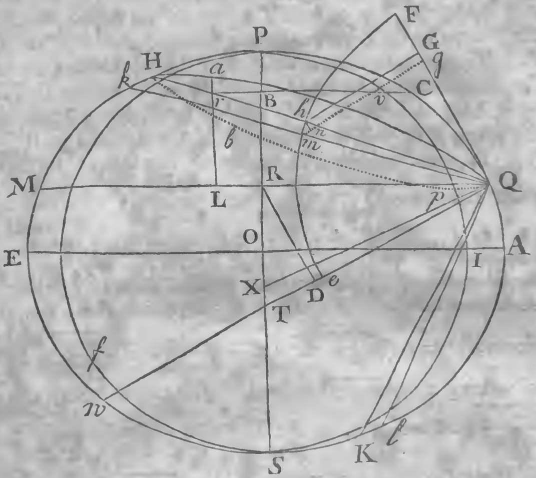
Hence, may the Fluent of $\frac{d\dot{s}}{\overline{1-ss}|^{\frac{1}{2}}} \times \overline{P v^{2n} s^2 + Q v^{2n-2} s^4 + R v^{2n-4} s^6}$, &c. when $1-ss$ is $=0$, and n any whole positive Number, be alſo determined; for let the Value of w (in the laſt Corollary) be ſucceſſively expounded by 1, 2, 3, &c. and that of t, at the ſame time, by n, $n-1$, $n-2$, &c. and then it is evident, the Fluents of $\frac{d\dot{s}}{\overline{1-ss}|^{\frac{1}{2}}} \times v^{2n} s^2$, $\frac{d\dot{s}}{\overline{1-ss}|^{\frac{1}{2}}} \times v^{2n-2} s^4$ &c. will come out $dE \times \frac{1.3.5.7 \ldots \overline{2n-1} \times 1}{2.4.6 \ldots \ldots 2n+2}$, $dE \times \frac{1.3.5 \ldots \overline{2n-3} \times 1.3}{2.4.6 \ldots 2n+2}$, $dE \times \frac{1.3.5 \ldots \overline{2n-5} \times 1.3.5}{2.4.6 \ldots \ldots 2n+2}$ &c. reſpectively; which therefore, being multiplied by their proper Coefficients P, Q, R, &c. and added together, give $dE \times \frac{1.3.5 \ldots \overline{2n-1} \times P + 1.3.5 \ldots \overline{2n-3} \times 3Q}{2.4.6.8 \ldots 2n+2}$ &c. $= \frac{E \times 1.3.5 \ldots \overline{2n-1}}{2.4.6.8 \ldots 2n+2} \times d \times \overline{P + \frac{3Q}{2n-1} + \frac{3.5R}{2n-1 \times 2n-3} + \frac{3.5.7.S}{2n-1 \times 2n-3 \times 2n-5}}$ &c. for the Fluent ſought.

LEMMA IV.

Suppoſing PASEPO to be any Spheroid, generated by the Rotation of an Ellipſis PASE about its leſſer Axis PS; to find the Attraction thereof exerted on a Corpuſcle, at any given Point Q in its Surface.

Let QRL and CBr be perpendicular, and rL parallel to PS; and let the Square of any Ordinate BC of the generating Ellipſis, be to the Square of the correſponding Ordinate Bv of the Circle PvISfP, deſcribed from the ſame Centre O, about the Axis PS, in any given Ratio of 1 + B : to

: to 1; let Q H be the Axis of a Section Q a H b Q of the proposed Solid, formed by the Intersection of a Plane passing through the Point Q perpendicularly to the Plane P A S O of the generating Ellipsis; and let P O $=$ R, O R $= b$, the



Sine of the Angle R Q H to the Radius 1, $= p$, its Cosine $= q$, Q $r = x$, its corresponding Ordinate $r\,a = y$; Q R $(= \sqrt{\overline{1+B} \times R^2 - b^2}) = a$, and R T $(= \overline{1+B} \times b) = A$: Therefore since Q L is $= qx$, and r L $= px$, we have O B $= b + px$, and B $r = qx - a$; whence B C$^2 = \overline{R^2 - bb - 2bpx - p^2x^2} \times \overline{1+B} = aa - 2Apx - p^2x^2 - p^2Bx^2$, and consequently $y^2\ (= BC^2 - Br^2) = 2aqx - 2Apx - p^2x^2 - q^2x^2 - Bp^2x^2$, or, because $p^2 + q^2$ is $= 1$, y^2 is $= \overline{aq - Ap} \times 2x - x^2 - Bp^2x^2$; which Equation being only of two Dimensions, shews the Curve Q a H b Q, whereto it pertains, to be an Ellipsis.

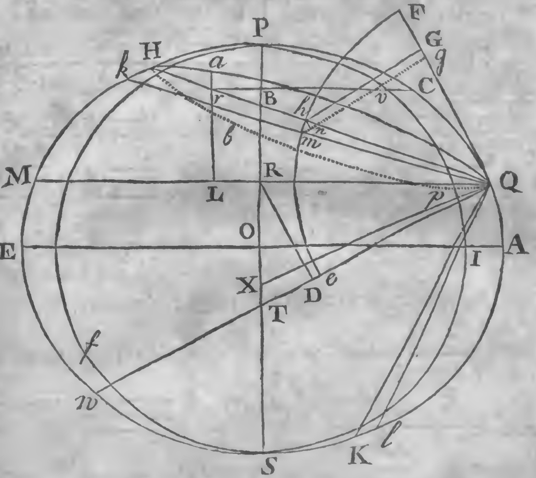
Let

Let now a Plane be ſuppoſed to revolve about Q as a Centre, always continuing perpendicular to the Plane of the generating Ellipſis P Q S E, and let Q b H and Q m k be two Poſitions of that Plane indefinitely near to each other; and ſuppoſing Q F to be a Tangent to the Ellipſis P Q S at the Point Q, Q T perpendicular thereto, and F b an Arch of a Circle whoſe Centre is Q, and Radius Unity; let G b, the Sine of that Arch, be denoted by s, its Coſine ($\sqrt{1-ss}$) by v, and, bm, the Fluxion of that Arch, by $\dot{e}$: Then, ſince the Angle B Q R is the Difference of the two Angles B Q T, R Q T, and the Sine and Coſine of the laſt of theſe two equal reſpectively to $\frac{A}{\sqrt{a^2+A^2}}$ and $\frac{a}{\sqrt{a^2+A^2}}$, we ſhall have $\frac{av-sA}{\sqrt{a^2+A^2}}=$ the Sine of B Q R, and $\frac{as+Av}{\sqrt{a^2+A^2}}=$ its Coſine, by the Elements of Trigonometry; which Values being therefore ſubſtituted inſtead of p and q in the Equation above found, it becomes $y^2=2s\sqrt{a^2+A^2}\times x-x^2-Bx^2\times\frac{\overline{av-sA}|^2}{a^2+A^2}$: Hence by writing $2s\sqrt{a^2+A^2}$ inſtead of f, $\dot{e}$ inſtead of d, and $B\times\frac{\overline{av-sA}|^2}{a^2+A^2}$ inſtead of g, in the Theorem at the End of the ſecond Lemma, we ſhall get $2\dot{e}s\times\overline{a^2+A^2}|^{\frac{1}{2}}\times\overline{\frac{2}{3}-\frac{2.\,4\,B}{3.\,5.}\times\frac{\overline{av-sA}|^2}{a^2+A^2}+\frac{2.\,4.\,6.\,B^2}{3.\,5.\,7}\times\frac{\overline{av-sA}|^4}{\overline{a^2+A^2}|^2}}$ &c. ſhewing the Ratio of the Force wherewith the Corpuſcle at Q is urged in the Direction H Q or k Q, by the Attraction of the Cuneus of Matter included between the two Ellipſes, whoſe Axes are Q H and Q k; from whence, by the Reſolution of Forces, the Attraction of that Cuneus in the Directions Q F and Q T will be had equal to $2\dot{e}sv\times\overline{a^2+A^2}|^{\frac{1}{2}}$

$\times\frac{2}{3}$

$\times \frac{2}{3} - \frac{2.4B}{3.5} \times \frac{\overline{av - sA}|^2}{a^2 + A^2}$ &c. and $2\dot{e}ss \times \overline{a^2 + A^2}|^{\frac{1}{2}}$

$\times \frac{2}{3} - \frac{2.4B}{3.5} \times \frac{\overline{av - sA}|^2}{a^2 + A^2}$ &c. reſpectively: which Expreſſions are, it is manifeſt, as the Fluxions of the whole Force, exerted by the Part Q H P Q of the Solid, in thoſe



Directions. Suppoſe, now, another Plane Q K to revolve about the ſame Point Q, and with the ſame Velocity as the former, but in a contrary Direction, ſo as to meet and coincide with it in the Perpendicular Q T; then v in this Caſe becoming $-v$, the Fluxion of the Attraction of the Part Q A K Q, in the foreſaid Directions Q F and Q T, (by writing $-v$ instead of $+v$) will be $-2\dot{e}sv \times \overline{a^2 + A^2}|^{\frac{1}{2}}$

C $\times \frac{2}{3}$

$\times \frac{2}{3} - \frac{2.4\,B}{3.5} \times \frac{\overline{-av-sA}|^2}{a^2+A^2}$ &c. and $2ss\dot{e} \times \overline{a^2+A^2}|^{\frac{1}{2}}$ $\times \frac{2}{3} - \frac{2.4\,B}{3.5} \times \frac{\overline{-av-sA}|^2}{a^2+A^2}$ &c. reſpectively: Wherefore, if theſe Fluxions be added to thoſe of the former Part in the like Directions, and $\overline{-av-sA}|^2$ be changed to $\overline{av+sA}|^2$, which is equal to it, we ſhall have $2\dot{e}sv \times \overline{a^2+A^2}|^{\frac{1}{2}}$ into $\frac{2.4\,B}{3.5.} \times \frac{\overline{av+sA}|^2 - \overline{av-sA}|^2}{a^2+A^2} - \frac{2.4.6\,B^2}{3.5.7.} \times \frac{\overline{av-sA}|^4 - \overline{av-sA}|^4}{\overline{a^2+A^2}|^2}$ &c. and $2\dot{e}s^2 \times \overline{a^2+A^2}|^{\frac{1}{2}}$ into $\frac{2}{3} - \frac{2.4B}{3.5} \times \frac{\overline{av+sA}|^2 + \overline{av-sA}|^2}{a^2+A^2} + \frac{2.4.6B^2}{3.5.7} \times \frac{\overline{av+sA}|^4 + \overline{av-sA}|^4}{\overline{a^2+A^2}|^2}$ &c. for the Fluxion of both Parts together, in theſe Directions. But, ſince the Triangles Q h G and $m\,h\,n$ are ſimilar, $\dot{e}$ will be to $\dot{s}$ $(= mn)$: as $1 : \sqrt{1-ss}$; therefore, by ſubſtituting $\frac{\dot{s}}{\sqrt{1-ss}}$ inſtead of $\dot{e}$, the foregoing Expreſſions will become $\frac{2v\dot{s}s \times \overline{a^2+A^2}|^{\frac{1}{2}}}{\overline{1-ss}|^{\frac{1}{2}}}$ into $\frac{2.4\,B}{3.5} \times \frac{\overline{av+sA}|^2 - \overline{av-sA}|^2}{a^2+A^2}$ &c. and $\frac{2s\dot{s} \times \overline{a^2+A^2}|^{\frac{1}{2}}}{\overline{1-ss}|^{\frac{1}{2}}}$ into $\frac{2}{3} - \frac{2.4\,B}{3.5} \times \frac{\overline{av+sA}|^2 + \overline{av-sA}|^2}{a^2+A^2}$ &c. The Fluents of which, when s becomes $= 1$, will, it is manifeſt, be as the whole required Forces, whereby the Corpuſcle is urged in the Directions Q F and Q T: Therefore, in order to find thoſe Fluents (which is by far the moſt difficult Part of the Propoſition) let r be put equal to the quadrantal

Arc, F b e, and let $\frac{2vs\dot{s}\times\overline{a^2+A^2}|^{\frac{1}{2}}}{\overline{1-ss}|^{\frac{1}{2}}} \times \frac{2.4.6\ldots\overline{2n+2}\times B^n}{3.5.7\ldots\ldots\overline{2n+3}}$

$\times \frac{\overline{av+sA}|^{2n}-\overline{av-sA}|^{2n}}{\overline{a^2+A^2}|^n}$, which is a general Term to the first of the two Expreſſions, be aſſumed; then the ſame, by expanding $\overline{av+sA}|^{2n}-\overline{av-sA}|^{2n}$ into a Series, &c. will become $4\times\frac{\overline{a^2+A^2}|^{\frac{1}{2}}}{\overline{a^2+A^2}|^n}\times\frac{2.4.6\ldots\overline{2n+2}\times B^n\dot{s}}{3.5.7\ldots\overline{2n+3}\times\overline{1-ss}|^{\frac{1}{2}}}$ into $2na^{2n-1}\times$ $Av^{2n}s^2+\frac{2n}{1}\times\frac{2n-1}{2}\times\frac{2n-2}{3}\times a^{2n-3}A^3v^{2n-2}s^4+\frac{2n}{1}\times$ $\frac{2n-1}{2}\times\frac{2n-2}{3}\times\frac{2n-3}{4}\times\frac{2n-4}{5}\times a^{2n-5}A^5v^{2n-4}s^6$, &c. But it is evident, from *Corol.* II. to *Lemma* III. that the Fluent of this Series (by writing $\frac{4\times\overline{a^2+A^2}|^{\frac{1}{2}}}{\overline{a^2+A^2}|^n}\times\frac{2.4.6\ldots\overline{2n+2}\times B^n}{3.5.7\ldots\overline{2n+3}}$, for d; r for E; $2na^{2n-1}A$ for P; $\frac{2n}{1}\times\frac{2n-1}{2}\times\frac{\overline{2n-2}}{3}$ $\times a^{2n-3}A^3$, for Q, &c.) will be $\frac{r\times1.3.5\ldots\overline{2n-1}}{2.4.6\ldots\overline{2n+2}}\times$ $\frac{4\times\overline{a^2+A^2}|^{\frac{1}{2}}}{\overline{a^2+A^2}|^n}\times\frac{2.4.6\ldots\overline{2n+2}\times B^n}{3.5.7\ldots\overline{2n+3}}$ into $2na^{2n-1}A+\frac{2n}{1}$ $\times\frac{2n-1}{2}\times\frac{2n-2}{3}a^{2n-3}A^3\times\frac{3}{2n-1}+\frac{2n}{1}\times\frac{2n-1}{2}\times$ $\frac{2n-2}{3}\times\frac{2n-3}{4}\times\frac{2n-4}{5}\times a^{2n-5}A^5\times\frac{3\times5}{\overline{2n-1}\times\overline{2n-3}}$ &c. $=$ $\frac{4rB^n}{\overline{a^2+A^2}|^{n-\frac{1}{2}}}$ into $2na^{2n-1}A+2n\times\overline{n-1}\times a^{2n-3}A^3+2n$ $\times\frac{n-1}{1}\times\frac{n-2}{2}a^{2n-5}A^5+2n\times\frac{n-1}{1}\times\frac{n-2}{2}\times\frac{n-3}{3}a^{2n-7}A^7$

&c.

&c. $= \frac{4rB^n \times 2nAa}{\overline{a^2+A^2}|^{n-\frac{1}{2}}}$ into $a^{2n-2} + \overline{n-1} \times a^{2n-4}A^2 + \frac{n-1}{1} \times \frac{n-2}{2} a^{2n-6}A^4$&c. $= \frac{4rB^n \times 2nAa}{\overline{a^2+A^2}|^{n-\frac{1}{2}}} \times \overline{a^2+A^2}|^{n-1} = \frac{4rAa}{\sqrt{a^2+A^2}} \times \frac{2nB^n}{\overline{2n+1} \times \overline{2n+3}}$. Let n be now expounded by 1, 2, 3, &c. successively, then will $\frac{4rAa}{\sqrt{a^2+A^2}}$ into $\frac{2nB^n}{\overline{2n+1} \times \overline{2n+3}}$ become $\frac{4rAa}{\sqrt{a^2+A^2}} \times \frac{2B}{3.5}$, $\frac{4rAa}{\sqrt{a^2+A^2}} \times \frac{4B^2}{5.7}$, $\frac{4rAa}{\sqrt{a^2+A^2}} \times \frac{6B^3}{7.9}$, &c. for the Fluents of the 1st, 2d, 3d, &c. Terms of the foresaid general Expression, respectively; and therefore the required Fluent of that whole Expression, or the Force whereby the Corpuscle is urged in the Direction Q F will be truly defined by $\frac{4raA}{\sqrt{a^2+A^2}}$ drawn into $\frac{2B}{3.5} - \frac{4B^2}{5.7} + \frac{6B^3}{7.9} - \frac{8B^4}{9.11} + \frac{10B^5}{11.13}$ &c. And, in the same Manner, the Force in the Direction Q T will come out $\frac{4ra^2}{\sqrt{a^2+A^2}} \times \overline{\frac{1}{1.3} - \frac{B}{3.5} + \frac{B^2}{5.7} - \frac{B^3}{7.9}}$ &c. $+ \frac{4rA^2}{\sqrt{a^2+A^2}} \times \overline{\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7} - \frac{B^3}{9} + \frac{B^4}{11}}$ &c. Which Values, by Writing $\overline{1+B} \times b$ and $\overline{1+B}|^{\frac{1}{2}} \times \overline{R^2-bb}|^{\frac{1}{2}}$, for their Equals A and a, will become $\frac{4rb \times \overline{1+B}| \times \overline{R^2-bb}|^{\frac{1}{2}}}{\overline{R^2+Bb^2}|^{\frac{1}{2}}} \times \frac{2B}{3.5} - \frac{4B^2}{5.7} + \frac{6B^3}{7.9}$ &c. and $\frac{4r \times \overline{R^2-b^2} \times \overline{1+B}|^{\frac{1}{2}}}{\overline{R^2+Bb^2}|^{\frac{1}{2}}} \times \frac{1}{1.3} - \frac{B}{3.5} + \frac{B^2}{5.7} - \frac{B^3}{7.9}$ &c.

&c. $+\frac{4rb^2 \times \overline{1+B}^{\frac{1}{2}}}{\sqrt{R^2+Bb^2}} \times \overline{\frac{1}{3}-\frac{B}{5}+\frac{B^2}{7}}$, &c. respectively. But, seeing $\overline{\frac{2B}{3 \cdot 5}-\frac{4B^2}{5 \cdot 7}+\frac{6B^3}{7 \cdot 9}-\frac{8B^4}{9 \cdot 11}}$, &c. is $=\frac{1}{2}B^{-\frac{3}{2}} \times \overline{-3B^{\frac{1}{2}}+\overline{3+B} \times \overline{B^{\frac{1}{2}}-\frac{B^{\frac{3}{2}}}{3}+\frac{B^{\frac{5}{2}}}{5}}}$, &c. and $\overline{\frac{1}{1 \cdot 3}-\frac{B}{3 \cdot 5}+\frac{B^2}{5 \cdot 7}-\frac{B^3}{7 \cdot 9}}$, &c. $=\frac{1}{2}B^{-\frac{3}{2}} \times \overline{-B^{\frac{1}{2}}+\overline{1+B} \times \overline{B^{\frac{1}{2}}-\frac{B^{\frac{3}{2}}}{3}+\frac{B^{\frac{5}{2}}}{5}}}$, &c. where $B^{\frac{1}{2}}-\frac{B^{\frac{3}{2}}}{3}+\frac{B^{\frac{5}{2}}}{5}-\frac{B^{\frac{7}{2}}}{7}$, &c. is a known Series expressing the Arch (Q) of a Circle, whose Radius is 1, and Tangent $B^{\frac{1}{2}}$, the said Forces will, it is manifest, be truly defined by $\frac{2rAa}{\sqrt{a^2+A^2}} \times \frac{\overline{3+B} \times Q-3B^{\frac{1}{2}}}{B^{\frac{3}{2}}}$, and $\frac{4rA^2}{\sqrt{a^2+A^2}} \times \frac{B^{\frac{1}{2}}-Q}{B^{\frac{3}{2}}}+\frac{2ra^2}{\sqrt{a^2+A^2}} \times \frac{\overline{1+B} \times Q-B^{\frac{1}{2}}}{B^{\frac{3}{2}}}$ respectively. Q. E. I.

COROLLARY I.

Hence, if RD be made perpendicular to QT, then, QR being $=a$, RT $=$ A, RD will be $=\frac{Aa}{\sqrt{a^2+A^2}}$, QD $=\frac{aa}{\sqrt{a^2+A^2}}$, and DT $=\frac{AA}{\sqrt{a^2+A^2}}$, and consequently the Attraction in the foresaid Directions QF and QT, as (RD) $\times \overline{\frac{2B}{3 \cdot 5}-\frac{4B^2}{5 \cdot 7}+\frac{6B^3}{7 \cdot 9}-\frac{8B^4}{9 \cdot 11}}$, &c. and (QD) $\times \overline{\frac{1}{1 \cdot 3}-\frac{B}{3 \cdot 5}+\frac{B^2}{5 \cdot 7}-\frac{B^3}{7 \cdot 9}}$, &c. + (DT) $\times \overline{\frac{1}{3}-\frac{B}{5}+\frac{B^2}{7}}$, &c. or as (RD) $\times \frac{\overline{3+B} \times Q-3B^{\frac{1}{2}}}{2B^{\frac{3}{2}}}$

 and

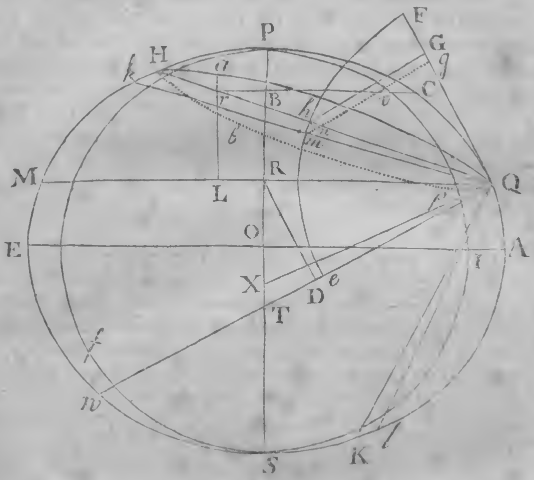
and (QD) $\times \frac{\overline{1+B} \times Q - B^{\frac{1}{2}}}{2 B^{\frac{3}{2}}}$ + (DT) $\times \frac{B^{\frac{1}{2}} - Q}{B^{\frac{3}{2}}}$, reſpectively. Therefore, ſince B is conſtant, it follows, that the Force whereby a Corpuſcle at any Point Q, in the Surface of a given Spheroid, is attracted in the Direction of the Tangent QF, will be ſimply as R D.

COROLLARY II.

If B be taken $= 0$, or the Spheroid be ſuppoſed to degenerate to a Sphere, the Attraction, perpendicular to the Surface, will become as $\frac{1}{3}$ QD + $\frac{1}{3}$ TD; or as $\frac{1}{3}$ of the Radius of that Sphere. Therefore it follows, that the Attraction at any Point Q, in the Surface of a Spheroid PASEP, in the Direction QF, of the Tangent, is to the Attraction at the Surface of a Sphere of any given Radius, as (R D) $\times \frac{\overline{3+B} \times Q - 3 B^{\frac{1}{2}}}{2 B^{\frac{3}{2}}}$ to $\frac{1}{3}$ of that Radius; and moreover, that the Attraction in the perpendicular Direction QT, is to the Attraction at the Surface of the ſame Sphere, as (T D) $\times \frac{B^{\frac{1}{2}} - Q}{B^{\frac{3}{2}}}$ + (QD) $\times \frac{\overline{1+B} \times Q - B^{\frac{1}{2}}}{2 B^{\frac{3}{2}}}$ to $\frac{1}{3}$ of the ſame Radius; or, becauſe (T D) $\times \frac{B^{\frac{1}{2}} - Q}{B^{\frac{3}{2}}}$ + (QD) $\times \frac{\overline{1+B} \times Q - B^{\frac{1}{2}}}{2 B^{\frac{3}{2}}}$ is $=$ (QT) $\times \frac{B^{\frac{1}{2}} - Q}{B^{\frac{3}{2}}}$ + (QD) $\times \frac{\overline{3+B} \times Q - 3 B^{\frac{1}{2}}}{2 B^{\frac{3}{2}}}$, as this laſt Quantity, to $\frac{1}{3}$ of that Radius.

COROLLARY III.

But when the given Spheroid is nearly Globular, B will be very ſmall, and therefore all the Terms in the foregoing Series, wherein two or more Dimenſions of B are concerned may be neglected, as inconſiderable; and then the Attraction in the Directions Q F and Q T, after proper Reduction,



will, in this Caſe, be as $\frac{4bcB}{30R}$ and $\frac{10R^2+3BR^2+Bb^2}{30R}$ reſpectively; from whence it is eaſy to determine, that the Poſition of the Line Q X, wherein the Corpuſcle gravitates or endeavours to deſcend, is ſuch that O x is every where, to O T, as 3 to 5, as Mr. *Stirling* has found.

C O-

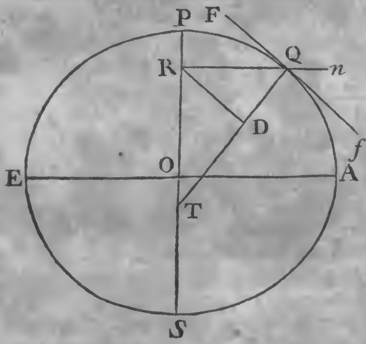
COROLLARY IV.

Hence it follows, that the Attraction at any Point Q, in the Surface of a Spheroid, not differing much from a Sphere, is to the Attraction of a Sphere upon the ſame Axis, as $10\,R^2 + 3\,BR^2 + Bb^2$, to $10\,R^2$ nearly. It alſo follows, that the Attraction of ſuch a Spheroid in going towards the Poles, increaſes or decreaſes in the duplicate Ratio of the Sine-Complement of the Diſtance from the Pole; and that at the Poles themſelves (where in an oblate Spheroid it is the greateſt, and in an Oblong the leaſt) it will be to the Attraction of a Sphere, having the ſame Axis as 4 Times the Diameter of the greateſt Circle of that Spheroid, increaſed by the Axis, to 5 Times that Axis; and laſtly, that the greateſt Difference of Attraction, on the Surface of ſuch a Spheroid, will be to the Difference between the Attraction at its Pole, and at the Surface of the foreſaid Sphere, as 1 to 4 very nearly.

PROPOSITION I.

If a Fluid or Body of homogeneous Matter, whoſe Particles are freely diſpoſed to move, and mutually attract each other in the duplicate Ratio of their Diſtances inverſely, revolves about an Axis, and all the Parts thereof retain the ſame Situation, with reſpect to each other; I ſay, the Form which that Fluid muſt be under, to preſerve this Equilibrium of its Parts, is that of an oblate Spheroid.

For, let PS be the Axis about which the propoſed Fluid PASEP revolves, and QT a Perpendicular to the Surface at any Point Q, making QR, and RD perpendicular to PS and QT, and F Q *f*, parallel to RD. Therefore, ſince the abſolute centrifugal Force, whereby a Corpuſcle at Q endeavours to recede from the Centre R, in the Direction Qn, is known to be as R Q, that Part of it by which the Corpuscle is urged in the Direction Qf, of the Tangent, or tends to slide along the Surface, will, by the Resolution of Forces,



be as R D. Therefore, as all the Particles remain quiescent with Regard to each other, the Attraction exerted on the Corpuscle in the contrary Direction Q F, to preserve this Equilibrium, must, it is manifest, be in the same Ratio of R D; but the Attraction of an oblate Spheroid, in this Direction appears, from *Corol.* I. to *Lem.* IV. to be as R D: therefore the Figure P Q A S E A, is an oblate Spheroid. Q. E. D.

PROPOSITION II.

THE same being supposed as in the last Proposition; and the time of Revolution, the Attraction at the Surface of the Fluid, when at Rest under a spherical Figure, together with the Diameter of that Sphere being given; to find the particular Spheroid which the Fluid retains by means of that Rotation, and also the Gravitation at any Point Q in the Surface thereof.

The foregoing Construction being retained, let the time of Rotation be denoted by m, and let the given Attraction at the Surface of the proposed Fluid, when at rest under the Form of a Sphere, be such, that a Projectile or revolving Body may thereby describe a circular Orbit, whose Radius is equal to the Radius of that Sphere, in a given Time n: Putting $PO = R$, $RO = b$, the Attraction at the Surface of the proposed Sphere $= f$, the Semi-Diameter of that Sphere $= d$, and the Proportion of the Square of the equatoreal Diameter AE, to the Square of the Axis PS, as $1 + B$ to Unity. Then, since the centrifugal Forces of equal Bodies, moving in Circles, are known to be universally as the Radii of those Circles, applied to the Squares of the Times of Revolution, we shall have as $\frac{d}{n^2} : \frac{(RQ)}{m^2} :: f : \frac{fn^2}{dm^2} \times (RQ)$ the Force with which a Particle of Matter at Q, thro' the Rotation of the Fluid, endeavours to recede from the Centre R, in the Direction Qn; from whence, by the Resolution of Forces, the Forces in the Directions QT and QF, arising from the same Cause, will be $-(QD) \times \frac{n^2 f}{dm^2}$, and $-(RD) \times \frac{n^2 f}{dm^2}$ respectively. But the Attraction in these two Directions, supposing Q to be the Arch of a Circle, whose Radius is 1, and

and Tangent $B^{\frac{1}{2}}$, will be to (f) the Attraction at the Surface of the Sphere, whose Semidiameter is d, as $(QT) \times \frac{B^{\frac{1}{2}} - Q}{B^{\frac{3}{2}}} + (QD) \times \frac{\overline{3+B} \times Q - 3B^{\frac{1}{2}}}{2B^{\frac{3}{2}}}$ to $\frac{d}{3}$, and as $(RD) \times \frac{\overline{3+B} \times Q - 3B^{\frac{1}{2}}}{2B^{\frac{3}{2}}}$ to $\frac{d}{3}$ respectively, by Corollary II. to the preceding Lemma: And therefore the whole compounded Force, whereby a Corpuscle at Q is urged in these Directions will be rightly defined by $\frac{3f}{d} \times \overline{(QT) \times \frac{B^{\frac{1}{2}} - Q}{B^{\frac{3}{2}}} + (QD) \times \frac{\overline{3+B} \times Q - 3B^{\frac{1}{2}}}{2B^{\frac{3}{2}}}} - (QD) \times \frac{n^2}{3m^2}$, and $\frac{3f}{d} \times \overline{(RD) \times \frac{\overline{3+B} \times Q - 3B^{\frac{1}{2}}}{2B^{\frac{3}{2}}}} - (RD) \times \frac{n^2}{3m^2}$. The last of which Expressions, that the Corpuscle may remain at rest, and all the Parts of the Fluid in Equilibrio, must, it is manifest, be equal to nothing; therefore $\frac{\overline{3+B} \times Q - 3B^{\frac{1}{2}}}{2B^{\frac{3}{2}}}$ is $= \frac{n^2}{3m^2}$, and consequently the Gravitation, or absolute Force in the perpendicular Direction QT, as $(QT) \times \frac{B^{\frac{1}{2}} - Q}{dB^{\frac{3}{2}}} \times f$, or barely as $(QT) \times \frac{B^{\frac{1}{2}} - Q}{B^{\frac{3}{2}}}$; from whence, by help of a Table of Sines and Tangents, *&c.* not only the Value of B, but the Gravitation answering to any assigned Value of $\frac{n^2}{m^2}$ may readily be determined. But when $\frac{n^2}{m^2}$ is small, the same Things may be effected in a more general Manner; for then our Equation $\frac{\overline{3+B} \times Q - 3B^{\frac{1}{2}}}{2B^{\frac{3}{2}}} = \frac{n^2}{3m^2}$ may be reduced to $\frac{2B}{3 \cdot 5} - \frac{4B^2}{5 \cdot 7} + \frac{6B^3}{7 \cdot 9}$, *&c.* $= \frac{n^2}{3m^2}$, where, the Series converging sufficiently swift, B will be found $= \frac{5n^2}{2m^2} + \frac{25 \times 6n^4}{4 \times 7m^4} + \frac{125 \times 37n^6}{8 \times 49m^6}$, *&c.* or $\frac{35n^2}{14m^2 - 30n^2}$ very nearly. Therefore the Ratio of the equatoreal Diameter to the Axis will, in this Case, be as $\overline{1+\frac{35n^2}{14m^2-30n^2}}\big|^{\frac{1}{2}}$ to 1, or, if $\frac{n^2}{m^2}$ be very small, barely as $1+\frac{5n^2}{4m^2}$ to Unity, the same as Sir *Isaac Newton* and Mr. *Stirling* have made it. Q. E. I.

COROLLARY I.

Because $\frac{\overline{3+B}\times Q-3B^{\frac{1}{2}}}{2B^{\frac{3}{2}}}$ (the Left-hand-side of our foregoing Equation) as appears from the Nature of the Expression, can never (let B be what it will) exceed a certain assignable Quantity, it is manifest that if $\frac{n^2}{3m^2}$ be so given, as to exceed that Quantity, the Problem will become impossible. Wherefore, to determine this Limit, let $B^{\frac{1}{2}}=x$, and the Fluxion of $\frac{\overline{3+x^2}\times Q-3x}{x^3}$ $\left(=\frac{\overline{3+B}\times Q-3B^{\frac{1}{2}}}{B^{\frac{3}{2}}}\right)$ which is

$$\frac{2xQ+\overline{3+x^2}\times\overline{1+xx}\big|^{-1}}{x^6}\times x^3-\frac{3x^3-9x^3+3x^2\,\overline{3+x^2}\times Q}{x^6},$$

be put $=0$; and we shall get $9x+7x^3-\overline{1+x^2}\times\overline{9+x^2}\times Q=0$, where x is found $=2.5293$; whence the corresponding Values of $\frac{n}{m}$ and $\sqrt{1+B}$, come out 0.58053, &c. and 2.7198, &c. respectively. Hence it appears, that it is impossible for the Parts of the Fluid to continue at Rest among themselves, when the Motion round the Axis is so great, that $\frac{n}{m}$ exceeds 0.58053, &c. or, that any Spheroid should be assumed whereof the Ratio of the equatoreal Diameter to the Axis is greater than that of 2.7198 to Unity. But if the Motion be greater than is here specified, the Fluid will contract its Axis, and continue rising higher and higher towards

towards the Equator, till, by increaſing its equatoreal Diameter and Time of Revolution, the Parts thereof either come to an equilibrium, or begin to fly off.

COROLLARY II.

If, inſtead of the time of Revolution, the Quantity of Motion of the Fluid about its Axis be given, ſo as to be to the Quantity of Motion in a ſolid Sphere of the ſame Maſs and Denſity, revolving in the forementioned Time n, in any given Ratio of r to s; then, becauſe the Quantities of Motion in equal Spheroids of the ſame Denſity about their Axis, are to one another in a Ratio compounded of the direct Ratio of the Radii of their greateſt Circles, and the inverſe Ratio of the Times of their Revolution, we ſhall have as $\frac{d}{n} : \frac{(AO)}{m} :: s : r$, and conſequently $AO = \frac{rmd}{ns}$. But (becauſe the Maſſes are equal) $\frac{AO^3}{\overline{1+B}|^{\frac{1}{2}}} (PO \times AO^2)$ is $= d^3$, and therefore $\frac{mrd}{ns} (=AO) = d \times \overline{1+B}|^{\frac{1}{6}}$; whence $m = n \times \frac{s \times \overline{1+B}|^{\frac{1}{6}}}{r}$, and $\frac{n^2}{3m^2} = \frac{r^2}{3s^2 \times \overline{1+B}|^{\frac{1}{3}}}$ which Value being ſubſtituted for $\frac{n^2}{3m^2}$ in the foregoing general Equation, we have $\frac{\overline{3+B} \times Q - 3B^{\frac{1}{2}}}{2B^{\frac{3}{2}}} = \frac{r^2}{3s^2 \times \overline{1+B}|^{\frac{1}{3}}}$, and therefore $\frac{\overline{\overline{3+B} \times Q - 3B^{\frac{1}{2}}} \times \overline{1+B}|^{\frac{1}{3}}}{B^{\frac{3}{2}}} = \frac{2r^2}{3s^2}$; from whence it will be eaſy to determine the Spheroid which a Fluid, whoſe Particles are at Reſt among themſelves, muſt aſſume when the Motion about its Axis is increaſed or decreaſed in any given Ratio; becauſe the abſolute Motion after ſuch Increaſe or Decreaſe is given, and will be no ways affected by the Action of the Particles upon one another while the Figure of the Fluid is changing.

COROLLARY III.

But (ſince $\frac{\overline{\overline{3+B}\times Q-3B^{\frac{1}{2}}\times\overline{1+B}}\,^{\frac{1}{3}}}{B^{\frac{3}{2}}}$, both when B is nothing and infinite, will be $=0$) it is evident that the Value of $\frac{\overline{\overline{3+B}\times Q-3B^{\frac{1}{2}}\times\overline{1+B}}\,^{\frac{1}{3}}}{B^{\frac{3}{2}}}$ can never, let B be what it will, exceed a certain finite Quantity; and therefore if the given Motion be ſuch that $\frac{2r^2}{3s^2}$ exceeds that Quantity, it will be impoſſible for the Parts of the Fluid ever to become quieſcent with regard to each other: Wherefore to determine this Limit, let x be put $=B^{\frac{1}{2}}$, and the Fluxion of $\frac{\overline{\overline{3+x^2}\times Q-3x\times\overline{1+x^2}}\,^{\frac{1}{3}}}{x^3}$ be taken and made $=0$, and the Equation, duly ordered, will be $\overline{x^4+24x^2+27}\times Q-15x^3-27x=0$; where x will come out $=7.5$ very nearly, and the correſponding Value of $\frac{r}{s}$ $=0.92705$. Hence it appears, that the Particles cannot poſſibly come to an Equilibrium among themſelves, when the Motion round the Axis is ſo great, that $\frac{r}{s}$ exceeds 0.92705, but will either fly off or continue to recede from the Axis *in Infinitum.*

COROLLARY IV.

Becauſe the Values of B and Q, when $\frac{n}{m}$ is given, are alſo given, it follows that the Gravitation $(QT)\times\frac{B^{\frac{1}{2}}-Q}{B^{\frac{3}{2}}}$, at any Point in the Surface of the given Spheroid or Fluid, will be, accurately, as a Perpendicular to the Surface at that Point, produced till it meets the Axis of the Figure. Therefore the Gravi-

Gravitation or Force wherewith a Corpuſcle tends to deſcend at the Equator, is to the Gravitation at either of the Poles, as the equatoreal Diameter to the Axis inverſely.

COROLLARY V.

Hence, if the Spheroid be nearly globular, then QT, which by the Property of the Ellipſis, is univerſally equal to $\overline{1+B}|^{\frac{1}{2}} \times \overline{R^2+Bb^2}|^{\frac{1}{2}}$, will here become $\overline{1+B}|^{\frac{1}{2}} \times \overline{R+\frac{Bb^2}{2R}}$, nearly. Whence it appears that the Increaſe of Gravitation from the Equator to the Pole, is in the Duplicate Ratio of the Sine Complement of the Diſtance from the Pole very nearly.

COROLLARY VI.

Moreover, becauſe the Ratio of the equatoreal Diameter to the Axis, when the Spheroid is nearly globular, becomes nearly as $1+\frac{5n^2}{4m^2}$ to 1, the Exceſs of that Diameter above the Axis, will, it is evident, be to the Axis as $\frac{5n^2}{4}$: to m^2, or (becauſe the Forces by which Bodies are retained in equal Circles, are in the duplicate Ratio of the Times inverſely) as $\frac{5}{4}$ of the centrifugal Force at the Equator to the mean Force of Gravity. Therefore, ſince the Ratio of the centrifugal Force, in different Circles, is compounded of the direct Ratio of the Diameter, and the inverſe-duplicate Ratio of the Time, it follows that the foreſaid Exceſs, in Figures nearly ſpherical, will be as the Diameter directly, and the Denſity and Square of the time of Revolution inverſely.

A TABLE *shewing the Time of Revolution, and the Momentum of Rotation of a Planet or given Fluid, according to the Ratio of its Axis and equatoreal Diameter.*

1: 1,01	11,236 *n*	0,08925
1: 1,05	5,137 *n*	0,1978
1: 1,5	2,056 *n*	0,5568 *s*
1: 2	1,814 *n*	0,6944 *s*
1: 4	1,810 *n*	0,8774 *s*
1: 10	2,338 *n*	0,9216 *s*
1: 20	3,110 *n*	0,8728 *s*
1: 40	4,275 *n*	0,8000 *s*
1: 100	6,600 *n*	0,7033 *s*
1: 1000	20,640 *n*	0,4845 *s*

Note. The firſt Column towards the Left-hand ſhews the Ratio of the Axis and equatoreal Diameter, the ſecond the Time of Revolution, and the third, the correſponding Momentum of Rotation, *n* being put for the Time in which a revolving Body or Satellite would deſcribe a circular Orbit juſt above the Surface of the Planet or Fluid, when at Reſt under a ſpherical Figure, and *s* for the Momentum of Rotation in an equal Sphere of the ſame Denſity, revolving about its Axis in that Time.

SCHOLIUM.

If the above Concluſions be made uſe of to determine the Ratio of the equatoreal Diameter and Axis, and the Variation of Gravitation at the Surface of the Earth, the Time *n* (in which a revolving Body would deſcribe a circular Orbit about the Earth, juſt above its Surface, by means of its own Gravity) will, it is known, be about $84\frac{3}{4}$ Minutes, and the Value

 of

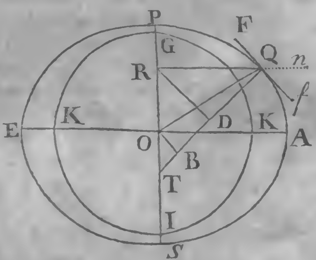
of *m* (one entire Revolution of the Earth about its Axis) 1436 Minutes; therefore by writing these Values in the Ratio of $\overline{1+\frac{35n^2}{14m^2-30n^2}}|^{\frac{1}{2}}$: 1 (as above found) it will become as 1.00435 : 1, or as 231 : 230 for the Ratio of the equatoreal Diameter and Axis of the Earth. Wherefore, as the former of these is about 8000 Miles, it must exceed the latter by $34\frac{1}{2}$ Miles, and the Gravitation at the Equator will be to the Gravitation at the Poles as 230 to 231. Hence it will not be difficult to determine how much Pendulum Clocks are accelerated or retarded from the Alteration of Gravitation when transported into different Latitudes; for the number of Vibrations performed by a given Pendulum, in any given Time being in the Sub-duplicate Ratio of the Force by which it is actuated, we have as $\sqrt{230} : \sqrt{231}$; or as 460 : 461; so is the Number of Vibrations of any Pendulum at the Equator, in any given Time, to the number of Vibrations of an equal Pendulum at either of the Poles in the same Time. Hence it will be as 460 : 1 : : so is 86400, the Seconds in 24 Hours : to 188, the Seconds which a Clock would gain *per Diem* (from the Cause under Consideration) when removed from the Equator to either of the Poles; and therefore, since it is proved that the Gravitation increases as the Square of the Sine of the Latitude, the Time which a Pendulum will gain or lose *per Diem*, by being transported out of any one given Latitude to another, is to 188 Seconds as the Difference of the Squares of the Sines of those Latitudes to the Square of the Radius.

The above Proportions, as likewise that of the Axis and equatoreal Diameter, are derived from a Supposition that all the Matter in the Earth is homogeneous (or nearly so;) but if the Parts next the Centre should be much denser than those nearer the Surface, the Conclusions will be pretty much affected thereby, as will appear from the following Propositions.

LEMMA.

I*N a Spheroid* ASEPA *nearly globular, whoſe Denſity about the Surface is every where nearly equal, but in the lower Parts thereof greater, according to any Law of the Diſtances from the Centre, if the Exceſs of its Quantity of Matter above the Quantity of Matter which it would contain, were all its Parts only of the ſame Denſity with thoſe near the Surface, be to this laſt ſpecified Quantity of Matter in any given Ratio of* p *to* 1; *'tis required to find the Attraction at any Place* Q *in the Surface of ſuch Spheroid.*

The foregoing Conſtruction being retained, join QO, and draw OB parallel to RD; then, ſince the Attraction which a Sphere, whoſe Denſity at equal Diſtances from the Centre is the ſame, exerts on a Corpuſcle above its Surface, is known to be as the Quantity of Matter in that Sphere apply'd to the Square of the Diſtance from its Centre, it is manifeſt, that if the Attraction at the Surface of a Sphere, whoſe uniform



Denſity is defined by Unity, be repreſented by $\frac{1}{3}$ of the Radius (as in the laſt Propoſition) the Attraction of the

 fore-

foresaid Excess of Matter, on a Corpuscle at Q, will be represented by $\frac{p \times OP \times AO^2}{3 \times OQ^2}$, or by $\frac{pR^3 \times \overline{1+B}}{3 \times \overline{R^2 + BR^2 - Bb^2}}$; or lastly, by $\frac{pR}{3} + \frac{pBb^2}{3R}$ nearly. Whence, by the Resolution of Forces, the Attraction of the said Matter, in the Directions QT and QF, will be $\frac{pR}{3} + \frac{pBb^2}{3R}$ and $\frac{1}{3}p \times (BO)$ nearly; which being therefore respectively added to $\frac{10R^2 + 3BR^2 + Bb^2}{30R}$ and $\frac{4B}{30} \times (RD)$ the Attraction in the same Directions, of the Spheroid considered as homogeneous, (see *Corol.* III. *Lem.* IV.) there will arise $\frac{pR}{3} + \frac{pBb^2}{3R} + \frac{R}{3} + \frac{BR}{10} + \frac{Bb^2}{30R}$, and $\frac{1}{3}p \times (BO) + \frac{2B}{15} \times (RD)$ for the whole Forces whereby the Corpuscle is urged in those Directions; but OB being to RD, as OT to TR, or as B to 1+B, the latter of these Forces will be as $(RD) \times \frac{pB}{3} + \frac{2B}{15}$ very nearly.

Q. E. I.

PROPOSITION III.

If a Fluid nearly globular, whose Density about the Surface is every where nearly equal, but in the lower Parts thereof, greater according to any Law of the Distances from the Centre, be revolving uniformly about an Axis; I say, the Figure of that Fluid under such a Rotation, is that of an oblate Spheroid nearly.

The Truth of this is manifest from the first Proposition and the preceding Lemma; for, since the Attraction of a Spheroid, whose Density varies according to the same Law, is, in the

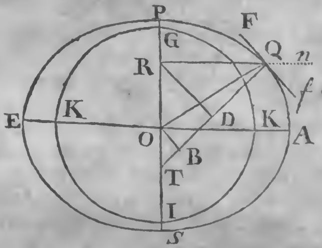
the Direction of the Tangent QF, nearly as RD, by the Lemma, what hath been proved in that Propoſition, with regard to an uniform Fluid, holds alſo in this Caſe.

PROPOSITION IV.

THE ſame being ſuppoſed as in the laſt Propoſition, and the Ratio of the centrifugal Force at the Equator AE, *to the Gravity being given (as* r : 1*); to find the Ratio of the equatorial Diameter to the Axis of the Spheroid or Fluid, and alſo the Gravitation at any Point* Q *in the Surface thereof.*

Let the ſame Conſtruction be ſtill retained: Then, ſince the abſolute centrifugal Force at Q, referred to the Centre R, is known to be as RQ, the Forces ariſing therefrom in the Directions Qf and QT, will, it is manifeſt, be to the Force of Gravity as (RD) $\times \frac{r}{R}$, to 1, and as (QD) $\times \frac{-r}{R}$ to 1 reſpectively. Wherefore it will be, as (RD) $\times \frac{r}{R}$: 1 : : ſo is (RD) $\times \overline{\frac{pB}{3} + \frac{2B}{15}}$ the Attraction in the Direction QF (*per Lemma*) to $\frac{pR}{3} + \frac{pBb^2}{3R} + \frac{R}{3} + \frac{BR}{10} + \frac{Bb^2}{30R}$, that in the Direction QT; whence by multiplying Extreams and Means, and rejecting all the Terms where more than one Dimenſion of B is found as inconſiderable (becauſe the Spheroid is ſuppoſed nearly globular) we ſhall get $B = \frac{5r \times \overline{1+p}}{2+5p}$; and conſequently the Proportion of the equatoreal Diameter to the Axis, as $1 + \frac{5r \times \overline{1+p}}{4+10p}$ to Unity. Moreover, by ſubſtituting this Value of B, in the Expreſſion for the Attraction in the Direction QT, we have $\frac{\overline{1+p} \times R}{3} + \frac{5r \times \overline{1+p}}{2+5p} \times \overline{\frac{pb^2}{3R} + \frac{R}{10} + \frac{b^2}{30R}}$, from

from which deducting $\overline{\frac{Rr}{3} - \frac{rb^2}{3R}} \times \overline{p+1}$ (= $QD \times \frac{r}{R} \times \frac{\overline{1+p}\times R}{3}$, &c.) the centrifugal Force in the opposite Direction,



there remains $\frac{\overline{1+p}\times R}{3} + \overline{1+p}\times r \times \frac{-R^2+5b^2-10pR^2+20pb^2}{6R\times\overline{2+5p}}$ for the Gravitation. Q. E. I.

COROLLARY I.

Hence it appears that the Gravitation, in going towards the Pole, increases as the Square of the Sine of the Latitude, and that the greatest Difference thereof, at the Pole and Equator, is to the centrifugal Force at the Equator, as $5+20p$: to $4+10p$. It also appears, that the greater the Density is towards the Centre, with Respect to that at the Surface, the nearer will the Figure approach to a Sphere, and the greater will be the Difference of the Gravitation at the Equator and Pole; and that if p be conceived to become infinite, or the Attraction to tend to the Centre of the Fluid only, and not to all the Parts thereof as some have supposed (with respect to the Earth) the Difference of Gravitation at the Pole and Equator, will be equal to twice the centrifugal Force at the Equator, and the Ratio of the equatoreal Diameter to the Axis of the Earth, only as 579 to 578.

COROLLARY II.

If the Ratio of the equatoreal Diameter to the Axis be given as $1+v$ to 1, there will be given $1+\frac{5r \times \overline{1+p}}{4+10p}=1+v$, and consequently $p=\frac{5r-4v}{10v-5r}$.

SCHOLIUM.

The Ratio of the greateſt and leaſt Diameters of *Jupiter* is, according to Mr. *Pound*'s Obſervations, as 13 to 12, and the centrifugal Force at the Equator of *Jupiter*, to the mean Force of Attraction, as 1 to 10; therefore, the Quantity of Matter in that Planet, will, according to the foregoing Hypotheſis, be greater by juſt one half, than it would if the Denſity was not greater towards the Centre, than it is nearer the Surface. There might, indeed, be other Hypotheſes aſſumed, that would bring out the Concluſions a little different, but as no Hypotheſis, for the Law of Variation of Denſity, can (from the Nature of the Thing) be verified either by Experiments, made on Pendulums in different Latitudes, or an actual Menſuration of the Degrees of the Meridian, I ſhall inſiſt no further on this Matter, but content myſelf with having proved in general, that the greater the Denſity is towards the Centre, the leſs will the Planet differ from a Sphere, and the greater will be the Variation of Gravitation at its Surface.

A GENERAL

INVESTIGATION

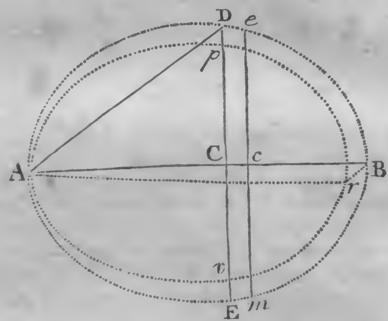
OF THE

ATTRACTION at the SURFACES of BODIES nearly ſpherical.

LEMMA.

SUppoſing the Planes of two Curves ABDEA, AprvA, *nearly circular, having both the ſame Equation* $y^2 = fx - x^2 + gx^2 + hx^3 + ix^4$, *&c. to be inclined to each other at their common Vertex* A, *in an indefinitely ſmall Angle* BAr, *ſo as to include between them the indefinitely ſmall Cuneus of uniformly denſe Matter* ADBEprvA; *to find the Attraction of that Cuneus exerted on a Corpuſcle at* A, *or the Ratio of the Force by which that Corpuſcle is urged in the Direction* BA.

Since the Equation of either Curve is $y^2 = fx - x^2 + gx^2 + hx^3 + ix^4$, &c. by putting $f - x + gx + hx^2$, &c. $= 0$ and reverting the Series, we ſhall get $x = \frac{f}{1-g} + \frac{hf^2}{\overline{1-g}|^3}$, &c. $= f + fg + fg^2$, &c. $+ hf^2 + 3hf^2g$, &c. equal to the Axis AB. But the Curve being ſuppoſed nearly circular, and the Equation of the Circle agreeing, in Curvature, with it at the Vertex being $fx - xx$, the reſt of the Terms gx^2, ix^3, kx^4, in the given Equation, muſt be ſmall in reſpect of the two firſt; and therefore all the Terms wherein two or more Dimenſions of the Quantities, g, i, k, &c. are found may be rejected, as inconsiderable, and then A B will become $= f + fg + hf^2 + if^3 + kf^4$, &c. which let be represented by a, and let D C and mc represent any two Ordinates indefinitely near to each other: Putting B C $(a-x) = z$, $Cc = \dot{z}$, and the Sine of the given Angle formed by the two Planes, to the Radius 1, $= e$; then it will be as $1 : e :: a-z : e \times \overline{a-z}$, the Distance of the Planes at the Ordinate D C, or the Thickness of the proposed Matter at that Ordinate; which drawn therefore into



$\frac{DC \times Cc}{AC \times AD}$, expressing the Force of the Particles in the Surface D C c m D (*vid.* p 1. gives $\frac{ey\dot{z}}{\sqrt{\overline{a-z}^2 + y^2}}$ for the Attraction of the Matter included between the Ordinates DC and mc, which, by writing $\overline{fx - x^2 + gx^2 + hx^3}, {}^{\frac{1}{2}}$&c. for its equal y, becomes $\frac{e\dot{z}x^{\frac{1}{2}} \times \overline{f - x + gx + hx^2 + ix^3, \&c.}|^{\frac{1}{2}}}{\overline{\overline{a-z}|^2 \times x \times \overline{f - x + gx + hx^2 + ix^3, \&c.}}|^{\frac{1}{2}}}$; wherein if $a - z$ and $a - ga - ha^2 - ia^3 - ka^4$, &c. be respectively substituted, for their Equals x and f, it will become

$e\dot{z}$

$$\frac{e\dot{z}\times\overline{a-z}^{\frac{1}{2}}\times\overline{a-ga-ha^{2},\&c.-a+z+g\times\overline{a-z}+h\times\overline{a-z}^{2},\&c.}^{\frac{1}{2}}}{\overline{\overline{a-z}^{2}+\overline{a-z}\times\overline{a-ga-ha^{2},\&c.-a+z+g\times\overline{a-z}+h\times\overline{a-z}^{2},\&c.}}^{\frac{1}{2}}}=$$

$$\frac{ez^{\frac{1}{2}}\dot{z}\times\overline{1-g-h\times\overline{2a-z}-i\times\overline{3aa-3az+z^{2}},\&c.}^{\frac{1}{2}}}{\overline{a-gz-hz\times\overline{2a-z}-iz\times\overline{3aa-3az+z^{2}},\&c.}^{\frac{1}{2}}}$$, which, being converted to an infinite Series, at length becomes $\frac{ez^{\frac{1}{2}}\dot{z}}{2a^{\frac{3}{2}}}\times\overline{2a+\overline{z-a}\times g+\overline{z-a}\times h\times\overline{2a-z}+\overline{z-a}\times i\times\overline{3aa-3az+zz},\&c.}$ The Fluent whereof, when z is $=a$, will be $ea\times\overline{\frac{2}{3}-\frac{2g}{1.3.5.}-\frac{22ha}{3.5.7.}-\frac{82ia^{2}}{5.7.9.},\&c.}$ where, if the Value of a, as above found, be ſubſtituted, there will ariſe $ef\times\overline{\frac{2}{3}+\frac{2.4g}{3.5.}+\frac{2.4.6hf}{3.5.7.}+\frac{2.4.6.8if^{2}}{3.5.7.9.}+\frac{2.4.6.8.10kf^{3}}{3.5.7.9.11.},\&c.}$ Q. E. I.

PROPOSITION.

Suppoſing PASEPO *to be a Solid nearly ſpherical, generated by the Rotation of any oval Figure* PAS, *whoſe Equation is included in this general Form* $y^2=a^2-Az-z^2-Bz^2-Cz^3-Dz^4$, *&c. To find the attractive Force of that Solid exerted on a Corpuſcle, at any given Point* Q *in its Surface.*

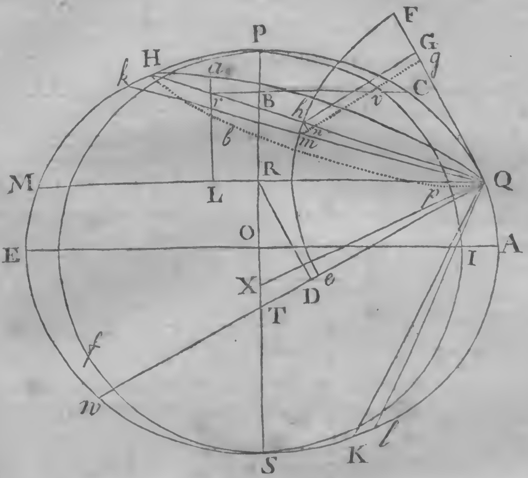
Let QRL and CBr be perpendicular, and rL parallel to the Axis PS, about which the Solid is generated; and let QH be the Axis of any Section, QaHbQ of that Solid, formed by the Interſection of a Plane paſſing thro' the given Point Q perpendicularly to the Plane PASO of the generating Curve: Putting RQ $=a$, RB $=z$, BC $=y$, the Sine of the Angle RQH, to the Radius $1=p$, its Coſine $=q$, Q$r=x$, and its correſponding Ordinate $ra=u$. Then, by plain Trigonometry, we ſhall have QL $=qx$, and rL$=px=z$;

 which

which Value of z being ſubſtituted in the Equation of the given Curve, it will become $y^2\ (=BC^2) = a^2 - Apx - p^2x^2 - Bp^2x^2 - Cp^3x^3$, &c. whence $u^2\ (=BC^2 - Br^2) = a^2 - Apx - p^2x^2 - Bp^2x^2 - Cp^3x^3$, &c. $- aa + 2aqx - q^2x^2 = \overline{2aq - Ap} \times x - \overline{1 + Bp^2} \times x^2 - Cp^3x^3 - Dp^4x^4$, &c. Let now a Plane be conceived to revolve about the Point Q as a Centre, continuing always perpendicular to the Plane PAS of the generating Curve; and let QhH, and Qmk, be two Poſitions of that Plane indefinitely near to each other; and, ſuppoſing Fh to be an Arch of a Circle whoſe Centre is Q, and Semi-diameter Unity, let hm, the Fluxion of that Arch be denoted by e: Then by writing $2aq - Ap$ for f, $-Bp^2$ for g, $-Cp^3$ for h, &c. in the above *Lemma*, we ſhall have e into $\frac{2 \times \overline{2aq - Ap}}{3} - \frac{2 \cdot 4 Bp^2 \times \overline{2aq - Ap}}{3 \cdot 5} - \frac{2 \cdot 4 \cdot 6 Cp^3 \times \overline{2aq - Ap}^2}{3 \cdot 5 \cdot 7} - \frac{2 \cdot 4 \cdot 6 \cdot 8 Dp^4 \times \overline{2aq - Ap}^3}{3 \cdot 5 \cdot 7 \cdot 9}$, &c. for the Force wherewith the Corpuſcle at Q is impelled in the Direction QH by the Attraction of the Cuneus of Matter included between the two Sections QH and Qk. But, to reduce this Expreſſion to a more commodious Form, let QF be a Tangent to the generating Curve at the Point Q, and QT perpendicular to it, and let the Sine Gb, of the Angle BQF $= s$, and its Coſine QG $= v$: Therefore, ſince the Fluxion of the Ordinate BC, when BR or z is $= 0$, is to the Fluxion of PB as $\frac{A}{2a}$ to Unity, the Tangent of the Angle RQT will be $= \frac{A}{2a}$; conſequently its Sine $= \frac{A}{\sqrt{4aa + A^2}}$ and its Coſine $= \frac{2a}{\sqrt{4aa + A^2}}$: Wherefore, as the Angle BQR is the Difference of the two Angles BQT, RQT, the Sine of that Angle will be $\frac{2av - sA}{\sqrt{4aa + A^2}}$, and its Coſine $\frac{2as + vA}{\sqrt{4aa + A^2}}$ (by the Elements of Trigonometry) which Values being there-

fore

fore respectively substituted for p and q in the foresaid Expression of the Force, it will become $\frac{2es \times \overline{4aa+AA}|^{\frac{1}{2}}}{3} - \frac{2.4esB \times \overline{2av-sA}|^{2}}{3.5 \times \overline{4aa+AA}|^{\frac{1}{2}}} - \frac{2.4.6es^{2}}{3.5.7.} \times \frac{C \times \overline{2av-sA}|^{3}}{\overline{4aa+AA}|^{\frac{1}{2}}} - \frac{2.4.6.8es^{3}D}{3.5.7.9} \times \frac{\overline{2av-sA}|^{4}}{\overline{4aa+AA}|^{\frac{1}{2}}}$, &c. But, by the Resolution of



Forces, as 1 (Qb) : s (Gb) : : so is the said Force to $\overline{\frac{s^{2}e}{\sqrt{4aa+AA}} \times \frac{2 \times \overline{4aa+AA}}{3} - \frac{2.4}{3.5} B \times \overline{2av-sA}|^{2} - \frac{2.4.6sC}{3.5.7} \times \overline{2av-sA}|^{3} - \frac{2.4.6s^{2}D}{3.5.7.9} \times \overline{2av-sA}|^{4}}$, &c. the Force in the Direction QT; and as 1 to v, so is the same Force to to $\frac{sev}{\sqrt{4aa+AA}} \times \frac{2\times\overline{4aa+A^2}}{3} - \frac{2.4B}{3.5} \times \overline{2av-sA}^{|2} - \frac{2.4.6sC}{3.5.7.} \times \overline{2av-sA}^{|3}$, &c. that in the Direction QF; which Quantities are, it is manifest, as the Fluxions of the whole Force exerted by the Part QHPQ, of the Solid in those Directions. Let now another Plane QK, be supposed to revolve about the same Point Q, and with the same Velocity as the former, but in a contrary Direction, so as to meet and coincide with it in the Perpendicular Qw; then v, in this Case, becoming Negative or — v, the Fluxion of the Part QKAQ in the said Directions, will be $\frac{s^2e}{\sqrt{4aa+AA}} \times \frac{2\times\overline{4a^2+A^2}}{3} - \frac{2.4B}{3.5} \times \overline{-2av-sA}^{|2} - \frac{2.4.6sC}{3.5.7} \times \overline{-2av-sA}^{3}$, &c. and $-\frac{sev}{\sqrt{4aa+AA}} \times \frac{2\times\overline{4a+A^2}}{3} - \frac{2.4B}{3.5} \times \overline{-2av-sA}^{2}$, &c. respectively; wherefore if these Fluxions be added to those of the former Part in the like Directions, and c be substituted instead of $2a$, we shall have $\frac{2s^2e}{\sqrt{cc+AA}} \times \frac{2\times\overline{cc+A^2}}{3} - \frac{2.4B}{3.5} \times \overline{c^2v^2+s^2A^2} + \frac{2.4.6sC}{3.5.7} \times \overline{3c^2v^2As+A^3s^3}$, &c. and $\frac{2sev}{\sqrt{cc+AA}} \times \frac{2.4B}{3.5} \times \overline{2Acvs} - \frac{2.4.6sC}{3.5.7.} \times \overline{c^3v^3+3cvA^2s^2}$, &c. for the Fluxions of the whole Force in those Directions: But, since the Triangles QbG, and mbn are similar, e ($= mb$) will be $= \frac{\dot{s}}{v}$ or $\frac{\dot{s}}{\sqrt{1-ss}}$; and consequently the said Expressions, by writing $\frac{\dot{s}}{\overline{1-ss}^{\frac{1}{2}}}$ instead of e, will become $\frac{2\dot{s}}{\sqrt{\overline{cc+AA}}^{|} \times \overline{1-ss}^{\frac{1}{2}}} \times \frac{2ccs^2+2AAs^2}{3} - \frac{2.4.B}{3.5} \times \overline{A^2s^4+c^2s^2v^2} + \frac{2.4.6C}{3.5.7} \times \overline{A^3s^6+3Ac^2s^4v^2}$, and $\frac{2\dot{s}}{\sqrt{\overline{c^2A^2}}^{|} \times \overline{1-ss}^{\frac{1}{2}}} \times \frac{2.4B}{3.5} \times \overline{2Acs^2v^2} - \frac{2.4.6C}{3.5.7} \times$

3

$\overline{3A^2cs^4v^2 + c^3s^2v^4}$, &c. respectively; the Fluents whereof, when $s = 1$, supposing the length (r) of the Arch Fhe given, may be easily had from the *Lemma* in Page 4. and will be

$$\frac{2r}{\sqrt{c^2+A^2}} \text{ into } \frac{1}{3}\times\overline{c^2+A^2} - \frac{B}{5}\times\overline{A^2+\frac{2\cdot 1}{2\cdot 3}\times c^2} + \frac{C}{7}\times\overline{A^3+\frac{3\cdot 2}{2\cdot 5}\times A^2c} - \frac{D}{9}\times\overline{A^4+\frac{4\cdot 3}{2\cdot 7}\times A^2c^2+\frac{4\cdot 3\cdot 2\cdot 1}{2\cdot 4\cdot 7\cdot 5}\times c^4} + \frac{E}{11}\times\overline{A^5+\frac{5\cdot 4}{2\cdot 9}\times A^3c^2+\frac{5\cdot 4\cdot 3\cdot 2}{2\cdot 4\cdot 9\cdot 7}\times Ac^4} - \frac{F}{13}\times\overline{A^6+\frac{6\cdot 5}{2\cdot 11}\times A^4c^2+\frac{6\cdot 5\cdot 4\cdot 3}{2\cdot 4\cdot 11\cdot 9}\times A^2c^4+\frac{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{2\cdot 4\cdot 6\cdot 11\cdot 9\cdot 7}\times c^6}, \text{ \&c.}$$

and

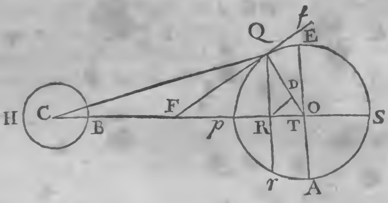
$$\frac{2r}{\sqrt{c^2+A^2}} \text{ into } \frac{2B}{5\cdot 3}\times Ac - \frac{3C}{7\cdot 5}\times\overline{A^2c+\frac{2\cdot 1}{2\cdot 3}\times c^3} + \frac{4D}{9\cdot 7}\times\overline{A^3c+\frac{3\cdot 2}{2\cdot 5}\times Ac^3} - \frac{5E}{11\cdot 9}\times\overline{A^4c+\frac{4\cdot 3}{2\cdot 7}\times A^2c^3+\frac{4\cdot 3\cdot 2\cdot 1}{2\cdot 4\cdot 7\cdot 5}\times c^5} + \frac{6F}{13\cdot 11}\times\overline{A^5c+\frac{5\cdot 4}{2\cdot 9}\times A^3c^3+\frac{5\cdot 4\cdot 3\cdot 2}{2\cdot 4\cdot 9\cdot 7}\times Ac^5}, \text{ \&c.}$$

where the Law of Continuation is manifest: And these Fluents do, it is evident, respectively express the Ratio of the absolute Forces, whereby the Corpuscle is urged in the Directions QT and QF; from which, by the Composition of Forces, both the Direction of Gravitation and the Force in that Direction, may be easily determined.

Q. E. I.

To determine the height of the TIDES *at any* PLANET, *caused by the Attraction of a* SATELLITE *or other remote* BODY.

Let P S be the Planet, taken as a perfect Sphere, except by so much as it differs therefrom through the Cause under Consideration (which will cause no sensible Error in the Solution) and let B H be the proposed Satellite; let the Distance O C of the two Bodies, in Semi-diameters of the former, be represented by d, and the Quantity of Matter in the former be to the Quantity of Matter in the latter, as 1 to m: Let P E S A P be a Section of the Planet formed by a Plane passing the Centres O and C, Q any Point in the Perimeter of that Section, F Q f a Tangent at that Point, and Q T perpendicular thereto; make Q R and O E perpendicular to O C, and R D



to Q T; putting f to represent the accelerative Force of the Planet at Q, in the Direction Q T, and $x = OR$: Therefore, since the Attractions or accelerative Forces of Bodies, are known to be as the Quantities of Matter in those Bodies directly, and the Squares of the Distances from their Centres inversely, we shall have as $\frac{1}{QO^2} : \frac{m}{CR^2}$, or as $1 : \frac{m}{\overline{d-x}|^2}$: : so is f : to to $\frac{mf}{\overline{d-x}|^2}$, the accelerative Force of the Satellite at the Point Q, because C Q and Q R may be taken as equal; and for the very ſame Reaſon, the accelerative Force at E will be $\frac{mf}{d^2}$; but $\frac{2fmx}{d^3} + \frac{3fmx^2}{d^4}$, &c. the Difference of thoſe two is, it is manifeſt, as the whole Force whereby a Particle of Matter at Q tends to recede from A E, or to alter its Situation, with reſpect to the Body of the Planet. Now this Force may be reſolved into two others, one in the Direction of the Tangent Q F, and the other Perpendicular thereto; whereof the former, which is nearly expounded by $\frac{2fm}{d^3} \times$ R D, ſhews how much that Particle, by the Attraction of the Satellite, is urged in the Direction Q F: Wherefore, this Force appearing to be in the ſimple Ratio of R D, the Attraction of the Planet in the contrary Direction Q *f*, as it is every where equal to it, muſt conſequently be in the Ratio of R D; and therefore the Figure of the Planet a Spheroid by what is proved in Page 14.

Let therefore the Square of the Diameter P S, to the Square of the Diameter A E, be now aſſumed as 1 : to 1 + B, then the Forces exerted, by the Planet in the Directions Q T and Q *f*, will be to one another, nearly as $\frac{1}{3}$: to $\frac{-2B}{3 \cdot 5} \times$ R D, as appears from Page 13.

Hence we have as $\frac{1}{3}$ to $\frac{-2B}{3 \cdot 5} \times \text{R D} :: f : \frac{2fm}{d^3} \times \text{R D}$; wherefore $B = \frac{-5m}{d^3}$, and conſequently $\text{O P} - \text{O A} = \frac{5m}{2d^3} \times \text{OP}$.

Q. E. I.

COROLLARY I.

Hence it appears, that the Forces of the Planets, or any remote Bodies, to produce Tides at the Earth's Surface, are

to

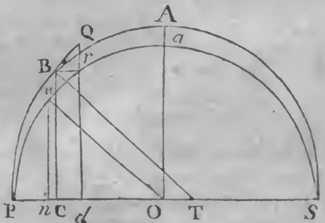
to one another as the Quantities of Matter in thoſe Bodies directly, and the Cubes of their Diſtances inverſely, or as their Denſities and the Cubes of their apparent Diameters, conjunctly; and this, it is evident, holds equally, whether the Earth be conſidered as partly covered with Water, or univerſally ſo.

COROLLARY II.

If d be taken $= 60$, $m = \frac{1}{40}$, and $OP = 21120000$ Feet, and theſe Values be ſubſtituted in the foregoing Theorem, there will come out 6.11 Feet, for the height of the Tides which would ariſe from the Attraction of the Moon, was the whole Body of the Earth quite covered with Water. Hence it follows, that tho' the Tides when forced up Rivers, and into narrow Inlets, are found in ſome Places, at certain particular Times, to riſe to a height greater than 40 Feet, yet in the Main Ocean, the greateſt Alteration of the height of the Surface of the Water that can poſſibly happen, when the Forces of the Sun and Moon are both united together to produce the Effect, and the Moon is in its Perige, will never exceed 11 Feet; nor can it be quite ſo much, ſince, even in the great Pacifick Ocean, it muſt be leſs than it would, was the whole Earth quite covered with Water.

To determine the Length of a Degree of the MERIDIAN, *and the meridional Parts anſwering to any given* LATITUDE, *according to the true ſpherodical* FIGURE *of the* EARTH.

Let POS be the Axis, AO the ſemi-equatoreal Diameter, and PBAS a Meridian of the Earth; and from any Point B in that Meridian, perpendicular to the Tangent BQ, draw BT meeting PS in T; and upon the Diameter PS deſcribe the Semi-circle P*v a* S, making O*v* parallel to TB, B*r* to PS, and *vn*, BC, and Q*rd*, each to AO; putting $PO = 1$, $AO = d$, $OC = x$, B*r* (C*d*) $= \dot{x}$, $d^2 = 1 + b$, the Sine (O*n*) of the Latitude of the Place B, to the Radius 1,



$= s$, and the meridional Diſtance anſwering to that Latitude, in Parts of the Axis PO, $= y$: Then by the Property of the Ellipſis, we have $BC = d\sqrt{1 - xx}$, $CT = d^2 x$, and $BT = d\sqrt{1 + bx^2}$; but as BT : CT : : O*v* (1) : O*n* (*s*); whence *x* (OC) $= \frac{s}{\sqrt{d^2 - bss}}$, CB $(d\sqrt{1 - xx}) = \frac{d^2\sqrt{1 - ss}}{\sqrt{d^2 - bs^2}}$ and B*r* ($\dot{x}$)

 =

$=\frac{d^2\dot{s}}{\overline{d^2-bs^2}|^{\frac{3}{2}}}$. Therefore, becauſe the Triangles O nv, B Q r are ſimilar, it will be as $\sqrt{1-ss}$ (nv) : 1 (O v) : : $\frac{d^2\dot{s}}{\overline{d^2-bs^2}|^{\frac{3}{2}}}$ (B r) : $\frac{d^2\dot{s}}{\overline{1-ss}|^{\frac{1}{2}}\times\overline{d^2-bs^2}|^{\frac{3}{2}}}$ = B Q, and as $\frac{d^2\sqrt{1-ss}}{\sqrt{d^2-bs^2}}$ (BC) : d(AO) : : $\frac{d^2\dot{s}}{\overline{1-ss}|^{\frac{1}{2}}\times\overline{d^2-bs^2}|^{\frac{3}{2}}}$ (B Q) : $\frac{d\dot{s}}{\overline{d^2-bs^2}\times\overline{1-ss}}=\dot{y}$; but $\frac{d\dot{s}}{\overline{d^2-bs^2}\times\overline{1-ss}}$ may be reduced to $\frac{d\dot{s}}{1-ss}-\frac{bd\dot{s}}{d^2-bs^2}$; of which Fluent being taken, we ſhall have y equal to $\frac{2.302585d}{2}$ into the (*Brigean*) Log. of $\frac{1+s}{1-s}$, $-\frac{2.302585b^{\frac{1}{2}}}{2}$ into the (*Brigean*) Log. of $\frac{d+b^{\frac{1}{2}}s}{d-b^{\frac{1}{2}}s}$: But as 3.14159, &c. × $2d$, the Meaſure of the whole Periphery of the Earth at the Equator, in Part of the Semi-axis P O, is to 21600, the Meaſure of the ſame Periphery in Geographical Miles, ſo is this Value of y, to 3958 × Log. $\frac{1+s}{1-s}$, $-\frac{3958b^{\frac{1}{2}}}{d}$ × Log. $\frac{d+b^{\frac{1}{2}}s}{d-b^{\frac{1}{2}}s}$, the Value of y in Geographical Miles, or the meridional Parts required.

Moreover, becauſe the Fluxion $\left(\frac{d^2\dot{s}}{\overline{d^2-bs^2}|^{\frac{3}{2}}\times\overline{1-ss}|^{\frac{1}{2}}}\right)$ of the Arch A B, is to the Fluxion $\left(\frac{\dot{s}}{\sqrt{1-ss}}\right)$ of the correſponding circular Arch av, whoſe Sine is s, as $\frac{d^2}{\overline{d^2-bs^2}|^{\frac{3}{2}}}$ to 1, it is evident that the length of that Degree of the Meridian, whoſe Middle is B, will be to $\frac{60}{d}$ the length of a Degree of the Circle P a S in the ſame Ratio of $\frac{d^2}{\overline{d^2-bss}|^{\frac{3}{2}}}$ to 1 very nearly, and

and therefore is equal to $\frac{60d}{\overline{d^2-bss}|^{\frac{3}{2}}}$ ſuch Parts (or Miles) whereof every Degree of the Equator contains 60. Q. E. I.

COROLLARY I.

If we conſider the Earth as (it really is) nearly ſpherical, d will be nearly $= 1$, and conſequently the Value of b very ſmall, in which Caſe $\frac{3958 b^{\frac{1}{2}}}{d} \times \text{Log.} \frac{d+b^{\frac{1}{2}}s}{d-b^{\frac{1}{2}}s}$ becomes $= 7916bs$ nearly; and conſequently $y = 3958 \times \text{Log.} \frac{1+s}{1-s}, - 7916bs$: But if we conſider it as a perfect Sphere, then $\frac{3958 b^{\frac{1}{2}}}{d} \times \text{Log.} \frac{d+b^{\frac{1}{2}}s}{d-b^{\frac{1}{2}}s}$ will be $= 0$, and therefore $y = 3958 \times \text{Log.} \frac{1+s}{1-s}$ which Value, it is eaſy to prove, is equal to 7916 multiplied by the Logarithmic Tangent of half the Diſtance from the remoteſt Pole (Radius being 1) Therefore, if this Product, or the meridional Parts anſwering to the given Latitude, when the Earth is conſidered as a perfect Sphere, be denoted by Q, it is manifeſt that the meridional Parts anſwering to the ſame Latitude, when the Earth is taken as a Spheroid, will be defined by $Q - 7916bs$, or $Q - 68.5s$; becauſe $\overline{1+b}|^{\frac{1}{2}}$ being to 1, as 231 to 230, (as has been before determined) $7916bs$ is $= 68.5s$.

COROLLARY II.

Moreover, becauſe the Earth is nearly ſpherical, $\frac{60d}{\overline{d^2-bs^2}|^{\frac{3}{2}}}$ will be nearly $= \frac{60 \times \overline{1+\frac{b}{2}}}{\overline{1+b-bss}|^{\frac{3}{2}}} = 60 \times \overline{1+\frac{b}{2}} \times \overline{1 - \frac{3}{2} \times \overline{b-bs^2}}$, &c. $= 60 \times \overline{1 - b + \frac{3bs^2}{2}}$, &c. whence it appears that the length

length of a Degree of the Meridian increases, from the Equator to the Pole, in the duplicate Ratio of the Sine of the Latitude very nearly.

EXAMPLE.

Let it be required to find the meridional Parts anſwering to 50° Latitude, every Degree of the Equator being ſuppoſed to contain 60 Geographical Miles. Here the artificial or logarithmic Tangent of (70°) half the Diſtance from the remoteſt Pole is 0,438934, which being multiply'd by 7916, gives 3474,6 for the meridional Parts anſwering to 50° Latitude, conſidering the Earth as a perfect Sphere: But as Radius to the Sine of 50, ſo is 68.5 to 52.5; which taken from 3474.6, leaves 3422.1 for the true Value required. The like of any other.

SCHOLIUM.

From the foregoing Concluſions, the Ratio of the Equatoreal Diameter and Axis of the Earth may be determined, by knowing (from Experiment) the Ratio of the Lengths of two Degrees of the Meridian: For if the Sines of the Latitudes in the Middle of thoſe Degrees, be denoted by s and S, and the Lengths of the Degrees themſelves be to one another, as 1 to n; then, from what has been found above, it will be as $1 : n :: \frac{60d}{\overline{dd-bs^2}|^{\frac{3}{2}}} : \frac{60d}{\overline{dd-bS^2}|^{\frac{3}{2}}}$; whence $n \times \overline{dd-bS^2}|^{\frac{3}{2}} = \overline{dd-bs^2}|^{\frac{3}{2}}$, and therefore $n^{\frac{2}{3}} \times \overline{dd-bS^2} = dd-bs^2$, but $d^2 = 1+b$; whence, by Subſtitution, $\overline{1+b} \times n^{\frac{2}{3}} - bS^2 \times n^{\frac{2}{3}} = 1+b-bs^2$, therefore $b = \frac{n^{\frac{2}{3}}-1}{1-n^{\frac{2}{3}}+n^{\frac{2}{3}}S^2-s^2}$, and $d (\overline{1+b}|^{\frac{1}{2}}) = \overline{1+\frac{n^{\frac{2}{3}}-1}{1-n^{\frac{2}{3}}+n^{\frac{2}{3}}S^2-s^2}}|^{\frac{1}{2}}$: From whence it appears, that the equatoreal Diameter will be to the Axis as $\overline{1+\frac{n^{\frac{2}{3}}-1}{1-n^{\frac{2}{3}}+n^{\frac{2}{3}}S^2-s^2}}|$ to 1. But when the Meaſures of the two Degrees are nearly equal, or the Figure differs but little

little from a Sphere, n will be nearly $= 1$, and therefore, if inſtead of n we ſubſtitute $1+m$, we ſhall have $n^{\frac{2}{3}} = 1 + \frac{2m}{3}$ nearly, and conſequently b $= \frac{2m}{3 \times \overline{S^2 - s^2}}$ (becauſe all the Terms of the Denominator, in which m enters, may be rejected as inconſiderable) Therefore, in this Caſe, the required Ratio will be as $\overline{1 + \frac{2m}{3 \times \overline{S^2 - s^2}}}\,|^{\frac{1}{2}}$ to 1, or as $1 + \frac{m}{3 \times \overline{S^2 - s^2}}$ to 1, very nearly.

A TABLE *ſhewing the length of a Degree of the Meridian, in ſuch Parts (or Miles) whereof every Degree of the Equator contains* 60.

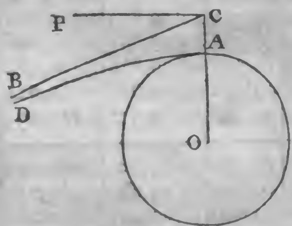
Deg.		Deg.		Deg.	
2	59,482	32	59,695	62	60,083
4	59,484	34	59,719	64	60,105
6	59,488	36	59,745	66	60,127
8	59,495	38	59,771	68	60,147
10	59,503	40	59,797	70	60,165
12	59,513	42	59,823	72	60,182
14	59,524	44	59,851	74	60,208
16	59,537	46	59,878	76	60,212
18	59,552	48	59,905	78	60,224
20	59,568	50	59,932	80	60,235
22	59,586	52	59,959	82	60,244
24	59,605	54	59,985	84	60,251
26	59,626	56	60,011	86	60,256
28	59,648	58	60,036	88	60,259
30	59,671	60	60,060	90	60,261

A DETERMINATION OF THE REFRACTION which a Ray of LIGHT suffers in its Passage to the EARTH.

PROPOSITION I.

SUpposing the Velocity of Light, in respect to the Velocity sufficient to retain a Body in a circular Orbit about the Earth just above its Surface, to be very great: I say, the greatest horizontal Refraction that would arise from the Attraction of the Earth, will be to 57° 17′ 44″, as the Square of the latter of those Velocities, to the Square of the former very nearly.

For, since the Earth's Attraction is in the inverse duplicate Ratio of the Distance from its Centre (O), the Curve DA



which a Particle of Light would describe thereby (setting all other Causes aside) will, it is known, be one of the Conic-Sections; and

and therefore, ſince the Velocity of Light is ſuppoſed very great in reſpect to the propoſed circular Velocity, it muſt be an Hyperbola; whoſe Semi-Tranſverſe, and Semi-Conjugate Axis (AC and CP) if the Ratio of the ſaid Velocities be put as n to 1, will be $\frac{AO}{n^2-2}$ and $\frac{n \times AO}{\sqrt{n^2-2}}$ reſpectively (as is proved in Page 153 of my Book of Fluxions.) Therefore, if the Aſymptote CB be deſcribed, it will be as $\frac{n \times AO}{\sqrt{n^2-2}}$ to $\frac{AO}{nn-2}$, or as 1 to $\frac{1}{n\sqrt{nn-2}}$, ſo is Radius, to the Tangent of PCB, the total Refraction of the Ray AD indefinitely produced. But ſince n is here very great, $\frac{1}{n\sqrt{nn-1}}$ is nearly $=$ $\frac{1}{nn}$; therefore, the Tangent of a very ſmall Arch being nearly equal to the Arch itſelf, $\frac{1}{nn}$ will be the Meaſure of the Angle BCP, to the Radius 1, in Parts of that Radius; hence we have as 1 to $\frac{1}{nn}$, or as nn to 1, ſo is 57° 17′ 44″ the Degrees, &c. in an Arch equal to the Radius, to the Refraction, or Degrees, &c. in the foreſaid Arch, whoſe length is $\frac{1}{nn}$. Q. E. D.

SCHOLIUM.

It is found, both from the Periodic Time of the Moon and from Experiments of Pendulums, that the Velocity ſufficient to retain a Body in a circular Orbit about the Earth, juſt above its Surface (ſetting aſide all Reſiſtance, &c.) muſt be ſuch as would carry it uniformly over a Space of 4.95 Miles per Second. Therefore, if Light, according to Obſervation, moves thro' a Space equal to the Semi-Diameter of the *Magnus Orbis* in 8 Minutes time, and the Sun's Parallax by 10 Seconds of a Degree, the Velocity of Light muſt be to the

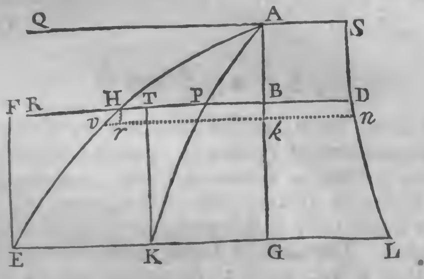
I Velocity

Velocity above named, nearly as 34090 to 1 : Hence we have as $\overline{34090}^2 : 1^2 :: 57° 17' 44'' : 35^{v}\ 18^{vi}$, for the Horizontal, or greateſt Refraction ariſing from Gravity. Whence it appears, that but a very ſmall Part of the Refraction obſerved in the Sun, Moon and Stars, can be owing to the foreſaid Cauſe, even ſhould the Velocity of Light, in reality, be much leſs than it is at preſent ſuppoſed. And therefore in all Practical Enquiries, about the Refraction of the Heavenly Bodies, the Conſideration of Gravity may be entirely neglected, as altogether too minute to cauſe any ſenſible Alteration.

PROPOSITION II.

To inveſtigate the Curve, which a Ray of Light, or any moving Body, will deſcribe by any given Force, continually urging it perpendicularly towards a given Plane.

Let EGL be the given Plane, AE the required Curve, H and v any two Points therein indefinitely near to each other; and let the Force by which the Body or Particle is urged towards EG, be repreſented by the Ordinates BD, &c. of any given Curve SDL, whoſe Axis AG is perpendicular to



EG; draw RHB, $vrkn$ and QAS parallel to EG, and put $AB = x$, $BH = y$, $BD = Q$, $AH = z$, Hr $(= Bk) = \dot{x}$, vr

$vr = \dot{y}$, the Area ASDB $= s$, the Sine of the Angle GAH to the Radius 1, $= b$, its Cosine $= c$, and the Velocity at the Point A $= g$: Therefore as $1 : b :: g : bg$, the Velocity at A in the Direction AQ; which, becauſe the Motion in the Direction of the Ordinate is not at all affected by the Force acting in the Direction Hr, muſt alſo be the Velocity at H in the Direction HR; wherefore that in the Direction Hr will be $\frac{bg\dot{x}}{\dot{y}}$, whoſe Fluxion $\frac{bg\ddot{x}}{\dot{y}}$ making $\dot{y}$ conſtant, will therefore be as $Q \times \frac{\dot{y}}{bg}$, that is, as the Force by which the Motion is accelerated at H, drawn into the time of deſcribing Hv: Hence, by putting $\frac{bg\ddot{x}}{\dot{y}} = \frac{Q\dot{y}}{bg}$ we have $\frac{b^2g^2\dot{x}\ddot{x}}{\dot{y}\dot{y}} = Q\dot{x}\ (= BDnk) = \dot{s}$, and conſequently, by taking the Fluent on both ſides, $\frac{b^2g^2\dot{x}^2}{2\dot{y}^2} = s +$ ſome conſtant Quantity d; which to determine, let B coincide with A; then s being $= 0$, $\frac{b^2g^2\dot{x}^2}{2\dot{y}^2}$ will become $= d$, but $\frac{\dot{x}}{\dot{y}}$ being there $= \frac{c}{b}$, d will be $= \frac{c^2g^2}{2}$, and conſequently $\frac{b^2g^2\dot{x}^2}{2\dot{y}^2} = s + \frac{c^2g^2}{2}$: Wherefore $\dot{y} = \frac{b\dot{x}}{\sqrt{cc + \frac{2s}{gg}}}$, and $\dot{z} = \frac{\dot{x}\sqrt{1 + \frac{2s}{gg}}}{\sqrt{cc + \frac{2s}{gg}}}$; from whence when s is given in Terms of x, the Values of y and z will be alſo given. Q. E. I.

COROLLARY I.

Becauſe the Value of s at all equal Diſtances from the given Plane EL is the ſame, and (b) the Sine of BAH is to $\left(\frac{b}{\sqrt{1 + \frac{2s}{gg}}} = \frac{\dot{y}}{\dot{z}}\right)$ the Sine of rHv, as $\sqrt{1 + \frac{2s}{gg}}$ to 1, it follows, that the Sines of Refraction, or of the Angles, which

N any

any two Rays A E, A K (having the ſame Velocity at A) make with the Perpendiculars F E, T K, at entering the given Plane or Surface E L, will be to one another as the Sines of the given Angles E A G, K A G. Therefore if the Refraction in any one Caſe, or anſwering to any one Angle K A G, be given from Experiment, the Refractions in all other Caſes will from hence be given, let the accelerating Force be what it will.

COROLLARY II.

But if the Force whereby the Particle, in its Paſſage between A Q and E G L is accelerated, be the Attraction of an interjacent Medium, whoſe Denſity in going from Q S increaſes according to ſome given Law, not only the ſame Thing, but the Curve itſelf will be had: For, let B *k* be ſuppoſed conſtant, or taken every where the ſame; then (B D) the accelerative Force of the Medium, or the indefinitely little Area B D *n k*, will, it is evident, be as the Difference of Denſities in B and *k*, and conſequently the Sum of all theſe indefinite little Areas, or the whole curvilinear Area A S D B, as the Difference of Denſities in A and B. Therefore ſince *s* is as this given Difference of Denſities, the Nature of the Curve will be readily had from the Equations foregoing. And hence it appears, that, if the Denſity in Q S be nothing, and that in E L given, the Refraction will alſo be given or remain invariable, let A G, the height of the Medium, and the Law of Denſity be what they will, and therefore is the ſame as it would be, was the Ray to be refracted immediately out of a Vacuum into the ſaid given Denſity.

COROLLARY III.

Hence may alſo be found the Law of Denſity, whereby a Ray of Light ſhall deſcribe a given Curve: For if *b* be taken

taken = 1, so that A may be the principal Vertex of the Curve, $\dot{y}$ will then become barely $= \frac{g\dot{x}}{\sqrt{2s}}$, and therefore $s = \frac{g^2\dot{x}^2}{2\dot{y}^2}$, which is as the Density required.

EXAMPLE I.

Let the given Curve be a Circle: Then y being $= \sqrt{2rx - xx}$, $\dot{y}$ is $= \frac{r\dot{x} - x\dot{x}}{\sqrt{2rx - xx}}$ and $s \left(= \frac{g^2\dot{x}^2}{2\dot{y}^2}\right) = \frac{g^2 \times 2rx - xx}{2 \times \overline{r-x}|^2}$: Therefore the Density is as $\frac{2rx - xx}{\overline{r-x}|^2}$, or as the Square of the Tangent of the Distance from the highest Point.

EXAMPLE II.

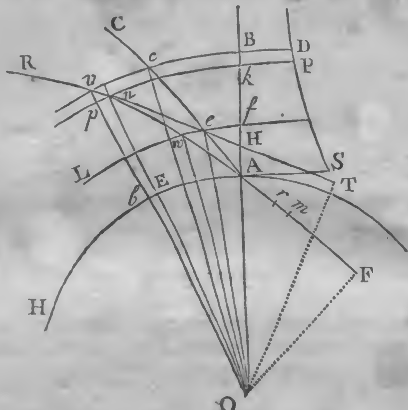
Suppose the Density to increase uniformly; to find the Curve: Here by writing x instead of s, in the former of the two Equations, in Cor. III. we have $\dot{y} = \frac{g\dot{x}}{\sqrt{2x}}$, and therefore $y = g\sqrt{2x}$; which answers to the common Parabolæ; the like of any other.

PROPOSITION III.

To find the Curve which a Particle of Light or any moving Body will describe by any given Force, continually urging it directly towards a given Centre.

Let O be the Centre to which the Body or Particle is urged, AR the required Curve, v and n any two Points therein indefinitely near to one another, and AF, vT Tangents at A and

and v; to which let O F and O T be perpendicular, and from the Centre O let the Circles A b H, $k n$, and B v be descri-



bed: Let the Velocity at A be represented by A m, and that at v by A r, and let the Force whereby the Body is urged towards the Centre O, at any Distance O B therefrom, be defined by the Ordinates B D, B D of any given Curve S P D: Putting $AO = a$, $Am = g$, $Ar = v$, $BD = Q$, $Ov = x$, $Pv = \dot{x}$, $vn = \dot{z}$, the Area $ABDS = s$, and the Sine of R A B or O A F to the Radius 1, $= b$. Then by the Resolution of Forces, it will be as $\dot{z} : \dot{x} :: Q : \frac{Q\dot{x}}{\dot{z}}$, the Force acting in the Direction vn, whereby the Motion at v is accelerated or retarded, which therefore drawn into $\frac{\dot{z}}{v}$ the Time of describing nv gives $\frac{Q\dot{x}}{v} = (-\dot{v})$ the Alteration of Velocity

locity in that Time: Hence we have $v\dot{v} = -Q\dot{x}$ (= BDPk) $= -\dot{s}$, and, by taking the Fluent on both sides, $\frac{v^2}{2} = \frac{g^2}{2} - s$; whence $v = \sqrt{g^2 - 2s}$. Wherefore because the Velocity, be the Law of Force what it will, is known to be inversely as a Perpendicular falling from the Centre of Force to the Tangent, we shall have ba (FO) : g (Am) : : $\sqrt{g^2-2s}$ (Ar) : OT $= \frac{bag}{\sqrt{g^2-2s}}$; $\therefore v\mathrm{T} = \frac{\sqrt{g^2x^2-2sx^2-b^2a^2g^2}}{\sqrt{g^2-2s}}$: But as v T : O T : : $vp : pn = \frac{\dot{x}bga}{\sqrt{g^2x^2-2sx^2-b^2a^2g^2}}$, and as OP : pn : : 1 (Radius) to $\frac{\dot{x}bga}{x\sqrt{g^2x^2-2sx^2-b^2a^2g^2}}$, the Fluxion or Decrement of the Angle AOv; from which, when the Relation of x and s is given, the Curve itself will be given. Q. E. I.

COROLLARY I.

If the Curve AR be that formed by a Ray of Light in passing thro' an elastic Medium, and the Refraction be required; then OT being $= \frac{bag}{\sqrt{g^2-2s}}$, its Fluxion will be $\frac{bag\dot{s}}{\overline{g^2-2s}|^{\frac{3}{2}}}$ which divided by $\frac{\sqrt{g^2x^2-2sx^2-b^2a^2g^2}}{\sqrt{g^2-2s}}$ ($=$ Tv) gives $\frac{abg\dot{s}}{\overline{g^2-2s} \times \sqrt{g^2x^2-2sx^2-a^2b^2g^2}} = \frac{\mathrm{OT}}{\mathrm{T}v} \times \frac{\dot{s}}{g^2-2s}$ for the Fluxion of the Refraction, where s is to be defined by the Difference of Densities in AE and Bv, for the very same Reason as in the last Proposition.

COROLLARY II.

Hence, because the Refraction $\left(\frac{\mathrm{OT}}{\mathrm{T}v} \times \frac{\dot{s}}{g^2-2s}\right)$ which a Ray of Light suffers in passing thro' any given Stratum of the Me-

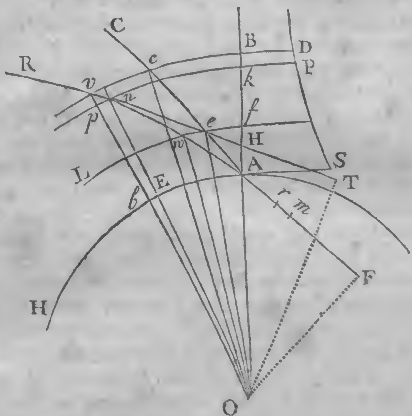
 dium,

dium, $Bvpk$, appears to be as, $\frac{OT}{Tv}$, the Tangent of Incidence on the Surface of that Stratum and $\frac{\dot{s}}{g^2-2s}$ conjunctly, it is manifest that if $\frac{OT}{Tv}$, as well as g^2-2s, be every where nearly the same, the whole Refraction or total Bending of that Ray, will be as the Tangent of the Angle RAB, and the Density of the Medium at the Surface AE very nearly.

SCHOLIUM.

The last Conclusion will be found to afford a short and very useful Theorem for determining the Refraction which the Light of the heavenly Bodies suffers in passing thro' the Earth's Atmosphere, by the help of one Observation only, in all Cases where the Zenith Distance is not very great: For let AE, *&c.* represent the Surface or a great Circle of the Earth; then, because the Atmosphere at a small Height AB, above that Surface, in Comparison of the Semi-Diameter AO, must be extreamly rarer than at the Surface itself, the Refraction beyond such Height will, at most, be but very small, and therefore the Curvature, which any Rays RvA, CcA, suffer below Bcv, may be considered as their total Refractions. But these Refractions being found by Experiment to be but small, the Angles vAB and AvO will be nearly equal, and therefore, if not very large, their Tangents will likewise be nearly equal; from whence, and what has been said in the last Corollary, it plainly appears that, let the Law of Density of the Atmosphere be what it will, the Refractions of the Sun, Moon and Stars, at all Altitudes except very small ones, will be nearly as the Tangents of their apparent Zenith Distances drawn into the respective Density of the Atmosphere, at the Places and Times, for which such Refractions are to be determined; and therefore if the Density be the

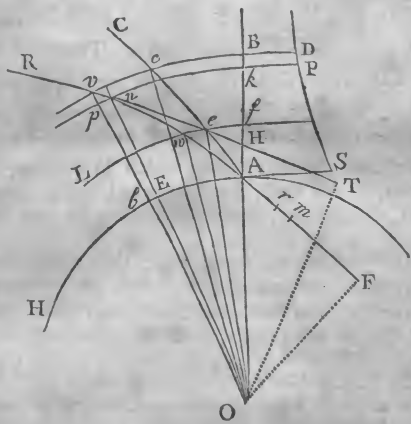
the ſame, are ſimply as the Tangents of their Zenith Diſtances. But now to eſtimate in ſome ſort, how near this Proportion comes to Truth, and how far it may be relied on; let any two convenient Altitudes, *i. e.* that are neither very ſmall nor very near each other be aſſumed. Suppoſe one of 20, and the other of 40 Degrees, and let *f e w* L, *&c.* be a Circle, or ſpherical Surface dividing the Atmoſphere into two Parts ſo that the Denſity at that Surface may be equal but to half the Den-



ſity at A. Now the Height of this Surface above A E, the Surface of the Earth, from the known Properties of Air, and Experiment made on the Tops of very high Hills, cannot be more than about 5 Miles, or $\frac{1}{800}$ of the Earth's Radius. Therefore we have, as $1 + \frac{1}{800}$ (O w) : 1 (A O) : : the Coſine of (R A E $=$ 20) the leaſt given Altitude to the Sine of 69° : 48′, which being

being increaſed by 1 Minute on Account of the Curvature of the Ray A *w*, gives 69° 49′, for the Angle O *w* A very nearly. And in the ſame manner the Angle O *e* A, correſponding to the other given Altitude, will be found 49° 55′. Now it hath been proved, that if the Angles of Incidence O *v* A, O *c* A, continued every where invariable, or equal to themſelves, the Refractions would be to one another exactly as the Tangents of thoſe Angles; therefore, becauſe the Difference of the Tangents of *v* A B and O *w* A, *&c.* is but little, and the Refraction above and below the Surface *f e w* L, nearly equal, therefore may 69° : 49′, and 49° : 55′, be taken as mean Incidences, and then the Refractions, anſwering thereto, will be to one another as the Tangents of thoſe Angles, or as 1 to 0.4372; which Proportion being much nearer the Truth than that of 1 to .4338, ariſing immediately from the Theorem, the Error, in the conſequent Term of this laſt Proportion, cannot, it is plain, be much greater than (.0034) the Difference between .4372 and .4388; which, ſhould it be even double that Quantity, would ſcarce cauſe an Error in the Refraction itſelf of a ſingle Second. Nor is it in this one particular Caſe only, that the Rule anſwers ſo exactly, the Error here being nearly as great, if not greater, than it can be in any other Caſe, where the leaſt of the two propoſed Altitudes is not leſs than 20 Degrees, as is eaſy to ſee from the Reaſons foregoing. Hence it appears, that if by any Means we can come at the true Refraction correſponding to any one given Altitude, not leſs than about 20°, the Refraction at all higher Altitudes, for the ſame Denſity of the Atmoſphere, may be had from the foreſaid Proportion, and that to a ſingle Second. And this is to be the more relied on in Practice, as it does not depend on any particular Hypotheſis, for the Law of Denſity of the Atmoſphere.

The Refractions in ſmall Altitudes, which remain to be conſidered, are not ſo certain and eaſy to come at, nor indeed, to be computed at all but by Virtue of ſome Hypotheſis. If the Denſity of the Atmoſphere, in going from the Earth, be ſuppoſed to decreaſe uniformly (which Law will be found to anſwer better to Experiment than the commonly received one, founded on the Elaſticity of Air) and b be put for A B the height of the whole Atmoſphere, in Parts of the Earth's Radius, and k be aſſumed equal to the greateſt Value of s, correſponding to this Height, then will $\dot{s}$ be to $\dot{x}$ as k to b, and the Fluxion of Refraction (found in Cor. I.) will become



$\frac{abgk\dot{x}}{b \times \overline{g^2 - 2s} \times \sqrt{g^2x^2 - 2sx^2 - a^2b^2g^2}}$, and is therefore to $\frac{abg\dot{x}}{x\sqrt{g^2x^2 - 2sx^2 - a^2b^2g^2}}$ the Fluxion of the Angle A O v, as $\frac{k}{b \times \overline{g^2 - 2s}}$ to $\frac{1}{x}$, or, if a be taken $= 1$, and $g = 1$, as k to $b \times \frac{1-2s}{x}$; which, becauſe x and $1-2s$ are always nearly the ſame, will be as k to b very nearly. Wherefore, becauſe the Angle v H B is equal to both the Angles OvH, HOv, the Fluxion thereof, or that of the Refraction, will be equal to the Fluxions of both them two, and is, therefore, to the Fluxion of $-OvH$, in the constant Ratio of k to $b-k$; therefore the Fluents themselves (corrected by their proper constant Quantities) must be in the same constant Ratio, that is, the Refraction will be to the Excess of OAF above OvH, as k to $b-k$. But, since OT is found above to be $=\frac{bag}{\sqrt{g^2-2s}}$, or $\frac{b}{\sqrt{1-2k}}$ (because $a=1$, $g=1$, &c.) the Sine of OvH is given $=\frac{b}{1+b\sqrt{1-2k}}$ $\left(=\frac{OT}{Ov}\right)=b\times\overline{1-b+k}$, very nearly. Therefore it will be as 1 to $1-b+k$: : so is the Sine of any apparent Zenith Distance, to the Sine of an Arc, the Difference between which Arc and the Zenith Distance, multiplied by $\frac{k}{b-k}$ will give the Refraction sought; from which Proportion, the Refraction may, in any Case, be determined, when b and k are given from Experiment; both which may be had from two Observations.

But if the Altitude be pretty large, then the Difference of the two Arcs measuring the Angles OAF, OvH, being nearly equal to the Difference of their Sines into Radius, applied to (c) the Cosine of the former, the Refraction will be barely $\frac{b}{c}\times k$ $\left(=\frac{b\times\overline{b-k}}{c}\times\frac{k}{b-k},\right)$ and therefore in any such Case, the Value of k may be found from one Observation only. For an Instance hereof, let us suppose the Refraction at the Altitude of 30 Degrees to be given from Experiment, $=1'\ 30''\frac{1}{2}$; then the length of an Arc of $1'\ 30''\frac{1}{2}$, in Parts of the Radius, being $=.00044$ and $\frac{b}{c}$, the Tangent of $60°$, $=1.732$, we have $1.732\,k=.00044$, and therefore $k=.000253$. By help of which, the Refractions at very small Altitudes may be also found, when the Refraction answering to any one ſuch Altitude is given. Suppoſe, for Example, the horizontal Refraction, correſponding to the above Value of k, to be given $= 33'$, and let the Refraction, at the apparent Altitude of 5°, at the ſame time, be required: Becauſe $b \times \overline{b-k}$, the Difference of the Sines of the Angles O A F, OvH, here becomes $= b-k =$ the verſed Sine of the Complement of OvH to a right Angle, the Arc correſponding to this verſed Sine, or, which is the ſame, the Difference of the Arcs meaſuring the ſaid Angles, will be nearly $= \sqrt{2 \times \overline{b-k}}$, by the Nature of the Circle; and therefore $\frac{k}{b-k} \times \sqrt{2 \times \overline{b-k}}$ will be equal to (.0096) the Arc meaſuring the given Refraction in Parts of the Radius; whence $\frac{2k^2}{b-k} = .00009216$, and $b-k = \frac{kk}{.00004608} = .00139$ (becauſe $k = 000253$) and therefore $\frac{k}{b-k}$ is $= \frac{253}{1390}$ or $\frac{2}{11}$ very nearly. Wherefore from the foregoing Proportion, we have this Rule: *As* 1 *to* .9986, *&c. or as Radius to the Sine of* 86° 58′½, *ſo is the Sine of any given Zenith Diſtance to the Sine of an Arc,* $\frac{2}{11}$ *of the Difference of which Arc and the Zenith Diſtance, is the Refraction ſought*; which in the Caſe above propoſed, comes out 9′ 10″. And in this Manner were the two following Tables computed, the firſt from the above Numbers, adapted to the mean Denſity of the Atmoſphere, and the other from Numbers ſomewhat larger, to anſwer when the Refractions are the greateſt.

Ap. Alt.	Refract.	Ap. Alt.	Refract.
°	′ ″	°	′ ″
0	33. 0	25	1.52
1	23.50	26	1.47
2	17.43	27	1.42
3	13.44	28	1.38
4	11.05	29	1.34
5	9.10	30	1.30
6	7.49	32	1.23
7	6.48	34	1.17
8	5.59	36	1.12
9	5.21	38	1.07
10	4.50	40	1.02
11	4.24	42	0.58
12	4.02	44	0.54
13	3.43	46	0.50
14	3.27	48	0 47
15	3.13	50	0.44
16	3.01	52	0.41
17	2.50	54	0.38
18	2.40	56	0.35
19	2.31	58	0.32
20	2.23	60	0.30
21	2.16	65	0.24
22	2.09	70	0.19
23	2.03	75	0.14
24	1.57	80	0. 9

Ap. Alt.	Refract.	Ap. Alt.	Refract.
°	′ ″	°	′ ″
0	35.30	25	1.58
1	25.30	26	1.53
2	18.51	27	1.48
3	14.36	28	1.43
4	11.43	29	1.39
5	9.44	30	1.35
6	8.18	32	1.28
7	7.12	34	1.22
8	6.20	36	1.16
9	5.39	38	1.11
10	5.07	40	1.06
11	4.40	42	1.01
12	4.16	44	0.57
13	3.56	46	0.53
14	3.39	48	0.49
15	3.24	50	0.46
16	3.11	52	0.43
17	2.59	54	0.40
18	2.48	56	0.37
19	2.39	58	0.34
20	2.31	60	0.32
21	2.23	65	0.26
22	2.16	70	0.20
23	2.09	75	0.15
24	2.03	80	0.10

The Numbers whereon theſe Tables are grounded, were deduced from Obſervations communicated by Dr. *Bevis*; which, by their near Agreement with each other, ſeem to be taken with great Care and Exactneſs: And the only material

rial Objection (that I foresee) the Tables are liable to, is their being founded on a Supposition, that *the Density of the Atmosphere decreases uniformly*; which is not only very different from what hath been hitherto commonly received, but seemingly contrary to Experiment, whereby it is proved, *that the Density of Air decreases as the compressing Force*: But it may be answered, that, tho' this is allowed to be true in Air containing the same Degree of Heat, yet it cannot be supposed to hold in the Earth's Atmosphere, since the upper Region thereof is known to be much colder, and consequently the Elasticity there much less than at the Earth's Surface: But, a convincing Proof that this Law of Density cannot obtain in our Atmosphere, is, that the mean horizontal Refraction computed therefrom, according to the known refractive Power, and specifick Gravity of Air, will be found to come out no less than 52 Minutes, which is greater by almost $\frac{1}{3}$ of a whole Degree than it ought to be; whereas, if the same Refraction be calculated from the Hypothesis of a Density decreasing uniformly, and compared with Observations, the Difference will not be near so considerable. This shews the Tables to be much exacter, than they could had they been computed from the common Hypothesis; I mean, in very small Altitudes; for the Refractions in high Altitudes, it has been proved, will be but little affected by different Laws of Density, and therefore come out very near the same, compute them according to what Hypothesis you will; even so near, that if the Refraction at any Altitude not less than about 7 Degrees be truly given from Experiment, the Refractions, computed from thence, according to the two Hypotheses forenamed, for any higher Altitude, will never differ from one another by more than about 2 Seconds. From whence we may infer, that as the Hypothesis on which the abovesaid Tables are founded is much the exacter of the two, the Error arising therefrom cannot in any such Altitude amount to more than a single Second.

OF THE

SUMMATION of SERIES.

PROPOSITION I.

IF $a^n + ba^{n-1}x + ca^{n-2}x^2 + da^{n-3}$, &c. *be any Power* (n) *of the Binomial* $a+x$, *either whole or broken, positive or negative, and the Terms thereof be respectively multiplied by any Series of Quantities* p, q, r, s, &c. *and the Differences of these Quantities be continually taken, and the first Difference* (q—p) *of the first Order, be denoted by* $\overset{\prime}{D}$, *and the first* (r—2q+p) *of the second Order, by* $\overset{\prime\prime}{D}$, &c. *I say, the Series* $pa^n + qba^{n-1}x + rca^{n-2}x^2$, &c. *thence arising, shall be* $= p\times\overline{a+x}|^{n} + \overset{\prime}{D}bx\times\overline{a+x}|^{n-1} + \overset{\prime\prime}{D}cx^2\times\overline{a+x}|^{n-2} + \overset{\prime\prime\prime}{D}dx^3\times\overline{a+x}|^{n-3}$, &c.

For, let $P\times\overline{a+x}|^{n} + Qx\times\overline{a+x}|^{n-1} + Rx^2\times\overline{a+x}|^{n-2} + Sx^3\times\overline{a+x}|^{n-3}$, *&c.* be assumed $= pa^n + qna^{n-1}x + rn\times\frac{n-1}{2}a^{n-2}x^2 + sn\times\frac{n-1}{2}\times\frac{n-2}{3}a^{n-3}x^3$, *&c.* $(= pa^n + qba^{n-1}x + rca^{n-2}x^2 + sda^{n-3}x^3$, *&c.*$)$; then, by converting the several Powers of $a+x$ to simple Terms, and transposing $pa^n + qna^{n-1}x + rn\times\frac{n-1}{2}a^{n-2}x^2$, *&c.* we shall have

+ P

$$\left.\begin{array}{l} +Pa^n + nPa^{n-1}x + n \times \frac{n-1}{2}Pa^{n-2}x^2, \&c. \\ \quad * + Qa^{n-1}x + \frac{n-1}{1}Qa^{n-2}x^2, \&c. \\ \quad * \quad * \quad + \quad Ra^{n-2}x^2, \&c. \\ \quad * \quad * \quad \quad * \quad \&c. \\ -pa^n - qna^{n-1}x - rn \times \frac{n-1}{2}a^{n-2}x^2, \&c. \end{array}\right\} = 0$$

From whence, by equating the homologous Terms, there will be $P = p$, $Q = n \times \overline{q - p}$, $R = n \times \frac{n-1}{2} \times \overline{r - 2q + p}$, $S = n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \overline{s - 3r + 3q - p}$, &c. But n is $= b$, $n \times \frac{n-1}{2} = c$, &c. $q - p = \overset{\prime}{D}$, $r - 2q + p = \overset{\prime\prime}{D}$, &c. and consequently $pa^n + qba^{n-1}x + rca^{n-2}x^2 + sda^{n-3}x^3$, &c. $= p \times \overline{a+x}|^n + \overset{\prime}{D}bx \times \overline{a+x}|^{n-1} + \overset{\prime\prime}{D}cx^2 \times \overline{a+x}|^{n-2} + \overset{\prime\prime\prime}{D}dx^3 \times \overline{a+x}|^{n-3} + \overset{\prime\prime\prime\prime}{D}ex^4 \times \overline{a+x}|^{n-4}$, &c. where it is evident, that the Value of $pa^n + qba^{n-1}$, &c. will be always had in finite Terms, when the last Differences of the Quantities p, q, r, &c. are equal. Q. E. D.

COROLLARY I.

Hence, if the Values of p, q, r, s, &c. be respectively expounded by the Terms of any Arithmetical Progression k, $k+m$, $k+2m$, $k+3m$, &c. then p being $= k$, $\overset{\prime}{D} = m$, and $\overset{\prime\prime}{D}$, $\overset{\prime\prime\prime}{D}$, $\overset{\prime\prime\prime\prime}{D}$, &c. each $= 0$, we shall have $ka^n + \overline{k+m} \times ba^{n-1}x + \overline{k+2m} \times ca^{n-2}x^2$, &c. barely $= k \times \overline{a+x}|^n + mbx \times \overline{a+x}|^{n-1}$, or $k \times \overline{a+x}|^n + mnx \times \overline{a+x}|^{n-1}$.

CO-

COROLLARY II.

But if the Values of p, q, r, &c. be defined by $\frac{1}{k}$, $\frac{1}{k+m}$, $\frac{1}{k+2m}$, &c. (the Reciprocals of that Progression) then $\acute{D}$ being $= \frac{-m}{k.\overline{k+m}}$, $\overset{''}{D} = \frac{m.2m}{k.\overline{k+m}.\overline{k+2m}}$, $\overset{'''}{D} = \frac{-m.2m.3m}{k.\overline{k+m}.\overline{k+2m}.\overline{k+3m}}$, $\overset{''''}{D} = \frac{m.2m.3m.4m}{k.\overline{k+m}.\overline{k+2m}.\overline{k+3m}.\overline{k+5m}}$, &c. we shall have $\frac{a^n}{k} + \frac{ba^{n-1}x}{k+m} + \frac{ca^{n-2}x^2}{k+2m} + \frac{da^{n-3}x^3}{k+3m}$, &c. $= \frac{\overline{a+x}|^n}{k} - \frac{mbx \times \overline{a+x}|^{n-1}}{k.\overline{k+m}} + \frac{m.2mcx^2 \times \overline{a+x}|^{n-2}}{k.\overline{k+m}.\overline{k+2m}} - \frac{m.2m.3mdx^3 \times \overline{a+x}|^{n-3}}{k.\overline{k+m}.\overline{k+2m}.\overline{k+3m}}$ $\frac{m.2m.3m.4mex^4 \times \overline{a+x}|^{n-4}}{k.\overline{k+m}.\overline{k+2m}.\overline{k+3m}.\overline{k+4m}}$, &c. $= \frac{\overline{a+x}|^n}{k} - \frac{nA}{k+m} \times \frac{mx}{a+x} - \frac{\overline{n-1} \times B}{k+2m} \times \frac{mx}{a+x} - \frac{\overline{n-2} \times C}{k+3m} \times \frac{mx}{a+x} - \frac{\overline{n-3} \times D}{k+4m} \times \frac{mx}{a+x}$, where A denotes the first Term, B the second, C the third, and so on.

COROLLARY III.

Therefore, if n be taken $=-1$, $a=1$, and $x=z^m$, then b being $=-1$, $c=+1$, $d=-1$, &c. we have $\frac{1}{k} - \frac{z^m}{k+m} + \frac{z^{2m}}{k+2m} - \frac{z^{3m}}{k+3m} + \frac{z^{4m}}{k+4m}$, &c. $= \frac{1}{k \times \overline{1+z^m}} + \frac{A}{k+m} \times \frac{mz^m}{1+z^m} + \frac{2B}{k+2m} \times \frac{mz^m}{1+z^m} + \frac{3C}{k+3m} \times \frac{mz^m}{1+z^m} + \frac{4D}{k+4m} \times \frac{mz^m}{1+z^m} + \frac{5E}{R+5m} \times \frac{mz^m}{1+z^m}$, &c. or $\frac{1}{k \times \overline{1+z^m}} + \frac{AQ}{k+m} + \frac{2BQ}{k+2m} + \frac{3CQ}{k+3m} + \frac{4DQ}{k+4m}$, &c. by putting $Q = \frac{mz^n}{1+z^m}$.

COROLLARY IV.

Moreover, when n is taken $=-1$ and $a=1$, we shall have $p-qx+rx^2-sx^3+tx^4$, &c. $=\frac{p}{1+x}-\frac{\acute{D}x}{\overline{1+x}|^2}+\frac{\overset{\prime\prime}{D}x^2}{\overline{1+x}|^3}-\frac{\overset{\prime\prime\prime}{D}x^3}{\overline{1+x}|^4}$, &c. and consequently (by writing $-x$ instead of $+x$) $p+qx+rx^2+sx^3$, &c. $=\frac{p}{1+x}+\frac{\acute{D}x}{\overline{1+x}|^2}+\frac{\overset{\prime\prime}{D}x^2}{\overline{1+x}|^3}+\frac{\overset{\prime\prime\prime}{D}x^3}{\overline{1+x}|^4}$, &c. shewing the Value of the Series $p+qx+rx^2$, &c. continued *in infinitum.*

COROLLARY V.

But if the Value of only a finite Number (n) of Terms be wanted, let the remaining Part of the Series be represented $Px^n+Qx^{n+1}+Rx^{n+2}$, and the Difference of the Coefficients P, Q, R, &c. be continually taken; and let the first Difference of the first Order be denoted by $\acute{E}$, the first Difference of the second Order by $\overset{\prime\prime}{E}$, &c. &c. Then, for the very same Reasons that $p+qx+rx^2$, &c. is $=\frac{p}{1-x}+\frac{\acute{D}x}{\overline{1-x}|^2}$, &c. will $P+Qx+Rx^2$, &c. be $=\frac{P}{1-x}+\frac{\acute{E}x}{\overline{1-x}|^2}+\frac{\overset{\prime\prime}{E}x^2}{\overline{1-x}|^3}$, &c. and therefore $Px^n+Qx^{n+1}+Rx^{n+2}$, &c. $=\frac{Px^n}{1-x}+\frac{\acute{E}x^{n+1}}{\overline{1-x}|^2}+\frac{\overset{\prime\prime}{E}x^{n+2}}{\overline{1-x}|^3}$, &c. which taken from $\frac{p}{1-x}+\frac{\acute{D}x}{\overline{1-x}|^2}$, &c. leaves $\frac{p-Px^n}{1-x}+\frac{\acute{D}x-\acute{E}x^{n+1}}{\overline{1-x}|^2}+\frac{\overset{\prime\prime}{D}x^2-\overset{\prime\prime}{E}x^{n+2}}{\overline{1-x}|^3}$, &c. equal to the n first Terms of the Series proposed.

COROLLARY VI.

Hence may the Sum of any Number (n) of Terms of the Series $\frac{p}{z} + \frac{q}{z^2} + \frac{r}{z^3} + \frac{s}{z^4}$, &c. where z is indeterminate, be also found; for since $px + qx^2 + rx^3$, &c. is $= \frac{px - Px^{n+1}}{1-x} + \frac{\acute{D}x^2 - \acute{E}x^{n+2}}{\overline{1-x}|^2}$, &c. $= \frac{p - Px^n}{\frac{1}{x} - 1} + \frac{\acute{D} - \acute{E}x^n}{\overline{\frac{1}{x} - 1}|^2}$, &c.

let z be written therein instead of $\frac{1}{x}$, and it will become $\frac{p}{z} + \frac{q}{z^2} + \frac{r}{z^3}$, &c. $= \frac{p - Pz^{-n}}{z-1} + \frac{\acute{D} - \acute{E}z^{-n}}{\overline{z-1}|^2} + \frac{\acute{\acute{D}} - \acute{\acute{E}}z^{-n}}{\overline{z-1}|^3}$, &c. equal to the Value sought, which therefore, when n is infinite, or the whole Series is taken, will be barely $= \frac{p}{z-1} + \frac{\acute{D}}{\overline{z-1}|^2} + \frac{\acute{\acute{D}}}{\overline{z-1}|^3} + \frac{\acute{\acute{\acute{D}}}}{\overline{z-1}|^4}$, &c.

EXAMPLE I.

Where $a^{\frac{1}{2}} + \frac{x}{2a^{\frac{1}{2}}} - \frac{x^2}{8a^{\frac{3}{2}}} + \frac{x^3}{16a^{\frac{5}{2}}}$, &c. being $= \overline{a+x}|^{\frac{1}{2}}$, 'tis proposed to find the Sum of the infinite Series $9a^{\frac{1}{2}} + \frac{16x}{2a^{\frac{1}{2}}} - \frac{25x^2}{8a^{\frac{3}{2}}} + \frac{36x^3}{16a^{\frac{5}{2}}}$, &c. Here p being $= 9$, $q = 16$, $r = 25$, $s = 36$, &c. the first Differences will be 7, 9, 11, 13, &c. the second 2, 2, 2, &c. and the third, fourth, &c. each equal to nothing: Therefore $\acute{D} = 7$, $\acute{\acute{D}} = 2$, $\acute{\acute{\acute{D}}}$, &c. $= 0$; whence by substituting these Values, with those of n and p, &c. in the general Equation, we have $9a^{\frac{1}{2}} + \frac{16x}{2a^{\frac{1}{2}}} - \frac{25x^2}{8a^{\frac{3}{2}}}$

+

$+\frac{36x^3}{16a^{\frac{5}{2}}}$, &c. $= 9\times\overline{a+x}|^{\frac{1}{2}} + \frac{7x\times\overline{a+x}|^{-\frac{1}{2}}}{2} - \frac{x^2\times\overline{a+x}|^{-\frac{3}{2}}}{4}$, which was to be found.

EXAMPLE II.

Where x being leſs than 1, 'tis required to find the Sum of the infinite Series $1+2x+3x^2+4x^3$, &c. In this Caſe, $p=1$, $q=2$, $r=3$, &c. $\overset{\prime}{D}=1$, $\overset{\prime\prime}{D}=0$, &c. and therefore (by Cor. IV.) $1+2x+3x^2$, &c. $=\frac{1}{1-x}+\frac{x}{\overline{1-x}|^2}=\frac{1}{\overline{1-x}|^2}$. In like Manner it will be found, that $1+4x+9x^2+16x^3$, &c. is $=\frac{1}{1-x}+\frac{3x}{\overline{1-x}|^2}+\frac{2x^2}{\overline{1-x}|^3}$, and that $1+8x+27x^2+64x^3$, &c. $=\frac{1}{1-x}+\frac{7x}{\overline{1-x}|^2}+\frac{12x^2}{\overline{1-x}|^3}+\frac{6x^3}{\overline{1-x}|^4}$, &c. &c.

SCHOLIUM.

The foregoing Concluſions are not only uſeful in finding the Values of Series, which are in their own Nature exactly ſumible, but may alſo be applied to very good Purpoſe in the Quadrature of Curves, and in approximating the Values of ſuch Series, whoſe exact Values cannot be determined. Let it, for example, be required to approximate the Value of the Series $\frac{4x^{\frac{3}{2}}}{3}+\frac{2x^{\frac{5}{2}}}{5}-\frac{2x^{\frac{7}{2}}}{4.7}+\frac{2.3x^{\frac{9}{2}}}{4.6.9}-\frac{2.3.5x^{\frac{11}{2}}}{4.6.8.11}+\frac{2.3.5.7x^{\frac{13}{2}}}{4.6.8.10.13}$, &c. expreſſing the Area of the rectangular Hyperbola, whoſe Abſciſſa is x, and principal Diameter Unity. In order to effect which, let a few of the leading Terms (ſuppoſe the four firſt) be collected into one Sum, and let the Differences of the Coefficients of a few of the firſt of the remaining Terms, which (in this Caſe) are $\frac{2.3.5}{4.6.8.11}$, $\frac{2.3.5.7}{4.6.8.10.13}$, $\frac{2.3.5.7.9}{4.6.8.10.12.15}$, &c.

or

or 0.0142, 0.00841, 0.00546, 0.00379, &c. be continually taken (as in the Margin)

.01420	.00841	.00546	.00379, &c.
	—.00579	—.00295	—.00167, &c.
		+.00284	+.00128, &c.
			—.00156, &c.

Then, the first Difference of the first Order being —0.00579, of the second Order +0.00284, of the third —0.00156, &c. if instead of p, q, r, s, &c. $\overset{\prime}{D}$, $\overset{\prime\prime}{D}$, $\overset{\prime\prime\prime}{D}$, &c. the above Values 0.01420, 0.00841, 0.00546, 0.00379, &c. — 0.00579, 0.0284, — 0.00156, &c. be respectively substituted in the general Equation $p-qx+rx^2-sx^3$, &c. $= \frac{p}{1+x} - \frac{\overset{\prime}{D}x}{\overline{1+x}|^2} + \frac{\overset{\prime\prime}{D}x^2}{\overline{1+x}|^3}$, &c. (as found in Corol. IV.) we shall get $0.01420-0.00841x + 0.00546x^2 - 0.00379x^3$, &c. $\left(\frac{2.3.5}{4.6.8.11} - \frac{2.3.5.7x}{4.6.8.10.13} + \frac{2.3.5.7.9x^2}{4.6.8.10.12.15}\right.$, &c.$\left.\right) = \frac{0.01420}{1+x} + \frac{0.00579x}{\overline{1+x}|^2} + \frac{0.00284x^2}{\overline{1+x}|^3} + \frac{0.00156x^3}{\overline{1+x}|^4}$, &c. and consequently $-\frac{2.3.5x^{\frac{11}{2}}}{4.6.8.11} + \frac{2.3.5.7x^{\frac{13}{2}}}{4.6.8.10.13} - \frac{2.3.5.7.9x^{\frac{15}{2}}}{4.6.8.10.12.15}$, &c. $= - x^4\sqrt{x} \times \overline{\frac{0.01420x}{1+x} + \frac{0.00.579x^2}{\overline{1+x}|^2} + \frac{0.00284x^3}{\overline{1+x}|^3}}$, &c. which added to $x\sqrt{x} \times \overline{\frac{4}{3} + \frac{4x}{10} - \frac{x^2}{14} + \frac{x^3}{36}}$, the Sum of the four first Terms, will give the Value of the whole propounded Series; which Value may now be easily found in Numbers, that of x being given; for let $x=1$, then will $-x^4\sqrt{x} \times \overline{\frac{0.01420x}{1+x} + \frac{0.00579x^2}{\overline{1+x}|^2}}$, &c. $= - 0.0090$, &c. and $x\sqrt{x} \times \overline{\frac{4}{3} + \frac{4x}{10} - \frac{x^2}{14} + \frac{x^3}{36}} = 1.6896$, and therefore the Value of the whole Series will be 1.6806, which is more exact, than if 20 Terms of the original Series had been taken. Again, let

let x be taken $= \frac{1}{2}$, then will $- x^4\sqrt{x} \times \overline{\frac{0.01420\,x}{1+x} + \frac{0.00579\,x^2}{\overline{1+x}|^2} + \frac{0.00284\,x^3}{\overline{1+x}|^3} + \frac{0.00156\,x^4}{\overline{1+x}|^4}}$, &c. $= \frac{-1}{16}\sqrt{\frac{1}{2}} \times \overline{\frac{0.01420}{3} + \frac{0.00579}{9} + \frac{0.00284}{27} + \frac{0.00156}{81}}$, &c. $= 0.000243$, and $x\sqrt{x} \times \overline{\frac{4}{3} + \frac{4x}{10} - \frac{x^2}{14} + \frac{x^3}{36}} = 0.537030$; and consequently the Value of the whole Series $= 0.536787$ very nearly. And in the same Manner may the Values of other Series be approximated; but the Advantages of this Method in many Cases are much more considerable; for let the Series propounded be $\frac{z}{1} - \frac{z^{m+1}}{m+1} + \frac{z^{2m+1}}{2m+1} - \frac{z^{3m+1}}{3m+1} + \frac{z^{4m+1}}{4m+1}$, &c. and let a few of the first Terms thereof be collected (as above) and let the Denominator of the first of the remaining Terms be denoted by k, and then it is evident, from the Law of Continuation, that the true Value of these Terms will be rightly defined by $\pm \frac{z^k}{k} \mp \frac{z^{k+m}}{k+m} \pm \frac{z^{k+2m}}{k+2m}$, &c. or $\pm z^k \times \overline{\frac{1}{k} - \frac{z^m}{k+m} + \frac{x^{2m}}{k+2m} - \frac{z^{3m}}{k+3m}}$, &c. But since $\frac{1}{k} - \frac{z^m}{k+m} + \frac{z^{2m}}{k+2m}$, &c. (as appears from Corollary III.) is universally equal to $\frac{1}{k \times \overline{1+z^m}} + \frac{AQ}{k+m} + \frac{2BQ}{k+2m} + \frac{3CQ}{k+3m} + \frac{4DQ}{k+4m} + \frac{5EQ}{k+5m}$, &c. where Q stands for $\frac{mz^m}{1+z^m}$, and A, B, C, &c. for the first, second, third &c. Terms, it is plain that $\pm z^k \times \overline{\frac{1}{k \times \overline{1+z^m}} + \frac{AQ}{k+m} + \frac{2BQ}{k+2m} + \frac{3CQ}{k+3m}}$, &c. will be equal to $\pm z^k \times \overline{\frac{1}{k} - \frac{z^m}{k+m} + \frac{z^{2m}}{k+2m} - \frac{z^{3m}}{k+3m}}$, &c. the Uses whereof shall immediately be ſhewn. And firſt, let $z = 1$, and $m = 2$, ſo that $\frac{z}{1} - \frac{z^{m+1}}{m+1} + \frac{z^{2m+1}}{2m+1}$, &c. may become $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$, &c. (expreſſing the length of $\frac{1}{8}$ of the Periphery of the Circle, whoſe Radius is Unity) then the Value (0.7440117) of ($1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11}$) the ſix firſt Terms thereof being collected there will remain $\frac{1}{13} - \frac{1}{15}$, &c. Therefore, ſeeing the Value of k (the Denominator of the firſt remaining Term) is here $= 13$, and $Q\left(= \frac{m z^m}{1+z^m}\right) = 1$, we ſhall, by writing theſe Values in the above Equation, have $\frac{1}{13} - \frac{1}{15} + \frac{1}{17}$, &c. $= 1 \times \overline{\frac{1}{2.13} + \frac{A}{15} + \frac{2B}{17} + \frac{3C}{19}}$, &c. $= 0.0384615 + 0.0025641 + 0.0003016 + 0.0000476 + 0.0000091 + 0.0000024 + 0.0000007 + 0.0000002 + 0.0000001 = 0.0413873$; and this added to 0.7440117, gives 0.785399 for the Value of the whole Series; which is true in the laſt Place, and more exact than if 100000 Terms of the original Series had been taken. Again, let $x = 1$, and $m = 1$, ſo that the propounded Series may be $\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$, &c. (expreſſing the hyperbolical Logarithm of 2) then the Sum (0.634523809) of the 8 firſt Terms being taken, the remaining Part of the Series will be $\frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \frac{1}{14}$, &c. Therefore k being here $= 9$, and $Q = \frac{1}{2}$, we ſhall have $\pm z^k \times \overline{\frac{1}{k \times \overline{1+z^m}} + \frac{AQ}{k+m} + \frac{2BQ}{k+2m} + \frac{3CQ}{k+3m}}$, &c. $= \frac{1}{18} + \frac{A}{20} + \frac{B}{11} + \frac{C}{8} + \frac{2D}{13} + \frac{5E}{28} + \frac{F}{5}$, &c. $= 0.055555555$ +

1

+ 0.002777777 + 0.000252525 + 0.000031565 + 0.000004856 + 0.000000867 + 0.000000173 + 0.000000038 + 0.000000009 + 0.000000002 = 0.058623367; and consequently 0.693147176 equal to the whole Value required; which errs but 4 in the last Place, and would have required, at least, 10000000 Terms of the original Series.——But after all it may not appear why a few initial Terms are always first to be taken, seeing the Series, for the Value of $\pm z^k \times \overline{\frac{1}{k} - \frac{z^m}{k+m} + \frac{z^{2m}}{k+2m}}$, &c. holds universally, let the Value of k be what it will; but the Reason is this, the more Terms there are first taken, the faster will the Series expressing the Value of the remaining Terms converge, so that by first collecting a proper Number of initial Terms (which will be greater or lesser, according as a greater or lesser Degree of Accuracy is required) the same Conclusion will be brought out with a great deal less Trouble, than if the Value of the whole Series was to be found by this Method, as upon Trial will plainly appear.

PROPOSITION II.

Supposing $a^n + ba^{n-1}x + ca^{n-2}x^2 + da^{n-3}x^3$, &c. *to be as in the last Proposition, and* r *any whole positive Number, and that* S *is equal to the* n+r *Power of the Binomial* a+x, *decreased by the* r *first Terms: I say, the Sum of the Series* $\frac{a^n}{1.2.3\ldots r} + \frac{ba^{n-1}x}{2.3.4\ldots r+1} + \frac{ca^{n-2}x^2}{3.4.5\ldots r+2} + \frac{da^{n-3}x^3}{4.5.6\ldots r+3}$ &c. (*whether finite or infinite*) *will be* $\frac{S}{\overline{n+1} \times \overline{n+2} \times \overline{n+3} \ldots \overline{n+r} \times x^r}$.

For the $n+r$ Power of $a+x$ being $a^{n+r} + \overline{n+r} \times a^{n+r-1}x + \overline{n+r} \times \frac{\overline{n+r-1}}{2} \times a^{n+r-2}x^2$, &c. if from the same the r first

Terms be taken, there will remain $\frac{\overline{n+r}\times\overline{n+r-1}\times\overline{n+r-2}\ldots\overline{n+1}\times a^{n}x^{r}}{1.2.2\ldots r}$ $+\frac{\overline{n+r}\times\overline{n+r-1}\ldots\ldots n a^{n-1}x^{r+1}}{1.2.3\ldots r+1}+\frac{\overline{n+r}\times\overline{n+r-1}\ldots\overline{n-1}\,a^{n-2}x^{2r+2}}{1.2.3\ldots r+2}$, &c. $=\overline{n+r}\times\overline{n+r-1}\ldots\ldots\overline{n+1}\times x^{r}\times\overline{\frac{a^{n}}{1.2.3\ldots r}+\frac{na^{n-1}x}{1.2.3.4\ldots r+1}}$ $\overline{+\frac{n\times\overline{n-1}\,a^{n-2}x^{2}}{1.2.3\ldots r+2}}=\overline{n+r}\times\overline{n+r-1}\ldots\overline{n+1}\times x^{r}\times\overline{\frac{a^{n}}{1.2.3\ldots r}}$ $\overline{+\frac{na^{n-1}x}{2.3.4\ldots r+1}+\frac{n\times\frac{n-1}{2}a^{n-2}x^{2}}{3.4.5\ldots r+2}}$, &c. $=S$; therefore $\frac{S}{\overline{n+1}\times\overline{n+2}\times\overline{n+3}\ldots\ldots\overline{n+r}\times x^{r}}$ is $=\frac{a^{n}}{1.2.3\ldots r}+\frac{na^{n-1}x}{2.3.4\ldots r+1}$, &c. $=\frac{a^{n}}{1.2.3\ldots r}+\frac{ba^{n-1}x}{2.3.4\ldots r+1}+\frac{ca^{n-2}x^{2}}{3.4.5\ldots r+2}$, &c. Q. E. D.

COROLLARY I.

Therefore it follows that $\frac{\overline{a+x}|^{n+1}-a^{n+1}}{\overline{n+1}\times x}$ is $=\frac{a^{n}}{1}+\frac{ba^{n-1}x}{2}$ $+\frac{ca^{n-2}x^{2}}{3}$, &c. that $\frac{\overline{a+x}|^{n+2}-a^{n+2}-\overline{n+2}\times a^{n+1}x}{\overline{n+1}\times\overline{n+2}\times x^{2}}$ is $=\frac{a^{n}}{1.2}+\frac{ba^{n-1}x}{2.3}+\frac{ca^{n-2}x^{2}}{3.4}$, &c. and that $\frac{\overline{a+x}|^{n+3}-a^{n+3}-\overline{n+3}\times a^{n+2}x-\overline{n+3}\times\frac{\overline{n+2}}{2}a^{n+1}x^{2}}{\overline{n+1}\times\overline{n+2}\times\overline{n+3}\times r^{3}}$ is $=\frac{a^{n}}{1.2.3}$ $+\frac{ba^{n-1}x}{2.3.4}+\frac{ca^{n-2}x^{2}}{3.4.5}$, &c. &c.

COROLLARY II.

Hence may the Value of $a^n + ba^{n-1}x + ca^{n-2}x^2 + da^{n-3}x^3$, &c. $(=\overline{a+x}|^n)$ when the Terms thereof are respectively divided by any Series of figurate Numbers (as 1, 3, 6, 10, 15, &c. or as 1, 4, 10, 20, 35, &c.) be also easily derived: For, since it is found that $\frac{a^n}{1.2.3\ldots r} + \frac{ba^{n-1}x}{2.3.4\ldots \overline{r+1}} + \frac{ca^{n-2}x^2}{3.4.5\ldots \overline{r+2}}$ &c. is $= \frac{S}{\overline{n+1} \times \overline{n+2} \times \overline{n+3} \ldots \overline{n+r} \times x^r}$, let the whole Equation be multiply'd by $1.2.3\ldots r$, and we shall have $\frac{a^n}{1} + \frac{ba^{n-1}x}{r+1} + \frac{ca^{n-2}x^2}{\overline{r+1} \times \frac{r+2}{2}} + \frac{da^{n-3}x^3}{\overline{r+1} \times \frac{r+2}{2} \times \frac{r+3}{3}}$, &c. $\left(= \frac{1.2.3\ldots r \times S}{\overline{n+1} \times \overline{n+2} \times \overline{n+3} \ldots \overline{n+r} \times x^r}\right) = \frac{S}{\frac{n+1}{1} \times \frac{n+2}{2} \times \frac{n+3}{3} \ldots \frac{n+r}{r} \times x^r}$; where $1,\ r+1,\ \overline{r+1} \times \frac{r+2}{2}$, &c. are known to represent universally any Series of figurate Numbers.

EXAMPLE I.

Let there be given $\frac{1}{c^{\frac{1}{2}}} + \frac{z}{2c^{\frac{3}{2}}} + \frac{3z^2}{8c^{\frac{5}{2}}}$, &c. $= \overline{c-z}|^{-\frac{1}{2}}$, to find the exact Value of $\frac{1}{1 \cdot c^{\frac{1}{2}}} + \frac{z}{2.2c^{\frac{3}{2}}} + \frac{3z^2}{3.8c^{\frac{5}{2}}}$, &c. Then, because r is $= 1$, $a = c$, $x = z$, and $n = -\frac{1}{2}$, we shall in this Case have $\frac{\overline{a+x}|^{n+r} - a^{n+r}}{\overline{n+1} \times x} = \frac{\overline{c-z}|^{\frac{1}{2}} - c^{\frac{1}{2}}}{\frac{1}{2} \times -z} = \frac{2c^{\frac{1}{2}} - 2 \times \overline{c-z}|^{\frac{1}{2}}}{z}$, which was to be found.

EXAMPLE II.

Let it be required to find the Valüe of $\frac{a^5}{1} + \frac{5a^4x}{4} + \frac{10a^3x^2}{10}$, &c. or of the fifth Power of $a+x$, when the Terms thereof are respectively divided by 1, 4, 10, 20, 35, &c. Now, in order that the above general Series (for figurate Numbers) may agree with (1, 4, 10, 20) the particular one here given, let the Value of $r+1$ be so taken, that the second Terms of both Series may be equal, and then the rest will be so of Course; therefore r, in this Case, being $=3$, $n=5$, and the three first Terms of $\overline{a+x}|^{n+r}$ expanded in a Series $=a^8 + 8a^7x + 28a^6x^2$, we have $S = \overline{a+x}|^8 - a^8 - 8a^7x - 28a^6x^2$, and $\frac{S}{\frac{n+1}{1} \times \frac{n+2}{2} \times \frac{n+3}{3} \ldots\ldots \frac{n+r}{r} \times x^r} = \frac{\overline{a+x}|^8 - a^8 - 8a^7x - 28a^6x^2}{56x^3}$ for the Value that was to be found.

PROPOSITION III.

S*Upposing* $ax^p + bx^{p+n} + cx^{p+2n} + dx^{p+3n}$, &c. *to be any Series, finite or infinite, whose Sum* A *is given; and the Terms thereof to be respectively multiply'd by the Terms* r, r+n, r+2n, &c. *of any arithmetical Progression, whose common Difference is* n; *to find the Sum* (B) *of all the Products, or the Value of the Series* $rax^p + \overline{r+n} \times bx^{p+n} + \overline{r+2n} \times cx^{p+2n}$, &c. *thence arising.*

Because $ax^p + bx^{p+n} + cx^{p+2n}$, &c. is given $=A$, there will be given $ax^r + bx^{r+n} + cx^{r+2n}$, &c. $= Ax^{r-p}$; whose Fluxion being taken and divided by $x^{r-p-1}\dot{x}$, we have $rax^p + \overline{r+n} \times bx^{p+n} + \overline{r+2n} \times cx^{p+2n}$, &c. $= \overline{r-p} \times A + \frac{x\dot{A}}{\dot{x}}$ =

=B; where, becauſe A is given in finite Terms, $\dot{A}$ will be always had in finite Terms, and conſequently the Value of B alſo. Q. E. I.

COROLLARY.

Hence, for the very ſame Reaſons that (B) the Sum of the Series $rax^{p}+\overline{r+n}\times bx^{p+n}+\overline{r+2n}\times cx^{p+2n}$, &c. is equal to $\overline{r-p}\times A+\frac{x\dot{A}}{\dot{x}}$, will (C) the Sum of the Series $rsax^{p}+\overline{r+n}.\overline{s+n}.bx^{p+n}+\overline{r+2n}.\overline{s+2n}.cx^{p+2n}$, &c. be $=\overline{s-p}\times B+\frac{x\dot{B}}{\dot{x}}$, and (D) the Sum of the Series $r.s.t.ax^{p}+\overline{r+n}.\overline{s+n}.\overline{t+n}.bx^{p+n}+\overline{r+2n}.\overline{s+2n}.\overline{t+2n}.cx^{p+2n}$, &c. $=\overline{t-p}\times C+\frac{x\dot{C}}{\dot{x}}$, &c. &c. &c.

EXAMPLE I.

Let there be given the Sum of the Series $x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}$, &c. expreſſing the Arc of a Circle, whoſe Radius is 1, and Tangent x, and let it be required to find the Sum (B) of the infinite Series $3x-\frac{5x^3}{3}+\frac{7x^5}{5}-\frac{9x^7}{7}+\frac{11x^9}{9}$, &c. Becauſe, in this Caſe, $p=1$, $n=2$, $r=3$, and $\dot{A}=\frac{\dot{x}}{1+x^2}$, we have B $\left(=\overline{r-p}\times A-\frac{x\dot{A}}{\dot{x}}\right)=2A+\frac{x}{1+x^2}$ for the Value ſought.

EXAMPLE II.

Having given the Sum of the infinite Series $x-\frac{x_2}{2}+\frac{x^3}{3}-\frac{x^4}{4}$, &c. equal to (A) the hyperbolical Logarithm of

$1+x$;

$1+x$; to find (C) the Sum of the infinite Series $\frac{2\cdot 2x}{1} - \frac{3\cdot 3x^2}{2} + \frac{4\cdot 4x^3}{3} - \frac{5\cdot 5x^4}{4}$, &c. Here p being $=1$, $n=1$, $r=2$, $s=2$ (*vid. Corol.*) and $\dot{A} = \frac{\dot{x}}{1+x}$, we have B $(=\overline{r-p}\times A + \frac{x\dot{A}}{\dot{x}}) = A + \frac{x}{1+x}$; therefore $\dot{B} = \frac{\dot{x}}{1+x} + \frac{\dot{x}}{\overline{1+x}|^2}$, and consequently C $(=\overline{s-p}\times B + \frac{\dot{B}x}{\dot{x}}) = A + \frac{2x}{1+x} + \frac{x}{\overline{1+x}|^2}$ which was to be found.

LEMMA.

To divide a Compound Fraction, as $\frac{\overline{a+x}\times\overline{\overset{\prime}{a}+x}\times\overline{\overset{\prime\prime}{a}+x}, \&c.}{\overline{b+x}\times\overline{\overset{\prime}{b}+x}\times\overline{\overset{\prime\prime}{b}+x}, \&c.}$ *into as many simple ones, as there are Factors in its Denominator; supposing* b, $\overset{\prime}{b}$, $\overset{\prime\prime}{b}$, &c. *to be unequal Numbers, and* a, $\overset{\prime}{a}$, $\overset{\prime\prime}{a}$, &c. *any Numbers, either equal or unequal, and that the Number of Factors in the Numerator is less than the Number of Factors in the Denominator.*

Let A, B, C, D, &c. represent the unknown invariable Numerators of the required Fractions to which $\frac{\overline{a+x}\times\overline{\overset{\prime}{a}+x}\times\overline{\overset{\prime\prime}{a}+x}, \&c.}{\overline{b+x}\times\overline{\overset{\prime}{b}+x}\times\overline{\overset{\prime\prime}{b}+x}, \&c.}$ may be reduced, or, which is the same in effect, let $\frac{A}{b+x} + \frac{B}{\overset{\prime}{b}+x} + \frac{C}{\overset{\prime\prime}{b}+x} + \frac{D}{\overset{\prime\prime\prime}{b}+x}$, &c. $= \frac{\overline{a+x}\times\overline{\overset{\prime}{a}+x}\times\overline{\overset{\prime\prime}{a}+x}, \&c.}{\overline{b+x}\times\overline{\overset{\prime}{b}+x}\times\overline{\overset{\prime\prime}{b}+x}, \&c.}$; then by multiplying the whole Equation by $b+x$, we shall have $A + \frac{\overline{b+x}\times B}{\overset{\prime}{b}+x} + \frac{\overline{b+x}\times C}{\overset{\prime\prime}{b}+x} + \frac{\overline{b+x}\times D}{\overset{\prime\prime\prime}{b}+x}$, &c. $= \frac{\overline{a+x}\times\overline{\overset{\prime}{a}+x}, \&c.}{\overline{\overset{\prime}{b}+x}\times\overline{\overset{\prime\prime}{b}+x}, \&c.}$. But since the Equation holds universally, be the Value of x what it it will, let x be taken $= -b$, or $b + x = 0$, and then, all the Terms affected by $b+x$ vanishing out of the Equation, we shall have $A = \frac{\overline{a-b} \times \overline{a'-b} \times \overline{a''-b}, \&c.}{\overline{b'-b} \times \overline{b''-b} \times \overline{b'''-b}, \&c.}$ for the Numerator of that of the required Fractions, whose Denominator is $b+x$: From whence the Numerators answering to the rest of the Denominators or Factors, $b'+x$, $b''+x$, &c. may be found by Inspection, and will be had (as the foregoing is) by taking the Quantity conjoined with x in the Factor proposed, and substituting the same, with its Sign changed, instead of x, in every other Factor of the Fraction given; and so we have

$$\frac{\overline{a-b} \times \overline{a'-b} \times \overline{a''-b}, \&c.}{\overline{b''-b} \times \overline{b''-b} \times \overline{b'''-b}, \&c.} \times \frac{1}{b+x} + \frac{\overline{a-b'} \times \overline{a'-b'} \times \overline{a''-b'}, \&c.}{\overline{b-b'} \times \overline{b''-b'} \times \overline{b'''-b'}, \&c.} \times \frac{1}{b'+x}$$

$$+ \frac{\overline{a-b''} \times \overline{a'-b''} \times \overline{a''-b''}, \&c.}{\overline{b-b''} \times \overline{b'-b''} \times \overline{b'''-b''}, \&c.} \times \frac{1}{b''+x}, \&c. = \frac{\overline{a+x} \times \overline{a'+x} \times \overline{a''+x} \times \overline{a'''+x}, \&c.}{\overline{b+x} \times \overline{b'+x} \times \overline{b''+x} \times \overline{b'''+x}, \&c.}.$$

Q. E. I.

COROLLARY.

If the Number of Factors in the Denominator of the proposed Fraction be greater by two, at least, than the Number of those in the Numerator, it will appear, by conceiving $\frac{A}{x+b} + \frac{B}{x+b'} + \frac{C}{x+b''} + \frac{D}{x+b'''}, \&c. - \frac{\overline{x+a} \times \overline{x+a'} \times \overline{x+a''}, \&c.}{\overline{x+b} \times \overline{x+b'} \times \overline{x+b''}, \&c.}$ $(= 0)$ to be reduced to one Denomination, that A+B+C+D, &c. will be the Sum of the Coefficients of the highest Power of x in the Numerator, and therefore equal to nothing.

PROPOSITION IV.

Having given (S) the Sum of the Infinite Series $\frac{z^m}{m} \pm \frac{z^{m+n}}{m+n} + \frac{z^{m+2n}}{m+2n} \pm \frac{z^{m+3n}}{m+3n} + \frac{z^{m+4n}}{4n}$, &c. *or the Fluent of* $\frac{z^{m-1}\dot{z}}{1 \mp z^n}$ *(which may be always had from the Quadrature of the Conic-Sections) 'tis proposed to find the Sum of the Infinite Series* $\frac{a+o \times b+o \times c+o, \&c.}{p+o \times q+o \times r+o, \&c.} \pm \frac{a+n \times b+n \times c+n, \&c.}{p+n \times q+n \times r+n, \&c.} \times z^n + \frac{a+2n \times b+2n \times c+2n, \&c.}{p+2n \times q+2n \times r+2n, \&c.} \times z^{2n} \pm \frac{a+3n \times b+3n \times c+3n, \&c.}{p+3n \times q+3n \times r+3n, \&c.} \times z^{3n}$, &c. *supposing* $\frac{p-m}{n}$, $\frac{q-p}{n}$, $\frac{r-p}{n}$, &c. *to represent any unequal whole Numbers* a, b, c, m, n, *any Numbers equal or unequal, and the Number of Factors in each Numerator of the Series to be the same, and less than the Number of Factors in the Denominator, this last Number being also supposed the same in every Term.*

Put $\frac{a-p \times b-p \times c-p, \&c.}{q-p \times r-p \times s-p, \&c.} = A$, $\frac{a-q \times b-q \times c-q, \&c.}{p-q \times r-q \times s-q, \&c.} = B$, $\frac{a-r \times b-r \times c-r, \&c.}{p-r \times q-r \times s-r, \&c.} = C$, $\frac{a-s \times b-s \times c-s, \&c.}{p-s \times q-s \times r-s, \&c.} = D$, &c. &c. then, from the preceding Lemma, it will appear that $\frac{A}{p+o}$, $\frac{B}{q+o}$, $\frac{C}{r+o}$, &c. are the simple Fractions to which the first Term of the Series is reducible, that is $\frac{abcd, \&c.}{pqrs, \&c.}$ will be $= \frac{A}{p} + \frac{B}{q} + \frac{C}{r} + \frac{D}{s}$, &c. And in the same Manner it will appear, that the 2d, 3d, 4th, &c. Terms of the said Series will be equal to

$\pm$

$$\pm \frac{Az^n}{p+n} \pm \frac{Bz^n}{q+n} \pm \frac{Cz^n}{r+n} \pm \frac{Dz^n}{s+n} \pm \frac{Ez^n}{t+n}, \text{ \&c.}$$

$$\frac{Az^{2n}}{p+2n} + \frac{Bz^{2n}}{q+2n} + \frac{Cz^{2n}}{r+2n} + \frac{Dz^{2n}}{s+2n} + \frac{Ez^{2n}}{t+2n}, \text{ \&c.}$$

$$\pm \frac{Az^{3n}}{p+3n} \pm \frac{Bz^{2n}}{q+3n} \pm \frac{Cz^{3n}}{r+3n} \pm \frac{Dz^{3n}}{s+3n} \pm \frac{Ez^{3n}}{t+3n}, \text{ \&c.}$$

Therefore the Sum of all these, or the whole Series,

$$\text{is} = \begin{cases} A \times \overline{\frac{1}{p} \pm \frac{z^n}{p+n} + \frac{z^{2n}}{p+2n} \pm \frac{z^{3n}}{p+3n} + \frac{z^{4n}}{p+4n}}, \text{ \&c.} \\ B \times \overline{\frac{1}{q} \pm \frac{z^n}{q+n} + \frac{z^{2n}}{q+2n} \pm \frac{z^{3n}}{q+3n} + \frac{z^{4n}}{q+4n}}, \text{ \&c.} \\ C \times \overline{\frac{1}{r} \pm \frac{z^n}{r+n} + \frac{z^{2n}}{r+2n} \pm \frac{z^{3n}}{r+3n} + \frac{z^{4n}}{r+4n}}, \text{ \&c.} \\ \text{\&c.} \end{cases}$$

But since $\overline{\frac{z^m}{m} \pm \frac{z^{m+n}}{m+n}} + \frac{z^{m+2n}}{m+2n}$, &c. is given $= S$, and $A \times \overline{\frac{1}{p} \pm \frac{z^n}{p+n} + \frac{z^{2n}}{p+2n}}$, &c. may be reduced to $\frac{A}{z^p} \times \overline{\frac{z^p}{p} \pm \frac{z^{p+n}}{p+n} + \frac{z^{p+2n}}{p+2n}}$, &c. it is evident that $A \times \overline{\frac{1}{p} \pm \frac{z^n}{p+n} + \frac{z^{2n}}{p+2n}}$, &c. will be equal to $\frac{A}{z^p} \times \overline{S - \frac{z^m}{m} \mp \frac{z^{m+n}}{m+n} - \frac{z^{m+2n}}{m+2n} \ldots\ldots \mp \frac{z^{p-n}}{p-n}}$; and in the same Manner will $B \times \overline{\frac{1}{q} \pm \frac{z^n}{q+n} + \frac{z^{2n}}{q+2n}}$, &c. appear equal to $\frac{B}{z^q} \times \overline{S - \frac{z^m}{m} \mp \frac{z^{m+n}}{m+n} \ldots \mp \frac{z^{q-n}}{q-n}}$, &c. and therefore the Sum of the whole Series propounded is

=

$$=\begin{cases} \frac{A}{z^p}\times\overline{S-\frac{z^m}{m}-\frac{z^{m+n}}{m+n}-\frac{z^{m+2n}}{m+2n}\cdots\cdots-\frac{z^{p-n}}{p-n}}. \\ \frac{B}{z^q}\times\overline{S-\frac{z^m}{m}-\frac{z^{m+n}}{m+n}-\frac{z^{m+2n}}{m+2n}\cdots\cdots-\frac{z^{q-n}}{q-n}}. \\ \frac{C}{z^r}\times\overline{S-\frac{z^m}{m}-\frac{z^{m+n}}{m+n}-\frac{z^{m+2n}}{m+2n}\cdots\cdots-\frac{z^{r-n}}{r-n}}, \text{ \&c.} \end{cases}$$

When all the Signs thereof are given affirmative, and equal to

$$\frac{A}{z^p}\times\overline{\pm S\mp\frac{z^m}{m}\pm\frac{z^{m+n}}{m+n}\mp\frac{z^{m+2n}}{m+2n}\cdots\cdots\cdots+\frac{z^{p-n}}{p-n}}.$$
$$\frac{B}{z^q}\times\overline{\pm S\mp\frac{z^m}{m}\pm\frac{z^{m+n}}{m+n}\mp\frac{z^{m+2n}}{m+2n}\cdots\cdots+\frac{z^{q-n}}{q-n}}.$$
$$\frac{C}{z^r}\times\overline{\pm S\mp\frac{z^m}{m}\pm\frac{z^{m+n}}{m+n}\mp\frac{z^{m+2n}}{m+2n}\cdots\cdots\cdots+\frac{z^{r-n}}{r-n}}, \text{ \&c.}$$

when they are given affirmative and negative alternately; in which Case, the last Term of every Line in the above Expression, must be taken affirmative. Q. E. I.

Note. When $p-n$ or $q-n$, &c. happens to be nothing or negative, the first Term (S) of the corresponding Line is *only* to be taken.

COROLLARY I.

Hence if the Series propounded be $\frac{a\times b\times c\times d, \text{ \&c.}}{m\times m+n\times m+2n, \text{ \&c.}}$ $\pm\frac{a+n\times b+n\times c+n}{m+n\times m+2n\times m+3n}$, &c. $\times z^n + \frac{a+2n\times b+2n\times c+2n, \text{ \&c.}}{m+3n\times m+4n\times m+4n, \text{ \&c.}}\times z^{2n}$ $\pm\frac{a+3n\times b+3n\times c+3n, \text{ \&c.}}{m+3n\times m+4n\times m+5n, \text{ \&c.}}\times z^{3n}$, &c. then p becoming $=m$, $q=m+n$, $r=m+2n$, &c. the Sum of the whole Series will

will be defined by $\frac{S}{z^n}\times\overline{A \pm Bz^{-n} + Cz^{-2n} \pm Dz^{-3n}}$, &c. $\mp \frac{1}{m}\times\frac{B}{z^n} - \overline{\frac{1}{m} \pm \frac{z^n}{m+n}}\times\frac{C}{z^{2n}} \mp \overline{\frac{1}{m} + \frac{z^n}{m+n} \pm \frac{z^{2n}}{m+2n}}\times\frac{D}{z^{3n}}$, &c.

COROLLARY II.

But if the Series to be ſummed, be compriſed in this Form $\frac{1}{p\times p+n\times p+2n, \&c.} \pm \frac{z^n}{p+n\times p+2n\times p+3n, \&c.}$, &c. and the number of Factors in the Denominator of each Term be denoted by $v+1$, we ſhall have A $\left(=\frac{1}{n\times 2n\times 3n\ldots\ldots vn}\right)=\frac{n^{-v}}{1.2.3.4\ldots v}$, $B=-vA$, $C=v\times\frac{v-1}{2}A$, $D=-v\times\frac{v-1}{2}\times\frac{v-2}{3}A$, &c. and therefore the Sum of the whole Series equal to $\frac{\overline{1-z^{-n}}|^v}{1.2.3.4\ldots v\times n^v z^p}\times\overline{S-\frac{z^m}{m}-\frac{z^{m+n}}{m+n}-\frac{z^{m+2n}}{m+2n}\ldots\ldots\frac{-z^{p-n}}{p-n}} + \frac{1}{1.2.3\ldots\ldots v\times n^v}\times\overline{\frac{vz^{-n}}{p}-v\times\frac{v-1}{2}\times\overline{\frac{z^{-2n}}{p}+\frac{z^{-n}}{p+n}}+v\times\frac{v-1}{2}\times\frac{v-2}{3}\times\overline{\frac{z^{-3n}}{p}+\frac{z^{-2n}}{p+n}+\frac{z^{-n}}{p+2n}}}$, &c. when all the Signs of the ſaid Series are given affirmative; but equal to $\pm\frac{\overline{1+z^{-n}}|^v}{1.2.3\ldots v\times n^v z^p}\times\overline{S-\frac{z^m}{m}+\frac{z^{m+n}}{m+n}+\frac{z^{m+2n}}{m+2n}\ldots\ldots\frac{z^{p-n}}{p-n}} - \frac{1}{1.2.3.4\ldots\ldots v\times n^v}\times\overline{\frac{vz^{-n}}{p}+v\times\frac{v-1}{2}\times\overline{\frac{z^{-2n}}{p}-\frac{z^{-n}}{p+n}}+v\times\frac{v-1}{2}\times\frac{v-2}{3}\times\overline{\frac{z^{-3n}}{p}-\frac{z^{-2n}}{p+n}+\frac{z^{-n}}{p+2n}}}$, &c. when they change alternately

alternately, in which Case the Sign + or — before $\overline{1+x^{-n}}|^{v}$ obtains according as $\frac{p-m}{n}$ is an even or an odd Number.

COROLLARY III.

Lastly, let z be taken = 1, and the Series propounded be $\frac{a\times b\times c\times d, \&c.}{p\times q\times r\times s, \&c.} + \frac{a+n\times b+n\times c+n, \&c.}{p+n\times q+n\times r+n, \&c.} + \frac{a+2n\times b+2n\times c+2n, \&c.}{p+2n\times q+2n\times r+2n, \&c.}$ &c. then, since the Sum of all the Coefficients A+B+C +D, &c. will be = 0 (*by the Corol. to the preceding Lemma*) when the Number of Factors in the Denominator of each Term is greater by two, at least, than the Number of those in the Numerator, we shall, in this Case, have $\overline{\frac{A}{z^p} + \frac{B}{z^q} + \frac{C}{z^r}, \&c.} \times \overline{S - \frac{z^m}{m} - \frac{z^{m+n}}{m+n} \cdots \frac{z^{p-n}}{p-n}}$ also equal to nothing; and therefore, by expunging this out of the general Expression, and substituting 1 instead of z, we have

$$\left.\begin{array}{l} -B\times\overline{\frac{1}{p}+\frac{1}{p+n}+\frac{1}{p+2n}\cdots\cdots\frac{1}{q-p}} \\ -C\times\overline{\frac{1}{p}+\frac{1}{p+n}+\frac{1}{p+2n}\cdots\cdots\frac{1}{r-p}} \\ -D\times\overline{\frac{1}{r}+\frac{1}{p+n}+\frac{1}{p+2n}\cdots\cdots\frac{1}{s-p}} \\ \&c. \end{array}\right\} = \begin{array}{l} \frac{a\times b\times c\times d,\ \&c.}{p\times q\times r\times s,\ \&c.} \\ +\frac{a+n\times b+n\times c+n,\ \&c.}{p+n\times q+n\times r+n,\ \&c.} \\ \&c. \end{array}$$

EXAMPLE I.

Let there be given the Series $\frac{z}{1} - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7}$, &c. or the Arc of a Circle, whose Radius is 1, and Tangent z, and let the Sum of the Series $\frac{2}{3\cdot5} - \frac{4z^2}{5\cdot7} + \frac{6z^4}{7\cdot9} - \frac{8z^6}{9\cdot11}$, &c. be be required. Here, by comparing these with the general Expressions, we have $m=1$, $n=2$, $a=2$, $p=3$, $q=5$, $p-n=1$, $q-n=3$, A $(=\frac{2-3}{5-3})=\frac{-1}{2}$, B$=(\frac{2-5}{3-5})=\frac{3}{2}$, C$=0$, &c. and consequently $\frac{-1}{2z^3}\times\overline{-S+\frac{z}{1}}+\frac{3}{2z^5}\times\overline{S-\frac{z}{1}+\frac{z^3}{3}}$, or $\frac{\overline{3+z^2}\times S-3z}{2z^5}$ equal to the Value sought.

EXAMPLE II.

Let there be given the Sum of the Infinite Series $\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}$, &c. expressing the hyperbolic Logarithm of $\frac{1}{1-x}$; to find the Sum of the Infinite Series $\frac{1}{1.2.3.4}+\frac{x}{2.3.4.5}+\frac{x^2}{3.4.5.6}+\frac{x^3}{4.5.6.7}+\frac{x^4}{5.6.7.8}$. Here $m=1$, $n=1$, $p=1$, $q=2$, $r=3$, $s=4$, $t=0$, &c. therefore (by Corol. II.) $\frac{\overline{1-x^{-1}}|^3}{1.2.3x}\times S+\frac{1}{1.2.3}\times\overline{\frac{3x^{-1}}{1}-3\times\frac{x^{-2}}{1}+\frac{x^{-1}}{2}+\frac{x^{-3}}{1}+\frac{x^{-2}}{2}+\frac{x^{-1}}{3}}$, or $-\frac{\overline{1-x}|^3}{6x^4}\times S+\frac{11}{36x}-\frac{5}{12x^2}+\frac{1}{6x^3}$ is the Value that was to be found; which therefore, when $x=1$, or the proposed Series is $\frac{1}{1.2.3.4}+\frac{1}{2.3.4.5}+\frac{1}{3.4.5.6}$, &c. will be barely equal to $\frac{11}{36}-\frac{5}{12}+\frac{1}{6}$, or $\frac{1}{18}$.

EXAMPLE III.

Where it is required to find the Sum of the Infinite Series $\frac{1}{1.2.4.7}+\frac{1}{2.3.5.8}+\frac{1}{3.4.6.9}+\frac{1}{4.5.7.10}$, &c. In this Case, we

have

have $p=1$, $q=2$, $r=4$, $s=7$, B $\left(\frac{1}{1-2\times4-2\times7-2}\right)=\frac{-1}{10}$, C $\left(\frac{1}{1-4\times2-4\times7-4}\right)=\frac{1}{18}$, D $\left(\frac{1}{1-7\times2-7\times4-7}\right)=-\frac{1}{90}$, E $=0$, &c. and therefore (by Corol. III.) $\frac{1}{10}-\frac{1}{18}\times\overline{1+\frac{1}{2}+\frac{1}{3}}+\frac{1}{90}\times\overline{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}}=\frac{137}{5400}$ is the Value required.

EXAMPLE IV.

Let there be given (S) the hyperbolical Logarithm of $\frac{1}{1-z}$, to find the Sum of the Series $\frac{19}{1.2.3}+\frac{28z}{2.3.4}+\frac{39z^2}{3.4.5}+\frac{52z^3}{4.5.6}+\frac{67z^4}{5.6.7}$, &c. This Series may be resolved into $\frac{4}{1.2.3}+\frac{4z}{2.3.4}+\frac{4z^2}{3.4.5}+\frac{4z^2}{4.5.6}$, &c. and $\frac{3.5}{1.2.3}+\frac{4.6z}{2.3.4}+\frac{5.7z^2}{3.4.5}$, &c. that is, into $4\times\overline{\frac{1}{1.2.3}+\frac{z}{2.3.4}+\frac{z^2}{3.4.5}}$, &c. and $\frac{5}{1.2}+\frac{6z}{2.3}+\frac{7z^2}{3.4}$, &c. Now the Sum of the former of these, by proceeding as in the foregoing Examples, will be $\overline{1-\frac{1}{z}}|^2\times\frac{2S}{z}+\frac{3}{z}-\frac{2}{z^2}$, and that of the latter equal to $\frac{4S}{z}-\frac{3\times\overline{S-z}}{z^2}$; which Values therefore, added together, give $\overline{\frac{2}{z^3}-\frac{7}{z^2}+\frac{6}{z}}\times S+\frac{6}{z}-\frac{2}{z^2}$ for the Sum of the whole Series propounded. In the same manner may the Sum of any other Series be determined, when the Denominators thereof come under the general Form in the Proposition, and the last Differences of its Numerators are equal, provided the number of Differences before you come to the last, be always less than the number of Factors in the Denominator of each Term.

EXAMPLE V.

Let there be given the Sum of the Series $z + \frac{z^5}{5} + \frac{z^9}{9} + \frac{z^{13}}{13}$, &c. (which is equal to half the Sum of the hyperbolic Logarithm of $\frac{\overline{1+z}|^{\frac{1}{2}}}{1-z}$ and the Length of the Arc whose Radius is 1, and Tangent z) and let it be required to find the Sum of the Infinite Series $\frac{y}{1.5.13} + \frac{3y^2}{5.9.17} + \frac{6y^3}{9.13.21} + \frac{10y^4}{13.17.25} + \frac{15y^5}{17.21.29}$, &c. Put $x^4 = y$, or $z = y^{\frac{1}{4}}$; then because the Numerators may be reduced to $\frac{z^4}{32} \times 4 \times 8$, $\frac{z^4}{32} \times 8 \times 12\, z^4$, $\frac{z^4}{32} \times 12 \times 16 z^8$, $\frac{z^4}{32} \times 16 \times 20\, z^{12}$, &c. the Series itself will be changed to $\frac{z^4}{32} \times \overline{\frac{4.8}{1.5.13} + \frac{8.12z^4}{5.9.17} + \frac{12.16z^8}{9.13.21}}$, &c. where m being $= 1$, $n = 4$, $p = 1$, $q = 5$, $r = 13$, $a = 4$, $b = 8$, we shall have $A = \frac{7}{16}$, $B = \frac{3}{32}$, $C = \frac{15}{32}$; and therefore $\frac{z^4}{32}$ into $\frac{7}{16z} \times S + \frac{3}{32z^5} \times \overline{S - z} + \frac{15}{32z^{13}} \times \overline{S - \frac{z}{1} - \frac{z^5}{5} - \frac{z^9}{9}} = \overline{14 + \frac{3}{z^4} + \frac{15}{z^{12}}} \times \frac{z^3 S}{1024} - \frac{14}{3072} - \frac{3}{1024z^4} - \frac{15}{1024z^8}$
$= \frac{\overline{14 + \frac{3}{y} + \frac{15}{y^3}} \times y^{\frac{3}{4}} S - \frac{14}{3} - \frac{3}{y} - \frac{15}{y^2}}{1024}$ is the exact Value which was to be found.

PROPOSITION V.

Suppofing the Sum of the Infinite Series $bx^{k} + cx^{k+l} + dx^{k+2l} + ex^{k+3l}$, *&c. when the Terms thereof are refpectively divided by thofe of any arithmetical Progreffion, whofe common Difference is* m, *to be given; 'tis propofed to find the Sum of the Infinite Series* $\frac{a+o \times \overset{\prime}{a}+o \times \overset{\prime\prime}{a}+o, \&c. \times bx^{k}}{p+o, q+o, r+o, \&c.} + \frac{a+m \times \overset{\prime}{a}+m \times \overset{\prime\prime}{a}+m, \&c. \times cx^{k+l}}{p+m \times q+m \times r+m, \&c.} + \frac{a+2m \times \overset{\prime}{a}+2m \times \overset{\prime\prime}{a}+2m, \&c. \times dx^{k+2l}}{p+2m \times q+2m \times r+2m, \&c.}$ *&c. fuppofing* p, q, r, s, *&c. to reprefent any unequal Numbers, and* a, $\overset{\prime}{a}$, $\overset{\prime\prime}{a}$, *&c. any Numbers equal or unequal, and that the Number of Factors in the Numerator of each Term is lefs than the Number of Factors in the Denominator.*

Let the Values of the Series $\frac{bx^{k}}{p} + \frac{cx^{k+l}}{p+m} + \frac{dx^{k+2l}}{p+2m}$, &c. $\frac{bx^{k}}{q} + \frac{cx^{k+l}}{q+m} + \frac{dx^{k+2l}}{k+2m}$, &c. $\frac{bx^{k}}{r} + \frac{cx^{k+l}}{r+m} + \frac{dx^{k+2l}}{r+2m}$, &c. (which are fuppofed given) be denoted by P, Q, R, &c. refpectively; and let $A = \frac{a-p.\overset{\prime}{a}-p.\overset{\prime\prime}{a}-p, \&c.}{q-p.r-p.s-p, \&c.}$, $B = \frac{a-q.\overset{\prime}{a}-q.\overset{\prime\prime}{a}-q, \&c.}{p-q.r-q.s-q, \&c.}$ &c. Then it will appear from the foregoing Lemma, that $\frac{A}{p+o}$, $\frac{B}{q+o}$, $\frac{C}{r+o}$, $\frac{D}{s+o}$, &c. are the fimple Fractions into which $\frac{a+o.\overset{\prime}{a}+o.\overset{\prime\prime}{a}+o, \&c.}{p+o.q+o.r+o, \&c.}$ may be refolved; and therefore it is evident, that the firft Term of the propofed Series will be equal to $\frac{Abx^{k}}{p+o} + \frac{Bbx^{k}}{q+o} + \frac{Cbx^{k}}{r+o}$, &c. or $\frac{Abx^{k}}{p} + \frac{Bbx^{k}}{q} + \frac{Cbx^{k}}{r}$, &c. and in the fame Manner it will appear, that the fecond and third Terms, &c. will be equal to $\frac{Acx^{k+l}}{p+m} + \frac{Bcx^{k+l}}{q+m} + \frac{Ccx^{k+l}}{r+m}$ &c. and $\frac{Adx^{k+2l}}{p+2m} + \frac{Bdx^{k+2l}}{q+2m} + \frac{Cdx^{k+2l}}{r+2m}$, &c. &c. Therefore the whole Series is

$$= \begin{cases} \frac{Abx^{k}}{p} + \frac{Bbx^{k}}{q} + \frac{Cbx^{k}}{r} + \frac{Dbx^{k}}{s}, \text{ \&c.} \\ \frac{Acx^{k+l}}{p+m} + \frac{Bcx^{k+l}}{q+m} + \frac{Ccx^{k+l}}{r+m} + \frac{Dcx^{k+l}}{s+m}, \text{ \&c.} \\ \frac{Adx^{k+2l}}{p+2m} + \frac{Bdx^{k+2l}}{q+2m} + \frac{Cdx^{k+2l}}{r+2m} + \frac{Ddx^{k+2l}}{s+2m}, \text{ \&c.} \\ \text{\&c.} \quad \text{\&c.} \quad \text{\&c.} \quad \text{\&c.} \end{cases}$$

Which, becauſe the Value $(A \times \overline{\frac{bx^{k}}{p} + \frac{cx^{k+l}}{p+m} + \frac{dx^{k+2l}}{p+2m} + \frac{ex^{k+3l}}{p+3m}}$, &c.) of the firſt Column towards the Left-Hand is $= AP$; of the ſecond, $= BQ$, &c. will therefore become $AP + BQ + CR + DS$, &c. Q. E. I.

COROLLARY I.

If the Series propoſed be $\frac{bx^{k}}{p.q.r, \text{ \&c.}} + \frac{cx^{k+l}}{p+m.q+m.r+m, \text{ \&c.}} + \frac{dx^{k+2l}}{p+2m.q+2m.r+2m, \text{ \&c.}}$, &c. then will A be $= \frac{1}{q-p.r-p.s-p, \text{ \&c.}}$, $B = \frac{1}{p-q.r-q.s-q, \text{ \&c.}}$, &c. and the Value of the Series as above exhibited.

COROLLARY II.

But if the Series proposed be $\frac{bx^k}{p.p+m.p+2m,\ \&c.}$ $\frac{cx^{k+1}}{p+m.p+2m.p+3m}+\frac{dx^{k+2l}}{p+2m.p+3m.p+4m}$, &c. and the Number of Factors in the Denominator of each Term be denoted by $n+1$; then q becoming $=p+m$, $r=p+2m$, $s=p+3m$, &c. we shall have $A=\frac{1}{m.2m.3m.4m\ldots.nm}$, $B=\frac{1}{-m.m.2m.3m\ldots.\overline{n-1}\times m}$, $C=\frac{1}{-2m.-m.m.2m\ldots\overline{n-2}\times m}$, &c. or $A=\frac{1}{1.2.3.4\ldots n\times m^n}$, $B=-\frac{nA}{1}$, $C=\frac{n}{1}\times\frac{n-1}{2}A$, $D=-\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}A$, &c. and therefore the whole Series $=\frac{P-nQ+\frac{n}{1}\times\frac{n-1}{2}R-\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}S,\ \&c.}{1.2.3.4.5\ldots..n\times m^n}$.

EXAMPLE I.

Where the Sum of the Series $\frac{x}{1}-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}$, &c. (expressing the Arch of a Circle, whose Radius is 1, and Tangent x) being given, it is required to find the Sum of the Series $\frac{x}{1.3.5}-\frac{x^3}{3.5.7}+\frac{x^5}{5.7.9}$, &c. In this Case we have $p=1$, $q=3$, $r=5$, $n=2$, $m=2$, $P=\frac{x}{1}-\frac{x^3}{3}+\frac{x^5}{5}$, &c. $Q=\frac{x}{3}-\frac{x^3}{5}+\frac{x^5}{7}$, &c. $\left(=\frac{x-P}{x^2}\right)$ $R=\frac{x}{5}-\frac{x^3}{7}+\frac{x^5}{9}$, &c. $\left(\frac{P-x+\frac{1}{3}x^3}{x^4}\right)$ and therefore by Cor. II. $\left(\frac{P-nQ+n\times\frac{n-1}{2}R,\ \&c.}{1.2.3\ldots n\times m^n}\right)$ =

$= \dfrac{P - 2 \times \frac{x-P}{x^2} + \frac{P - x + \frac{1}{3}x^3}{x^4}}{8} = \dfrac{\overline{1+x^2}|^2 \times 3P - 3x - 5x^3}{24x^4}$ equal to the Value sought.

EXAMPLE II.

Let the Sum of the Series $x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7}$, &c. = the Hyp. Log. of $\sqrt{\frac{1+x}{1-x}}$ be given, and let it be required to find the Sum of the Series $\frac{2x}{1 \cdot 3 \cdot 5} + \frac{4x^3}{3 \cdot 5 \cdot 7} + \frac{6x^5}{5 \cdot 7 \cdot 9} + \frac{8x^7}{7 \cdot 9 \cdot 11}$, &c. Here, a being $= 2$, $p = 1$, $q = 3$, $r = 5$, A $\left(= \frac{a-p}{q-p \times r-p}\right) = \frac{1}{8}$, B $\left(= \frac{a-q}{p-q \times r-q}\right) = \frac{1}{4}$, C $\left(= \frac{a-r}{p-r \times q-r}\right) = -\frac{3}{8}$, $P = x + \frac{x^3}{3} + \frac{x^5}{5}$, &c. $Q = \frac{x}{3} + \frac{x^3}{5} + \frac{x^5}{7}$, &c. $\left(= \frac{P-x}{x^2}\right)$ and $R = \frac{x}{5} + \frac{x^3}{7} + \frac{x^5}{9}$, &c. $\left(\frac{P - x - \frac{1}{3}x^3}{x^4}\right)$ we have $\frac{2x}{1 \cdot 3 \cdot 5} + \frac{4x^3}{3 \cdot 5 \cdot 7} + \frac{6x^5}{5 \cdot 7 \cdot 9}$, &c. $(= AP + BQ + CR)$ equal to $\frac{\overline{x^4 + 2x^2 - 3} \times P + 3x - x^3}{8x^4}$; which therefore, when $x = 1$, will become barely $= \frac{1}{4}$.

PROPOSITION VI.

Supposing n − r *to represent any whole positive Number; 'tis required to find the Sum of the Infinite Series* $1 \pm \frac{n \cdot px}{1 \cdot r} + \frac{n \cdot n+1 \cdot p \cdot p+1 \cdot x^2}{1 \cdot 2 \cdot r \cdot r+1} \pm \frac{n \cdot n+1 \cdot n+2 \cdot p \cdot p+1 \cdot p+2 \cdot x^3}{1 \cdot 2 \cdot 3 \cdot r \cdot r+1 \cdot r+2} + \frac{n \cdot n+1 \cdot n+2 \cdot n+3 \cdot p \cdot p+1 \cdot p+2 \cdot p+3 \cdot x^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot r \cdot r+1 \cdot r+2 \cdot r+3}$, *&c. where* r, p *and* x *denote any Quantities at pleasure.*

The Solution of this Problem is easily derived from Proposition I. For the proposed Series may be considered as generated

nerated by a reſpective Multiplication of the Terms of $\overline{1+x}|^{-p}$ expanded in a Series, by thoſe of the Series 1, $\frac{n}{r}$, $\frac{n}{r} \times \frac{n+1}{r+1}$, $\frac{n}{r} \times \frac{n+1}{r+1} \times \frac{n+2}{r+2}$, &c. therefore if the Differences of theſe laſt Quantities be continually taken, according to the Method there propoſed; then the firſt Difference of the firſt Order being $= \frac{n-r}{r}$, of the ſecond $= \frac{n-r}{r} \times \frac{n-r-1}{r+1}$, of the third $= \frac{n-r}{r} \times \frac{n-r-1}{r+1} \times \frac{n-r-2}{r+2}$, &c. it is evident, from what is there proved, that the Sum of the whole propoſed Series, putting $n-r=q$, will be truly defined by $\overline{1 \mp x}|^{-p}$ $\pm \frac{q \cdot px \cdot \overline{1 \mp x}|^{-p-1}}{1 \cdot r} + \frac{q \cdot q-1 \cdot p \cdot p+1 \cdot x^2 \cdot \overline{1 \mp x}|^{-p-2}}{1 \cdot 2 \cdot r \cdot r+1}$, &c. or $\frac{1}{\overline{1 \mp x}|^{p}} \times \overline{1 \pm \frac{pqx}{r \cdot \overline{1 \mp x}|^{1}} + \frac{q \cdot q-1 \cdot p \cdot p+1 \cdot x^2}{1 \cdot 2 \cdot r \cdot r+1 \cdot \overline{1 \mp x}|^{2}}}$ $\overline{\pm \frac{q \cdot q-1 \cdot q-2 \cdot p \cdot p+1 \cdot p+2 \cdot x^3}{1 \cdot 2 \cdot 3 \cdot r \cdot r+1 \cdot r+2 \cdot \overline{1 \mp x}|^{3}}}$, &c. which, when q, or $n-r$ is a whole poſitive Number, will always terminate in $q+1$ Terms; and therefore in all ſuch Caſes, its exact Value will from hence be obtained.

Q. E. I.

Note. When two Signs are prefixed to one Term, as above, the upper takes place when all the Signs of the propoſed Series are given affirmative; but the lower when they are given + and — alternately.

CO-

COROLLARY I.

Therefore if p be taken $=1$, we shall have $1 \pm \frac{nx}{r} + \frac{n.\overline{n+1}.x^2}{r.\overline{r+1}}$ $\pm \frac{n.\overline{n+1}.\overline{n+2}.x^2}{r.\overline{r+1}.\overline{r+2}}$, &c. $= \frac{1}{1 \mp x} \pm \frac{q.x}{r.\overline{1 \mp x}|^2} + \frac{q.\overline{q-1}.x^2}{r.\overline{r+1}.\overline{1 \mp x}|^3}$ $\pm \frac{q.\overline{q-1}.\overline{q-2}.x^3}{r.\overline{r+1}.\overline{r+2}.\overline{1 \mp x}|^4} + \frac{q.\overline{q-1}.\overline{q-2}.\overline{q-3}.x^4}{r.\overline{r+2}.\overline{r+2}.\overline{r+3}.\overline{1 \mp x}^5}$, &c.

COROLLARY II.

But if p be taken $=n$, and $r=1$, then we shall have $1 \pm n^2 x + \frac{n^2}{1} . \frac{\overline{n+1}|^2}{4} \times x^2 \pm \frac{n^2}{1} . \frac{\overline{n+1}|^2}{4} . \frac{\overline{n+2}|^2}{9} \times x^3 + \frac{n^2}{1}$ $\frac{\overline{n+1}|^2}{4} . \frac{\overline{n+2}|^2}{9} . \frac{\overline{n+3}|^2}{16} \times x^4$, &c. $= \frac{1}{\overline{1 \mp x}^n}$ into $1 \pm \frac{n.\overline{n-1}}{1} \times$ $\frac{x}{1 \mp x} + \frac{n.\overline{n-2}.\overline{n^2-1}}{1.4} \times \frac{x^2}{\overline{1 \mp x}|^2} \pm \frac{n.\overline{n-3}.\overline{n^2-1}.\overline{n^2-4}}{1.4.9} \times \frac{x^3}{\overline{1 \mp x}|^3}$ $+ \frac{n.\overline{n-4}.\overline{n^2-1}.\overline{n^2-4}.\overline{n^2-9}}{1.4.9.16} \times \frac{x^4}{\overline{1 \mp x}|^4}$, &c.

COROLLARY III.

Also from the foregoing general Expression, the Sum of any Infinite Series as $1 \pm \frac{n}{r} \times \frac{px}{v} + \frac{n}{r} \times \frac{n+m}{r+m} \times \frac{p}{v} \times \frac{p+v}{2v} \times x^2$ $\pm \frac{n}{r} \times \frac{n+m}{r+m} \times \frac{n+2m}{r+2m} \times \frac{p}{v} \times \frac{p+v}{2v} \times \frac{p+2v}{3v} \times x^3$, &c. where $\frac{n-r}{m}$ is a whole positive Number may be easily derived: For this Series may be changed to $1 \pm \frac{n}{r} \times \frac{px}{v} + \frac{n}{r} \times \frac{\frac{n}{m}+1}{\frac{r}{m}+1} \times$

$\frac{p}{v}$

$\frac{p}{v} \times \frac{\frac{p}{v}+1}{2}$, &c. and therefore by writing $\frac{p}{v}$, $\frac{n}{m}$ and $\frac{r}{m}$ instead of p, n and r in that Expreſſion, we ſhall have $\frac{1}{\overline{1 \mp x}|^{\frac{p}{v}}} \times$

$$\overline{1 \pm \frac{q.p}{r.v} \times \frac{x}{1 \mp x} + \frac{q.q-m.p.p+v.x^2}{r.r+m.v.2v.\overline{1 \mp x}|^2}}$$

$$\overline{\pm \frac{q.q-m.q-2m.p.p+v.p+2v.x^3}{r.r+m.r+2m.v.2v.3v.\overline{1 \mp x}|^3}}, \text{ \&c.} = 1 \pm \frac{n.px}{r.v}$$

$$+ \frac{n.n+m.p.p+v.x^2}{r.r+m.v.2v} \pm \frac{n.n+m.n+2m.p.p+v.p+2v.x^3}{r.r+m.r+2m.v.2v.3v}, \text{ \&c.}$$

EXAMPLE I.

Let it be required to find the Sum of the Infinite Series $1 + 2^2 \times x + \frac{2^2}{1} \times \frac{3^2}{4} \times x^2 + \frac{2^2}{1} \times \frac{3^2}{4} \times \frac{4^2}{9} \times x^3$, &c. or $1 + 4x + 9x^2 + 16x^3$, &c. then by comparing this Series with that in Corollary II. we ſhall have $n = 2$, and conſequently $\frac{1}{\overline{1-x}|^2} \times \overline{1 + \frac{2x}{1-x}} = \frac{1+x}{\overline{1-x}|^3}$ equal to the Value ſought.

EXAMPLE II.

Let it be propoſed to find the Sum of the Infinite Series $1 - \frac{1.10z^2}{2.4} + \frac{1.3.10.13z^4}{2.4.4.7} - \frac{1.3.5.10.13.16z^6}{2.4.6.4.7.10} + \frac{1.3.5.7.10.13.16.19z^8}{2.4.6.8.4.7.10.13}$, &c. then p being $=1$, $v=2$, $n=10$, $r=4$, $m=3$, $x=z^2$, and q $(n-r)=6$, we ſhall, by ſubſtituting theſe Values in Corollary III. have $\frac{1}{\overline{1+z^2}|^{\frac{1}{2}}} \times$

$$\overline{1 - \frac{6.1}{4.2} \times \frac{z^4}{1+z^2} + \frac{6.3.1.3}{4.7.2.4} \times \frac{z^4}{\overline{1+z^2}|^2}} = \frac{1}{\overline{1+z^2}|^{\frac{1}{2}}} \times$$

$$\overline{1 - \frac{3}{4} \times \frac{z^2}{1+z^2} + \frac{27}{112} \times \frac{z^4}{\overline{1+zz}|^2}}$$

for the Value that was to be found.

PRO-

PROPOSITION VII.

TO determine the Sum of any Infinite Series, as $1 + \frac{n.p}{r.v} \times x + \frac{n.\overline{n+1}.p.\overline{p+1}}{r.\overline{r+1}.v.\overline{v+1}} \times x^2 + \frac{n.\overline{n+1}.\overline{n+2}.p.\overline{p+1}.\overline{p+2}}{r.\overline{r+1}.\overline{r+2}.v.\overline{v+1}.\overline{v+2}} \times x^3$, *&c. where* (v) *one of the two Divisors* r, v, *is a whole positive Number, and the Difference* (n—r) *between the other* (r) *and one of the Multiplyers* (n) *a whole positive Number also.*

Put $\frac{p+1-v}{1} \times \frac{p+2-v}{2} \times \frac{p+3-v}{3} \ldots\ldots \times \frac{p-1}{v-1} = d$, $\frac{p}{v} \times d = e$, $\frac{p+1}{v+1} \times e = f$, $\frac{p+2}{v+2} \times f = g$, $\frac{p+3}{v+3} \times g = h$, &c. and let $S = 1 + \frac{n.p}{r.v} \times x + \frac{n.\overline{n+1}.p.\overline{p+1}}{r.\overline{r+1}.v.\overline{v+1}} \times x^2$, &c. and then, the whole Equation being multiplied by dx^{v-1}, we ſhall have $dSx^{v-1} = dx^{v-1} + \frac{n}{r} \times ex^{v} + \frac{n.\overline{n+1}}{r.\overline{r+1}} \times fx^{v+1} + \frac{n.\overline{n+1}.\overline{n+2}}{r.\overline{r+1}.\overline{r+2}} \times gx^{v+2}$, &c. where it is evident that dx^{v-1}, ex^{v}, fx^{v+1}, gx^{v+2}, &c. expreſs the Terms that remain of the Binomial $\overline{1-x}^{-p-1+v}$ expanded, after the $v-1$ firſt Terms are taken away. Wherefore, to deduce the true Value of the Series $dx^{v-1} + \frac{n}{r} \times ex^{v}$, &c. (according to Prop. I.) let the $v-1$ Terms of the Binomial, which are not above expreſſed, be denoted by $A + Bx + Cx^2 + Dx^3 \ldots\ldots bx^{v-3} + cx^{v-2}$, and let the Series $dx^{v-1} + \frac{n}{r} \times ex^{v} + \frac{n.\overline{n+1}}{r.\overline{r+1}} \times fx^{v+1}$, &c. be continued downwards by the ſame Law that it is continued upwards, ſo that all thoſe Terms may be taken in; and then, from ſuch Continuation, there will ariſe $\frac{r-v+1}{n-v+1} \times \frac{r-v+2}{n-v+2} \times \frac{r-v+3}{n-v+3} \ldots\ldots \frac{r-1}{n-1} \times$

$A \ldots\ldots + \frac{r-2}{n-2} \times \frac{r-1}{n-1} \times b x^{v-3} + \frac{r-1}{n-1} \times c x^{v-2}$, equal to $QA + \frac{n-v+1}{r-v+1} \times QBx + \frac{n-v+1}{r-v+1} \times \frac{n-v+2}{r-v+2} \times QCx^2 \ldots\ldots \frac{r-1}{n-1} \times c x^{v-2}$, by writing $Q = \frac{r-v+1}{n-v+1} \times \frac{r-v+2}{n-v+2} \times \frac{r-v+3}{n-v+3} \ldots\ldots \frac{r-1}{n-1}$: Which being therefore added to both sides of the Equation, we shall have $dSx^{v-1} + Q \times \overline{A + \frac{n-v+1}{r-v+1} \times Bx + \frac{n-v+1}{r-v+1} \times \frac{n-v+2}{r-v+2} \times Cx^2}$, &c. $= QA + \frac{n-v+1}{r-v+1} \times QBx + \frac{n-v+1}{r-v+1} \times \frac{n-v+2}{r-v+2} \times QCx^2 \ldots\ldots \frac{r-1}{n-1} \times c x^{v-2} + d x^{v-1} + \frac{n}{r} \times e x^{v} + \frac{n.n+1}{r.r+1} \times f x^{v+1}$, &c. which last Expression, on the Right-hand-side, may be considered as generated by a respective Multiplication of the Terms of the two following Lines

$$A + \quad Bx \qquad + \ldots\ldots b x^{v-3} \quad + c x^{v-2}, \text{\&c.}$$
$$Q + \frac{n-v+1}{r-v+1} \times Q + \ldots\ldots \frac{r-2}{n-2} \times \frac{r-1}{n-1} + \frac{r-1}{n-1}, \text{\&c.}$$

whereof the former is the Binomial $\overline{1-x}|^{-p-1+v}$ expanded, and the latter one regular Infinite Series, continued throughout by the same Law. Let, therefore, the Differences of the Terms of this last Series be continually taken (*according to the forementioned Proposition*) and then, the first Difference of the first Order being $\frac{n-r}{r-v+1} \times Q$, of the second, $\frac{n-r.n-r-1}{r-v+1.r-v+2} \times Q$, &c. we shall have $\frac{Q}{\overline{1-x}|^{p+1-v}} \times \overline{1 + \frac{n-r}{r-v+1} \times \frac{Bx}{1-x} + \frac{n-r.n-r-1}{r-v+1.r-v+2} \times \frac{Cx^2}{\overline{1-x}|^2}}$, &c. equal to $QA + \frac{n-v+1}{r-v+1} \times QBx + \frac{n-v+1.n-v+2}{r-v+1.r-v+2} \times Qcx^2 \ldots\ldots$

$r-1$

$\frac{r-1}{n-1} \times cx^{v-2} + dx^{v-1} + \frac{n}{r} \times ex^{v}$, &c. (*by what is there demonstrated*) which, therefore, is also equal to $dSx^{v-1} + Q \times \overline{A + \frac{n-v+1}{r-v+1} \times Bx + \frac{n-v+1.n-v+2}{r-v+1.r-v+2} \times Cx^2}$, &c. Whence S comes out equal to $\frac{Q}{dx^{v-1}}$ multiply'd into the Difference of the two following Series $\frac{1}{\overline{1-x}|^{p+1-v}} \times \overline{1 + \frac{n-r}{r-v+1} \times \frac{Bx}{1-x} + \frac{n-r.n-r-1}{r-v+1.r-v+2} \times \frac{Cx^2}{\overline{1-x}|^2}}$, &c. $A + \frac{n-v+1}{r-v+1} \times Bx + \frac{n-v+1.n-v+2}{r-v+1.r-v+2} \times Cx^2$, &c. whereof the latter is to be continued to as many Terms, as there are Units in $v-1$, and the former till it terminates; which, as $n-r$ is a whole positive Number, will always come to pass in $n-r+1$ Terms.

Q. E. I.

COROLLARY I.

Hence may the Sum of the Series $1 - \frac{n.p}{r.v} \times x + \frac{n.n+1.p.p+1}{r.r+1.v.v+1} x^2 - \frac{n.n+1.n+2.p.p+1.p+2}{r.r+1.r+2.v.v+1.v+2} \times x^3$, &c. where the Signs change alternately, be easily deduced; for let $-x$ be substituted instead of x, in the foregoing general Expression, and then we shall have $\pm \frac{Qx^{1-v}}{d \times \overline{1+x}|^{p+1-v}} \times \overline{1 - \frac{n-r}{r-v+1} \times \frac{Bx}{1+x} + \frac{n-r.n-r-1}{r-v+1.r-v+2} \times \frac{Cx_2}{\overline{1+x}|^2}}$, &c. $\mp \frac{Q}{dx^{v-1}} \times \overline{A - \frac{n-v+1}{r-v+1} \times Bx + \frac{n-v+1.n-v+2}{r-v+1.r-v+2} \times Cx^2}$, &c. for the required Value in this Case. Therefore, generally, the Sum of the Series

$1 \pm \frac{n.p}{r.v} \times x + \frac{n.\overline{n+1}.p.\overline{p+1}}{r.\overline{r+1}.v.\overline{v+1}} \times x^2 \pm \frac{n.\overline{n+1}.\overline{n+2}.p.\overline{p+1}.\overline{p+2}}{r.\overline{r+1}.\overline{r+2}.v.\overline{v+1}.\overline{v+2}} \times$ x^3, &c. will be truly represented by $\frac{bx^{1-v}}{\overline{1 \mp x}|^{p+1-v}} \times$

$$\overline{1 \pm \frac{\overline{n-r}.\overline{p-v+1}}{\overline{r-v+1}\,.\,1} \times \frac{x}{1 \mp x} + \frac{\overline{n-r}.\overline{n-r-1}.\overline{p-v+1}.\overline{p-v+2}}{\overline{r-v+1}.\overline{r-v+2}\,.\,1\,.\,2} \times \frac{x^2}{\overline{1 \mp x}|^2}}$$

$$\overline{\pm \frac{\overline{n-r}.\overline{n-r-1}.\overline{n-r-2}.\overline{p-v+1}.\overline{p-v+2}.\overline{p-v+3}}{\overline{r-v+1}.\overline{r-v+2}.\overline{r-v+3}\,.\,1\,.\,2\,.\,3} \times \frac{x^3}{\overline{1 \mp x}|^3}}, \text{\&c.} - bx^{1-v} \times$$

$$\overline{1 \pm \frac{\overline{n-v+1}.\overline{p-v+1}}{\overline{r-v+1}\,.\,1} \times x + \frac{\overline{n-v+1}.\overline{n-v+2}.\overline{p-v+1}.\overline{p-v+2}}{\overline{r-v+1}.\overline{r-v+2}\,.\,1\,.\,2} \times x^2}, \text{\&c.}$$

where b is equal to $\pm \overline{\frac{1.\overline{r-v+1}}{\overline{n-v+1}.\overline{p-v+2}}} \times \overline{\frac{2.\overline{r-v+2}}{\overline{n-v+2}.\overline{p-v+2}}} \times$ $\overline{\frac{3.\overline{r-v+3}}{\overline{n-v+3}.\overline{p-v+3}}}$, &c. continued to $v-1$ Factors, in which last Value the Sign — prevails only when the Signs of the proposed Series are + and — alternately, and v, at the same time, is an even Number.

COROLLARY II.

Moreover from hence the Sum of any Infinite Series as $1 \pm \frac{n.p}{r.v} \times x + \frac{n.\overline{n+m}.p.\overline{p+w}}{r.\overline{r+m}.v.\overline{v+w}} \times x^2 \pm \frac{n.\overline{n+m}.\overline{n+2m}.p.\overline{p+w}.\overline{p+2w}}{r.\overline{r+m}.\overline{r+2m}.v.\overline{v+w}.\overline{v+2w}} \times$ x^3, &c. where $\frac{v}{w}$ and $\frac{n-r}{m}$ are whole positive Numbers, may be easily derived; for since this Series may be reduced to $1 \pm \frac{\frac{n}{m}.\frac{p}{w}}{\frac{r}{m}.\frac{v}{w}} \times x + \frac{\frac{n}{m}.\overline{\frac{n}{m}+1}.\frac{p}{w}.\overline{\frac{p}{w}+1}}{\frac{r}{m}.\overline{\frac{r}{m}+1}.\frac{v}{w}.\overline{\frac{v}{w}+1}} \times x^2$, &c. let $\frac{n}{m}$, $\frac{r}{m}$, $\frac{p}{w}$, and $\frac{v}{w}$ be respectively substituted for n, r, p and v, in the preceding Corollary, and let $n - r = q$, $r + m - \frac{mv}{w} = s$, $p - v + w = t$, and $n + m - \frac{vm}{w} = k$; then will bx

$$\frac{bx^{\frac{w-v}{w}}}{\overline{1 \mp x}|^{\frac{t}{w}}} \times \overline{1 \pm \frac{q.t}{s.w} \times \frac{x}{1 \mp x} + \frac{q.q-m.t.t+w}{s.s+m.w.2w} \times \frac{x^2}{\overline{1 \mp x}|^2}}$$
$$\overline{\pm \frac{q.q-m.q-2m.t.t+w.t+2w}{s.s+m.s+2m.w.2w.3w} \times \frac{x^3}{\overline{1 \mp x}|^3}}, \text{ &c.} - bx^{\frac{w-v}{w}} \times \overline{1 \pm \frac{k.t}{s.w} \times x}$$
$$\overline{+ \frac{k.k+m.t.t+w}{s.s+m.w.2w} \times x^2 \pm \frac{k.k+m.k+2m.t.t+w.t+2w}{s.s+m.s+2m.w.2w.3w} \times x^3}, \text{ &c.}$$

be the true Value required; where the firſt Series is to be continued till it terminates, and the ſecond to $\frac{v-w}{w}$ Terms (in Caſe it does not terminate before) b being equal to $\pm \frac{\overline{w.s}}{t.k} \times \frac{\overline{2w.s+m}}{t+w.k+m} \times \frac{\overline{3w.s+2m}}{t+2w.k+2m}$, &c. continued to $\frac{v-w}{w}$ Factors, in which Value, the Sign $-$ obtains only when the Signs of the propoſed Series change alternately, and $\frac{v}{w}$ is, at the ſame time, an even Number.

EXAMPLE I.

Let it be required to find the Sum of the Infinite Series $1 - \frac{2.10}{1.12} \times x + \frac{2.3.10.13}{1.2.12.15} \times x^2 - \frac{2.3.4.10.13.16}{1.2.3.12.15.18} \times x^3$, &c. Then, by comparing this Series with that in the laſt Corollary, we ſhall have $n=2$, $r=1$, $p=10$, $v=12$, $m=1$, and $w=3$; therefore $\frac{v}{w}=4$, and $\frac{n-r}{m}=1$; which two laſt being whole poſitive Numbers is an Indication that the Series is exactly ſumible: Therefore let all theſe ſeveral Values be now ſubſtituted above, and we ſhall have $q=1$, $s=-2$, $t=1$, $k=-1$, and $b\ (-\frac{3 \times -2}{1 \times -1} \times \frac{6 \times -1}{4 \times 0} \times \frac{9 \times 0}{7 \times 1}) = \frac{81}{7}$; and therefore $\frac{81x^{-3}}{7 \times \overline{1+x}|^{\frac{1}{3}}} \times \overline{1 - \frac{1.1}{-2.3} \times \frac{x}{1+x}} - \frac{81x^{-3}}{7} \times \overline{1 - \frac{-1.1}{-2.3} \times x} = \frac{27 \times \overline{6+7x}}{14x^3 \times \overline{1+x}|^{\frac{4}{3}}} - \frac{27 \times \overline{6-x}}{14x^3}$ is the true Value required.

EXAMPLE II.

Where it is propofed to find the Sum of the Series $a^e + \frac{3.5}{1.6} \times a^{e-1} y + \frac{3.4.5.7}{1.2.6.8} \times a^{e-2} y^2 + \frac{3.4.5.5.7.9}{1.2.3.6.8.10} \times a^{e-3} y^3 + \frac{3.4.5.6.5.7.9.11}{1.2.3.4.6.8.10.12} \times a^{e-4} y^4$, &c. This Series may be reduced to $a^e \times \overline{1 + \frac{3.5}{1.6} \times \frac{y}{a} + \frac{3.4.5.7}{1.2.6.8} \times \frac{y^2}{a^2}}$, &c. or to $a^e \times \overline{1 + \frac{3.5}{1.6} \times x + \frac{3.4.5.7}{1.2.6.8} \times x^2}$, &c. by writing $x = \frac{y}{a}$. Therefore, by comparing this laft Series with $1 + \frac{n.p}{r.v} \times x + \frac{n.\overline{n+m}.p.\overline{p+w}}{r.\overline{r+m}.v.\overline{v+w}} \times x^2$, &c. we have $n=3$, $r=1$, $p=5$, $v=6$, $m=1$, $w=2$, $q=2$, $s=-1$, $t=1$, $k=1$, $b\left(\frac{2\times-1}{1\times1} \times \frac{4\times0}{2\times3}\right) = \frac{8\times0}{6}$, and therefore $\left(\frac{8\times0\times x^{-2}}{6\times\overline{1-x}|^{\frac{5}{2}}} \times \overline{1 + \frac{2\times1}{-1\times2} \times \frac{x}{1-x} + \frac{2\times1\times1\times3}{-1\times0\times2\times4} \times \frac{x^2}{\overline{1-x}|^2}} - \frac{8\times0\times x^{-2}}{6} \times \overline{1 + \frac{1\times1}{-1\times2} \times x}\right)$ $\frac{1}{\overline{1-x}|^{\frac{5}{2}}} = 1 + \frac{3.5}{1.6} \times x + \frac{3.4.5.7}{1.2.6.8} \times x^2$, &c. which laft Equation being multiplied by a^e, and the Value of x reftored, we fhall have $\frac{a^e}{\overline{1-\frac{y}{a}}|^{\frac{5}{2}}} = a^e + \frac{3.5}{1.6} \times a^{e-1} y$, &c. which was to be found.

In this Example $(s+m)$ one of the Factors in the Value of b being $= 0$, it may feem, at firft Sight, as if both the Expreffions multiplied by b would intirely vanifh; but upon confidering that in the former of thefe Expreffions there is a Term which has the fame $s+m$ in its Denominator, it is evident that that Term, after an actual Multiplication by the Value of b, will be no ways affected by $s+m$, tho' all the reft of the Terms intirely vanifh; fo that the Sum of the Series is as truly de-

 termined,

termined as if $s+m$ was a real Quantity; which will manifestly appear if that Series be first reduced to $a^e \times \overline{1+\frac{5x}{2}+\frac{5\cdot 7}{2.4}\times x^3}$ &c. whose Sum is found almost by bare Inspection. Hence at the same time as we see the extent of this Method, we also see how needful it is (for avoiding Trouble) to first reduce every Series to the most commodious Form, before we set about to determine its Value.

Of the Values of SERIES *by Approximation.*

CASE I.

LET the Series $ax^m + bx^{m+n} + cx^{m+2n} + dx^{m+3n}$, &c. be propounded, and let $\frac{ax^m}{1+Px^n}$ be assumed as an Approximation for the Value thereof. Then, by writing $\frac{ax^m}{1+Px^n} = ax^m + bx^{m+n}$, &c. and reducing the whole Equation to one Denomination, &c. we shall have $\left.\begin{matrix} ax^m + bx^{m+n}, \&c. \\ -ax^m + Pax^{m+n}, \&c. \end{matrix}\right\} = 0$; and therefore $P = \frac{-b}{a}$, and consequently, when the Series converges sufficiently swift, $ax^m + bx^{m+n} + cx^{m+2n}$, &c. is equal to $\frac{ax^m}{1-\frac{bx^n}{a}}$ or $\frac{a^2}{a-bx^n}$ nearly.

CASE

CASE II.

THE Series propounded being as above, let $\frac{ax^m + Ax^{m+n}}{1+Px^n}$ be assumed as nearly equal thereto; then, by proceeding as in the last Case, we shall have

$$\left.\begin{array}{l} ax^m + \ bx^{m+n} + \ cx^{m+2n}, \&c. \\ \quad * \ + aPx^{m+n} + bPx^{m+2n}, \&c. \\ -ax^m - \ Ax^{m+n} \qquad\quad * \end{array}\right\} = 0;$$ whence, by comparing the like Terms, $P = \frac{-c}{b}$, $A = b - \frac{ac}{b}$, and therefore $\frac{ax^m + Ax^{m+n}}{1+Px} = \frac{ax^m + \overline{b - \frac{ac}{b}} \times x^{m+n}}{1 - \frac{cx}{b}} = x^m \times \frac{ab + \overline{bb - ac} \times x^n}{b - cx^n}$.

CASE III.

LET $\frac{ax^m + Ax^{m+n}}{1+Px^n+Qx^{2n}}$ be assumed as nearly equal to ax^m $bx^{m+n} + cx^{m+2n}$, &c. (the Series first propounded) then by following the above Method of Operation, there will will come out $\frac{bc-ad}{ac-bb} = P$, $\frac{bd-cc}{ac-bb} = Q$, $\frac{b \times \overline{ac-bb} + a \times \overline{bc-ad}}{ac-bb}$ $= A$, and therefore $\frac{ax^m + Ax^{m+n}}{1+Px^n+Qx^{2n}} =$ $\frac{a \times \overline{ac-bb} \times x^m + \overline{a \times \overline{bc-ad} + b \times \overline{ac-bb}} \times x^{m+n}}{ac-bb + \overline{bc-ad} \times x^n + \overline{bd-cc} \times x^{2n}} = \frac{ax^m + \overline{b+aP} \times x^{m+n}}{1+Px^n+Qx^{2n}}$ where P and Q are as above specified.

C O-

COROLLARY.

Hence it appears, that the true Value of the Series $ax^{m} + bx^{m+n} + cx^{m+2n} + dx^{m+3n}$, &c. is nearly equal to $\frac{a^2x^m}{a-bx^n}$, more nearly equal to $x^m \times \frac{ab+\overline{bb-ac}\times x^n}{b-cx^n}$, and ſtill nearer equal to $\frac{ax^m+\overline{b+aP}\times x^{m+n}}{1+Px^n+Qx^{2n}}$ or $\frac{\overline{a\times ac-bb}\times x^m+\overline{\overline{a\times bc-ad}+b\times\overline{ac-bb}}\times x^{m+n}}{ac-bb+\overline{bc-ad}\times x^n+\overline{bd-cc}\times x^{2n}}$.

EXAMPLE I.

Let the Series $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$, &c. expreſſing the length of the Arc whoſe Radius is 1 and Tangent x, be propoſed; then, in this Caſe, m being $=1$, $n=2$, $a=1$, $b=-\frac{1}{3}$, $c=\frac{1}{5}$, &c. the Value of the Series, by writing theſe Values in the ſecond of the foregoing Expreſſions, will come out equal to

$$\left(x \times \frac{\overline{-\frac{1}{3}+\overline{\frac{1}{9}-\frac{1}{5}}}\times x}{-\frac{1}{3}-\frac{x}{5}}\right) x \times \frac{15+4x^2}{15+9x^2}$$ nearly.

EXAMPLE II.

Suppoſe $x+\frac{n-1}{2}x^2+\frac{n-1}{2}\times\frac{n-2}{3}x^3+\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}x^4$, &c. to be the Series propounded; then a being $=1$, $b=\frac{n-1}{2}$, $c=\frac{n-1}{2}\times\frac{n-2}{3}$, &c. the Value of the Series will be $\frac{x+\frac{n+1}{6}x^2}{1-\frac{n-2}{3}x}$ nearly, or $\frac{x+\frac{1}{2}x^2}{1-\frac{n-2}{2}x+\frac{n-1}{2}\times\frac{n-2}{6}x^2}$ more nearly.

EXAMPLE III.

Lastly, let $\frac{x^2}{1.2} - \frac{x^4}{2.3.4} + \frac{x^6}{2.3.4.5.6} - \frac{x^8}{2.3.4.5.6.7.8}$, &c. expressing the versed Sine of the Arc x to the Radius 1, be proposed; then the Value thereof, by proceeding as above, will come out $\frac{x^2}{2} \times \frac{60-3x^2}{60+2x^2}$ nearly, or $\frac{x^2}{2} \times \frac{15120-600x^2}{15120+660x^2+13x^4}$ more nearly.

Of the Roots of EQUATIONS *by Approximation.*

Let there be given the Equation $ax+bx^2+cx^3+dx^4$, *&c.* $=y$, *where* $ax+bx^2+cx^3+dx^4$, *&c. represents any Infinite or converging Series; to find the Value of* x.

IT appears from the preceding Pages, that the Value of the proposed Series will be nearly defined by $\frac{a^2}{a-bx}$ but more nearly by $\frac{abx+\overline{bb-ac}\times x^2}{b-cx}$, and still nearer by $\frac{ax+\overline{b+aP}\times x^2}{1+Px+Qx^2}$; wherein P and Q stand for $\frac{bc-ad}{ac-bb}$ and $\frac{bd-cc}{ac-bb}$ respectively; therefore, if these three Expressions be put, successively, equal to y, we shall have, first, $a^2x=ay-byx$; and therefore $x=\frac{ay}{aa+by}=\frac{y}{a+\frac{by}{a}}$ nearly. Secondly $abx+\overline{bb-ac}\times x^2=by-cyx$, or $\overline{b-\frac{ac}{b}}\times x^2+ax+\frac{cyx}{b}=y$; whence, by putting $b-\frac{ac}{b}=A$, and $\frac{a}{2}+\frac{cy}{2b}=B$, it will be Ax^2+2Bx

=

$=y$, and therefore $x = \pm\sqrt{\frac{y}{A} + \overline{\frac{B}{A}}\big|^2} - \frac{B}{A}$ more nearly. Thirdly, $ax + \overline{b+aP} \times x^2 - Qx^2y - Pxy = y$, or, by writing $b + aP - Qy = C$ and $\frac{a-Py}{2} = D$, it will be $Cx^2 + 2Dx = y$; therefore $x = \pm\sqrt{\frac{y}{C} + \overline{\frac{D}{C}}\big|^2} - \frac{D}{C}$ ſtill nearer: Which three Expreſſions will ſerve as ſo many general Theorems for the Value of x, and may be uſed at Pleaſure, according as a leſſer or greater Degree of Accuracy is required.

COROLLARY I.

If there be given $\frac{z^2}{2} - \frac{z^4}{2.3.4} + \frac{z^6}{2.3.4.5.6} - \frac{z^8}{2.3.4.5.6.7.8}$, &c. $=y$ (expreſſing the Relation between the verſed Sine (y) and the Arc (z) of a Circle, whoſe Radius is Unity) then, by putting $z^2 = x$, it will be $\frac{x}{2} - \frac{x^2}{2.3.4} + \frac{x^3}{2.3.4.5.6}$, &c. $=y$, therefore a being here $=\frac{1}{2}$, $b = -\frac{1}{2.3.4}$, &c. we ſhall, by ſubſtituting theſe Values above, have $x\,(=z^2) = 10 - \frac{2}{3}y - \sqrt{\overline{10 - \frac{2}{3}y}\big|^2 - 40y}$ nearly. Or $x\,(=z^2) = \frac{378 - 33y - \sqrt{\overline{378-33y}\big|^2 - 1512y \times \overline{30 + 1{,}3y}}}{30 + 1{,}3y}$ more nearly.

COROLLARY II.

But if the Equation given be $x - \frac{n+2}{3}x^2 + \frac{n+2}{3} \times \frac{n+3}{4}x^3 - \frac{n+2}{3} \times \frac{n+3}{4} \times \frac{n+4}{5}x^4$, &c. $= \frac{2n-2s}{n \times \overline{n+1}}$ (where s is the Value of an Annuity, n the Number of Years, and $1+x$ the Rate of Intereſt correſponding) we ſhall, by writing 1, $-\frac{n+2}{3}$,

$\frac{n+2}{3} \times \frac{n+3}{4}$, *&c.* instead of a, b, c, *&c.* in the last of the three general Expressions, and putting $\frac{2n-2s}{n \times \overline{n+1}} = y$, get $\frac{n+8}{15} - \frac{\overline{n+2} \times \overline{n+3} \times y}{20} = C$, $\frac{1}{2} - \frac{\overline{n+3} \times y}{5} = D$, and $x = \sqrt{\overline{\frac{D}{C}}\big|^2 + \frac{y}{C}} - \frac{D}{C}$ very nearly.

COROLLARY III.

Lastly, if the given Equation be $x + \frac{n-1}{2}x^2 + \frac{n-1}{2} \times \frac{n-2}{3}x^3 + \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}x^4$, *&c.* $= y$; then will $a = 1$, $b = \frac{n-1}{2}$, $c = \frac{n-1}{2} \times \frac{n-2}{3}$, *&c.* which Values being substituted above, we shall have $A = \frac{n+1}{6}$, $B = \frac{3 + \overline{n-2} \times y}{6}$, $C = \frac{1}{2} - \frac{\overline{n-1} \times \overline{n-2} \times y}{12}$, $D = \frac{2 + \overline{n-2} \times y}{4}$, and x equal to $\frac{y}{1 + \frac{n-1}{2}y}$ nearly, but more nearly equal to $\sqrt{\overline{\frac{B}{A}}\big|^2 + \frac{y}{A}} - \frac{B}{A}$, and still nearer equal to $\sqrt{\overline{\frac{D}{C}}\big|^2 + \frac{y}{C}} - \frac{D}{C}$. Hence we have a ready Method for finding the Cube, Biquadrate, *&c.* Root of any given Quantity; for let Q represent that Quantity, and let k be taken nearly equal to the required Root thereof, and let n be the given Index, that is, if the Cube-Root be required, let n be 3, if the Biquadrate, 4, *&c.* and let the true Root be represented by $k+z$, that is, let $\overline{k+z}\big|^n$ be $= Q$. Therefore $\overline{1 + \frac{z}{k}}\big|^n = \frac{Q}{k^n}$, and, by expanding $\overline{1 + \frac{z}{k}}\big|^n$, we have $1 + n \times \frac{z}{k} + n \times \frac{n-1}{2} \times \frac{z^2}{k^2}$, *&c.* $= \frac{Q}{k^n}$; therefore $\frac{z}{k} + \frac{n-1}{2} \times \frac{z^2}{k^2} + \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{z^3}{k^3}$, *&c.* $= \frac{Q - k^n}{nk^n}$, and, by putting $\frac{z}{k} = x$, and $\frac{Q - k^n}{nk^n} = y$, we have $x + \frac{n-1}{2}$

x^2.

$x^2 + \frac{n-1}{2} \times \frac{n-2}{3} x^3$, &c. the very ſame as above, and conſequently the Value of x is as there determined; whence the required Value of $k + kx(k+z)$ is alſo given.

EXAMPLE I.

Let it be required to find the length (z) of the circular Arc, whoſe Radius is 1, and verſed Sine $\frac{1}{2}$: Here, by writing $\frac{1}{2}$ inſtead of y, in Corollary I. we ſhall have z equal to $\left(\sqrt{\frac{29}{3} - \sqrt{\frac{29 \times 29}{9} - 20}}\right)$ 1.0472 nearly, or $\left(\sqrt{\frac{723 - \sqrt{723 \times 723 - 1512 \times 613}}{61.3}}\right)$ 1.047198 more nearly. Therefore, ſince the Arc, whoſe verſed Sine is $\frac{1}{2}$, is equal to $\frac{1}{3}$ of the Semicircle, it is manifeſt that the length of the whole Semicircle, according to the foregoing Numbers, ought to be 3.1416 nearly, or 3.141594 more nearly. Now the true length of the Semicircle is known to be 3.1415926, &c. therefore the Error in the former of theſe Values, is leſs than $\frac{1}{40000}$ Part of the whole, and in the latter leſs than $\frac{1}{2000000}$ Part. And if the verſed Sine firſt taken had been leſs, as for Inſtance, that of 15 Degrees ($= 1 - \sqrt{\frac{3}{8}} - \sqrt{\frac{1}{8}}$) the Concluſions would, ſtill, have been much more exact, and true, at leaſt, to 9 or 10 Places, which is ſo very near, that I believe it ſcarcely poſſible to find out more eaſy and exact Approximations for the Arc of a Circle, than thoſe above given.

EXAMPLE II.

Where ſuppoſing the Value of an Annuity for 10 Years to be 8 Years Purchaſe, 'tis required to find the Rate of Intereſt. By comparing the Values here given with thoſe in Corol. II. we have $n = 10$, $s = 8$, therefore $y \left(\frac{2n-2s}{n \times \overline{n+1}}\right) = \frac{4}{110}$, $C \left(\frac{n+8}{15} - \frac{\overline{n+2} \times \overline{n+3} \times y}{20}\right) = \frac{504}{550}$, $D \left(\frac{1}{2} - \frac{\overline{n+3} \times y}{5}\right) = \frac{223}{550}$, $\frac{D}{C} = \frac{223}{504}$, $\frac{y}{C} = \frac{5}{126}$, and conſequently $1 - \frac{D}{C} + \sqrt{\overline{\frac{D}{C}}\Big|^2 + \frac{y}{C}}$ $= 1.042775$ for the Rate of Intereſt required very nearly. Now according to Dr. *Halley*'s Theorem, the Rate will be 1.042798, which is alſo very near the Truth, but not ſo exact as the former, which is right in all its Places.

EXAMPLE III.

Where it is required to extract the Cube-Root of 10. Becauſe it appears that the Root required is a little greater than 2, let the Value thereof be repreſented by $2+z$, or the Value of k, in Corol. III. be taken $= 2$; then, n being $= 3$, y will be $= \frac{1}{12}$, $C = \frac{35}{72}$, and $D = \frac{25}{48}$; therefore (by the firſt of the Approximations there given) x will come out $\left(\frac{1}{13}\right)$ $= .0768$, &c. nearly; or (by the laſt) equal to $\left(\sqrt{\frac{225}{196} + \frac{12}{70}} - \frac{15}{14}\right)$ .077217 ſtill nearer: Hence $(k + kx)$ the Value ſought, will be 2.154, or 2.154434 more nearly.

EXAMPLE IV.

Let it be required to extract the firſt ſurſolid Root of 125000. Here, as the required Root appears, by Inſpection, to be ſomething greater than 10, let the ſame be denoted by $10 + z$; that

that is, let $\overline{10+z}|^3 = 125000$; then, by proceeding as in the laſt Example, we ſhall have $y = 0.05$, $A = 1$, $B = 0.525$, from whence, by the ſecond Theorem (in Corol. III.) the required Value will come out 10.45636, which is very near the Truth, but if the laſt Theorem had been uſed, the Anſwer would have, ſtill, been more exact.

EXAMPLE V.

Where there is given the Equation $z^3 + z^2 + z = 90$; to find the Root thereof. Since the Value of z, it is eaſy to perceive, is not much greater than 4, let it be denoted by $4+x$, and let this Value be ſubſtituted inſtead of z in the given Equation, and it will become $84 + 57x + 13x^2 + x^3 = 90$, or $57x + 13x^2 + x^3 = 6$; which being compared with the general Equation $ax + bx^2 + cx^3 + dx^4$, &c. $= y$, we thence have $a = 57$, $b = 13$, $c = 1$, $d = 0$, &c. and $y = 6$; wherefore, by the firſt Approximation, the Value of x $\left(= \frac{ay}{aa+by}\right)$ will be 0.1028, and therefore that of $z = 4.1028$ nearly: But if a greater Degree of Exactneſs be deſired, then, according to the laſt of the three Approximations, laid down in the general Propoſition, we ſhall have $P\left(\frac{bc-ad}{ac-bb}\right) = -\frac{13}{112}$, $Q\left(\frac{bd-cc}{ac-bb}\right) = \frac{1}{112}$, $C\,(b+aP-Qy) = \frac{709}{112}$, $D\left(\frac{a-Py}{2}\right) = \frac{3231}{112}$, and therefore $x\left(\sqrt{\frac{D}{C}|^2 + \frac{y}{C}} - \frac{D}{C}\right) = 0.10283235$; which is true to the laſt Place.

EXAMPLE VI.

Let $300z - z^3$ be given $= 1000$; to find a Value of z. Here, as it appears by Inſpection, that $300z - z^3$, when $z = 3$, will be leſs, and when $z = 4$, greater than 1000, let z be put $= 3.5 + x$, and then by writing this Value inſtead of of z in the given Equation, we ſhall have $263.25x - 10.5x^2 - x^3 = -7.125$, or $2106x - 84x^2 - 8x^3 = -57$; therefore a being here $= 2106$, $b = -84$, and $y = -57$, we have $x \left(= \frac{y}{a + \frac{by}{a}}\right) = -.002703633$, and conſequently $z = 3.472963$ very nearly.

Note. When the Root of any high Equation is ſought according to this Method, it will be convenient, and ſhorten the Operation very much, to neglect all the Powers of the converging Quantity x, which, in ſubſtituting for the true Root (z) would riſe higher than the 2d, 3d, or 4th Dimenſion, according as you would work by the 1ſt, 2d, or 3d Theorem, or as a leſſer or greater Degree of Accuracy is required.

Note alſo, That if, after the Value of the Root is once approximated, a greater Exactneſs be ſtill deemed neceſſary, the Operation may be repeated till you arrive as near the Truth as you deſire, as will appear from the following.

EXAMPLE VII.

Wherein $z^5 + 2z^4 + 3z^3 + 4z^2 + 5z$ being given $= 54321$; 'tis required to find the Value of z, according to the firſt Approximation. Here, becauſe it is eaſy to perceive that z is greater than 8, and leſs than 9, write $8 + x = z$; and then by involving $8 + x$, and neglecting all the Powers of x above the 2d, we ſhall have $z^5 = 32768 + 20480x + 5120x^2$, $z^4 = 4096 + 2048x + 384x^2$, $z^3 = 512 + 192x + 24x^2$, $z^2 = 64 + 16x + x^2$, and therefore $42792 + 25221x + 5964x^2 = 54321$, that is, $25221x + 5964x^2 = 11529$; which, by ſtriking off two Figures in each Term (to ſhorten the Operation) will be $252x + 59x^2 = 115$ nearly, conſequently $x \left(\frac{252 \times 115}{252 \times 252 + 115 \times 59}\right) = 0.41$, and $z = 8.41$ nearly. Let, therefore, $8.41 + x$ be

 now

now aſſumed $= z$; then, by repeating the Operation, we ſhall have $30479x + 6876x^2 = 135.92$: whence (according to the foreſaid Theorem) we have $\frac{30479 \times 135.92}{30479|^2 + 6876 \times 135.92}$ $\left(\frac{ay}{aa + by}\right)$ $= .004454$ for a new Value of x; which, therefore, added to 8.41, gives 8.414454, equal to the true Value of z very nearly.

Of the Areas *of* Curves, &c. *by Approximation.*

PROPOSITION I.

Suppoſing a b c *to be a ſmall Portion of any Curve* a b f i, *and* A a, B b, C c, *three equidiſtant Ordinates; to find an Expreſſion in Terms of thoſe Ordinates, and the common Diſtance* A B, *that ſhall nearly exhibit the included Area* A C c b a A.

LET a common Parabola, having its Axis parallel to the given Ordinates, be deſcribed thro' the three Points *a*, *b*, *c*, of the propoſed Curve, or rather, to avoid confuſing the Figure, let that Curve itſelf repreſent a Portion of ſuch a Parabola; join A and C with a Right-Line, and make S *b* T parallel thereto, producing A *a*, and C *c* to meet S *b* T in S and T, and drawing *v m* from any Point *v*, in the Parabola, parallel to A S. Then *v m*, by the Property of the Parabola being to S *a*, as bm^2, to bS^2, or, in the duplicate Ratio of *b m*, the Space *b a* S *b*, included by the Parabola and the Right-Lines S *a*,

 and

and Sb, will be $\frac{1}{3}$ of the Parallelogram $bra Sb$, for the ſame Reaſon that a Pyramid, whoſe Sections made by a Plane parallel to the Baſe, are in a duplicate Ratio of their Diſtances from the Vertex, is known to be $\frac{1}{3}$ of its circumſcribing Priſm. Wherefore, ſeeing $Bb \times 2AB$ is equal to the Area of $ACTbSA$, and $\overline{Aa+Cc} \times AB$ to that of $ACcraA$, the former of theſe Quantities muſt exceed the Parabolic Area $ACcbaA$ by juſt half what the latter wants of it; and therefore twice the former added to once the latter, will be juſt three times this Area, and conſequently the Area itſelf equal to $\frac{Aa+4Bb+Cc}{3} \times AB$; which Quantity, ſince a Parabola admits of infinite Variation of Curvature, ſo as to nearly coincide with any Curve for a ſmall Diſtance, muſt be equal alſo to the Area ſought very nearly.

Q. E. I.

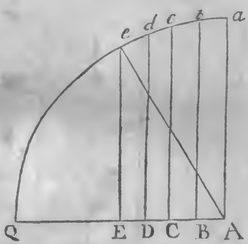
COROLLARY.

Hence may the Area of the whole Curve be alſo nearly found; for let the Abſciſſa be divided into any even Number of equal Parts, at the Points B, C, D, &c. according as a leſſer or greater Degree of Accuracy is required, and let Bb, Cc, Dd, &c. be Ordinates to the Curve at thoſe Points; then, for the ſame Reaſon that $\frac{Aa+4Bb+Cc}{3} \times AB$ is the Area of $ACcbaA$, will $\frac{Cc+4Dd+Ee}{3} \times CD$, be the Area of $CEedcC$, and $\frac{Ee+4Ff+Gg}{3} \times EF$, that of $EGgfeE$, &c. &c. But the Sum of all theſe Areas taken together, or $\frac{AB}{3} \times \overline{Aa+4Bb+2Cc+4Dd+2Ee+4Ff+2Gg, \&c.}$ is the Area of the whole Curve: Hence it appears, that if to four times the Sum of the 2d, 4th, and 6th Ordinates, &c. be added the Double of all the reſt, but the firſt and laſt, and the Sum be in-

increased by these two single Ordinates, and multiply'd by $\frac{1}{3}$ of the common Distance, the Product will be the Area sought.

EXAMPLE I.

Supposing AQa to be a Quadrant of a Circle, whose Radius AQ is 8, and Aa, Bb, Cc, Dd, Ee, five Ordinates thereto, whose common Distance AB is Unity; to find the Area $AaceEA$.



Here, by the Property of the Curve, Aa being $= 8$, $Bb = \sqrt{63} = 7.93725$, $Cc = \sqrt{60} = 7.74596$, $Dd = \sqrt{55} = 7.4162$, $Ee = \sqrt{48} = 6.9282$, we have $\overline{4 \times \overline{Bb + Dd} + 2Cc + Aa + Ee} \times \frac{AB}{3} = 30.6113$; which, by the foregoing Corollary, is the Area sought. From whence the Area of the whole Quadrant may be easily found; for, taking 13.8564 $(= Ee \times \frac{1}{2} AE)$ the Area of the Triangle AeE, from 30.6113 there remains 16.7549 for the Area of the Sector $AaeA$; the treble whereof, since AE is $= EQ$, will be the Area of the whole Quadrant, which therefore is to its circumscribing Square, as 0.78538, &c. to 1 nearly, the same as it is known to be by other Methods.

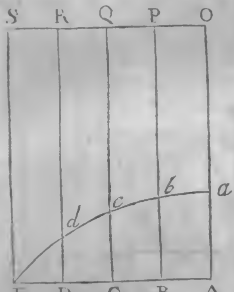
Note. When the Area of any Part of a Curve near the Vertex, where the Ordinates are very oblique to the Curve, is proposed to be found, the Solution by this Method will be the least exact: Therefore, in all such Cases it will be conve-

 nient

nient, inſtead of ſuch Ordinates, to make uſe of Lines parallel to the Axis, as in the following.

EXAMPLE II.

Where *abcde* being a Semi-Hyperbola, whoſe Abſciſſa A*a* is 10, Ordinate AE 20, and Semi-Tranſverſe O*a* 20; 'tis required to find an Expreſſion for the Area A*ac*EA, in Numbers, that ſhall be true, at leaſt, to 3 or 4 Places.



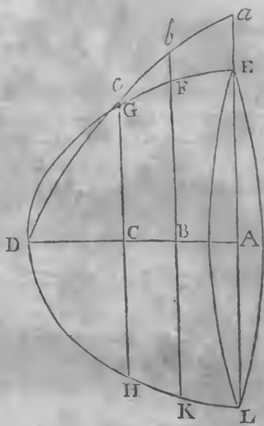
Firſt, I ſuppoſe AE divided into 4 equal Parts, at the Points B, C, and D, and B*b*, C*c*, D*d*, &c. parallel to the Axis AO, produced to meet the conjugate Axis of the Hyperbola in the Points P, Q, R, S. Therefore, by the Property of the Curve, AE^2 (400) : $ES^2 - Oa^2$ (500) :: OP^2 (25) : $Pb^2 - Oa^2$ :: OQ^2 (100) : $Qc^2 - Oa^2$ :: OR^2 (225) : $Rd^2 - Oa^2$; whence $Pb = 20.766$, $Qc = 22.913$, $Rd = 26.100$; therefore $Aa = 10$, $Bb = 9.234$, $Cc = 7.087$, $Dd = 3.900$, $Ee = 0$; and conſequently $4 \times \overline{9.234 + 3.900} + 2 \times 7.087 + 10 + 0$ into $\frac{5}{3} = 127.8$ equal to the Area ſought.

EXAMPLE III.

Suppoſe EGDHLE to be a Solid, generated by the Rotation of any Conic-Section about its Axis DA, *viz.* either a Cone, Sphere, Spheroid, or Conoid; and let the Content of any Fruſtum EFGHKLE of that Solid be required.

Here if p be put for the Area of a Circle, whoſe Diameter is Unity, and a Curve as *abc* be ſuppoſed, whoſe Ordinates

A*a*,



Aa, Bb, &c. ſhall every where be as the Areas $p \times EL^2$, $p \times FK^2$, &c. of the correſponding Sections, then the Area of that Curve will, it is manifeſt, be as the required Content of the propoſed Fruſtum: But this Curve is always a Portion of the common Parabola, except in the parabolic Conoid, where it degenerates to a Right-line, and therefore its Area, ſuppoſing $AB = BC$, will be, exactly, equal to $\overline{Aa+4Bb+Cc} \times \frac{AC}{6}$; and conſequently the Content of the Fruſtum, equal to $\overline{EL^2+4FK^2+GH^2} \times \frac{1}{6}p \times AC$; which is therefore to $EL^2 \times p \times AC$, the Content of the circumſcribing Cylinder, as $EL^2+4FK^2+GH^2$ to $6EL^2$. This Proportion, if AC be ſuppoſed $= DC$, and the whole Solid $EGDHLE$ be taken, will become as EL^2+4GH to $6EL^2$; where $4GH^2$ is equal to EL^2, $2EL^2$, or $3EL^2$, according as the Solid is a Cone, parabolic Conoid, or Semi-ſpheroid. Hence it appears that a Cone, a parabolic Conoid, a Semi-ſpheroid and a Cylinder, having the ſame common Baſe and Altitude, are to one another as 2, 3, 4 and 6 reſpectively.

LEMMA.

If in any Series of Quantities a, b, c, d, e, *&c. there be taken* $A=b-a$, $B=c-2b+a$, $D=d-3c+3b-a$, $E=e-4d+6c-4b+a$, $F=f-5e+10d-10c+5b-a$, *&c. ſo that the Unciæ of the Values of* A, B, C, D, *&c. may be thoſe of a Binomial raiſed to the* 1*ſt*, 2*d*, 3*d*, 4*th*, *&c. Powers; I ſay, the Value of any Term in that Series, whoſe Diſtance from the firſt is denoted by* n, *will be* $a+nA+n\times\frac{n-1}{2}B+n\times\frac{n-1}{2}\times\frac{n-2}{3}C+n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}D$, *&c.*

For, ſince $b-a$ is $=A$, $c-2b+a=B$, *&c.* we ſhall, by Tranſpoſition, have

$$b=a+A$$
$$c=2b-a+B$$
$$d=3c-3b+a+C$$
$$e=4d-6c+4b-a+D$$
$$f=5e-10d+10c-5b+a+E,$$
&c. *&c.* *&c.*

where, by taking the Value of b, as found in the firſt Equation, and ſubſtituting it in the reſt, there comes out

$$c=a+2A+B$$
$$d=3c-3A-2a+C$$
$$e=4d-6c+4A+3a+D$$
$$f=5e-10d+10c-5A-4a+E,$$
&c. *&c.* *&c.*

in which the Value of c, here found, being ſubſtituted, we next have

$d=$

$$d = a + 3A + 3B + C$$
$$e = 4d - 3a - 8A - 6B + D$$
$$f = 5e - 10d + 6a + 15A + 10B + E,$$
&c. &c. &c.

In like manner the Values of e, f, &c. are found to be $a+4A+6B+4C+D$ and $a+5A+10B+10C+5D+E$, &c. Where the Unciæ, in the Value of each of the Terms b, c, d, &c. are, it is manifeſt, thoſe of a Binomial raiſed to that Power, whoſe Index is equal to the Number denoting the Diſtance of that Term from the firſt in the Series; therefore the Value of that Term, whoſe Diſtance from the firſt is denoted by n, will be $a + nA + \frac{n}{1} \times \frac{n-1}{2} \times B$, &c. Q. E. D.

PROPOSITION II.

Suppoſing a b c d e, *&c. to be a Curve of any kind, and* A a, B b, C c, D d, E e, *&c. given Ordinates thereto, at equal Diſtances, but not very far from each other; to approximate the Area of the Curve by means of thoſe Ordinates.*

LET $Aa=a$, $Bb=b$, $Cc=c$, $Dd=d$, &c. $AB=BC$, &c. $=p$, and the number of given Ordinates, Aa, Bb, &c. equal to $n+1$; putting $b-a=A$, $c-2b+a=B$, $d-3c+3b-a=C$, $e-4d+6c-4b+a=D$, &c. Then that Ordinate, whoſe Place from the firſt is denoted by n, or whoſe

Diſtance

Diſtance from the firſt is n times the common Diſtance, will, by the foregoing Lemma, be $a + nA + n \times \frac{n-1}{2} B + n \times \frac{n-1}{2} \times \frac{n-2}{3} C + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} D$, &c. Wherefore, if np, the Diſtance of this Ordinate from the Point A, be put $= x$, and $\frac{x}{p}$ be ſubſtituted above inſtead of its equal (n) we ſhall have $a + \frac{Ax}{p} + \frac{Bx}{p} \times \frac{x-p}{2p} + \frac{Cx}{p} \times \frac{x-p}{2p} \times \frac{x-2p}{3p}$, &c. equal to that Ordinate, whoſe correſponding Abſciſſa is x; which reduced to ſimple Terms, will be $a + \frac{Ax}{p} + \frac{Bx^2}{2p^2} - \frac{Bx}{2p} + \frac{Cx^3}{6p^3} - \frac{Cx^2}{2p^2} + \frac{Cx}{3p} + \frac{Dx^4}{24p^4} - \frac{Dx^3}{4p^3} + \frac{11Dx^2}{24p^2} - \frac{Dx}{4p}$, &c. Hence it is manifeſt that $a + \frac{Ax}{p} + \frac{Bx^2}{2p^2} - \frac{Bx}{2p} + \frac{Cx^3}{6p^3}$, &c. $= y$, is the Equation of a parabolic Curve, which, being deſcribed to the Abſciſſa A G, will paſs thro' all the given Points a, b, c, d, &c. Therefore the Area of this Curve, which by the common Methods is found to be $x \times \overline{a + \frac{Ax}{2p} + \frac{Bx^2}{6p^2} - \frac{Bx}{4p} + \frac{Cx^3}{24p^3} - \frac{Cx^2}{6p^2} + \frac{Cx}{6p} + \frac{Dx^4}{120p^4} - \frac{Dx^3}{16p^3} + \frac{11Dx^2}{72p^2} - \frac{Dx}{8p} + \frac{Ex^5}{720p^5}}$, &c. muſt be equal, very nearly, to the Area of the propoſed Curve A a b c d g G. Q. E. I.

COROLLARY.

Hence, if x be $= 2p$, or the Ordinates given be only a, b, and c; then A being $= b - a$, B$= c - 2b + a$, C$=$o, &c. the included Area $x \times \overline{a + \frac{Ax}{2p} + \frac{Bx^2}{6p^2}}$, &c. will be $\frac{a+4b+c}{6} \times x$: But, if $x = 3p$, or 4 Ordinates a, b, c and d be given, C will be $= d - 3c + 3b - a$, D$=$o, &c. and therefore the included

3 Area

Area, equal to $\frac{a+3b+3c+d}{8} \times x$: Moreover, if 5, 6, 7, &c. be the Number of Ordinates given, then the required Area, by proceeding in the same manner, will come out $\frac{7a+32b+12c+32d+7e}{90} \times x$, $\frac{19a+75b+50c+50d+75e+19f}{288} \times x$, and $\frac{41a+216b+27c+272d+27e+216f+41g}{840} \times x$, &c. respectively; where x, in each Case, denotes the Distance of the first and last Ordinates.

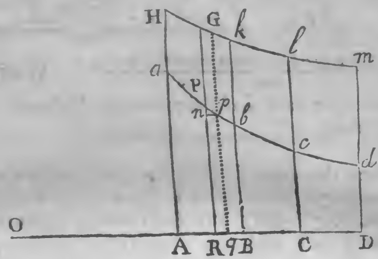
Note. When the Equation of the given Curve is comprehended in this Form, *viz.* $y = Q + Rx + Sx^2 + Tx^3$, &c. where Q, R, S, &c. may signify any determinate Quantities whatever, the Curve described thro' the given Points a, b, c, &c. (as above) will be the very Curve given: Therefore, in this Case, the required Area may be exactly had, by making use only of as many Ordinates as there are Terms in the Value of y, or as there are are Units in the Index of the highest Power of x in that Equation, increased by one.

EXAMPLE I.

Let O be the Centre, a the Vertex, and O D an Assymptote of the equilateral Hyperbola $a\,b\,c\,d$; and, supposing O A, A a, and A D, each equal to Unity; let it be required to find the Area comprehended by the Curve, the Assymptote, and the Ordinates A a and D d.

Here, if only three Ordinates be used, then a being $= 1$, $b = .6666$, $c = .5$, by the Property of the Curve, we shall have $\frac{a+4b+c}{6} \times 1 = .6944$, &c. for the Area sought. But if four

Ordinates be taken, then a being $=1$, $b=.75$, $c=.6$, $d=.5$, $\frac{a+3b+3c+d}{8}\times 1=.6937$, &c. will be the Area ſought;



which Value is ſomething nearer than the former, the true Area being .69314, &c.

EXAMPLE II.

The ſame being ſuppoſed as in the preceding Example, 'tis required to find the length of the Arch $a\,d$.

Let R P (y) be any Ordinate to the Hyperbola, O R (x) its correſponding Abſciſſa, and R $q=\dot{x}$; then y being $=\frac{1}{x}$ by the Property of the Curve, we ſhall have $\dot{y}=\frac{-\dot{x}}{xx}$, and $(\sqrt{\dot{x}^2+\dot{y}^2}=)\,\dot{x}\sqrt{1+\frac{1}{x^4}}$ equal to the Fluxion of the Curve. Wherefore, if we now ſuppoſe another Curve HGklm, whoſe Abſciſſa is x, and Ordinate $\sqrt{1+\frac{1}{x^4}}$, the Meaſure of the Area of this Curve, will, it is manifeſt, alſo expreſs the Length of the required Arch $a\,d$, the Fluxions of both being the ſame. Hence, if A D be divided, as in the foregoing Example, into any Number of equal Parts (ſuppoſe 3) by the Ordi-

Ordinates B *k*, C *l*, &c. and the Values of theſe Ordinates be ſubſtituted in the foregoing Corollary, there will come out 1.134 for the Arch required. By proceeding in this Manner, to find a Curve whoſe Area ſhall be as the Value ſought, not only the Lengths of Curves, but any other Quantities, whoſe Fluxions are given, may be approximated, even when the given Fluxions are ſo complicated, as to render a Solution by Infinite Series very troubleſome, if not impracticable.

Of QUADRATURES *and the Compariſon of* FLUENTS.

PROPOSITION I.

SUppoſing $a+cz^n$ equal to x, I ſay, the Fluent of $\overline{a+cz^n}|^m \times dz^{pn-1}\dot{z}$, or the Area of the Curve, whoſe Abſciſſa is z, and Ordinate $\overline{a+cz^n}|^m \times dz^{pn-1}$, will be $\frac{dx^m z^{pn}}{pn} \times \overline{1 - \frac{m}{p+1} \times \frac{cz^n}{x} + \frac{m \cdot m-1}{p+1 \cdot p+2} \times \frac{c^2z^{2n}}{x^2} - \frac{m \cdot m-1 \cdot m-2}{p+1 \cdot p+2 \cdot p+3} \times \frac{c^3z^{3n}}{x^3}}$ $\overline{+ \frac{m \cdot m-1 \cdot m-2 \cdot m-3}{p+1 \cdot p+2 \cdot p+3 \cdot p+4} \times \frac{c^4z^{4n}}{x^4}}$, &c.

For $\overline{a+cz^n}|^m \times dz^{pn-1}\dot{z}$, reduced to a Series, will become $dz^{pn-1}\dot{z} \times \overline{a^m + ma^{m-1}cz^n + \frac{m}{1} \times \frac{m-1}{2} a^{m-2} c^2z^{2n}}$, &c. of which the Fluent is $dz^{pn} \times \overline{\frac{a^m}{pn} + \frac{ma^{m-1}cz^n}{pn+n} + \frac{m \cdot m-1}{1 \cdot 2} \times \frac{a^{m-2}c^2z^{2n}}{pn+2n}}$, &c. or $\frac{dz^{pn}}{n} \times \overline{\frac{a^m}{p} + \frac{ma^{m-1}cz^n}{p+1} + \frac{m \cdot m-1}{1 \cdot 2} \times \frac{a^{m-2}c^2z^{2n}}{p+2}}$, &c. (by the common Method) But the Series

$\frac{a^m}{p} + \frac{m a^{m-1} c z^n}{p+1}$, &c. is (*by Corol.* II. *Prop.* I. *of the Summation of Series*) equal to $\frac{\overline{a+cz^n}|^m}{p} \times \overline{1 - \frac{mcz^n}{p+1 \times a+cz^n} + \frac{m.m-1 \times c^2 z^{2n}}{p+1.p+2 \times \overline{a+cz^n}|^2}}$, &c. that is $= \frac{x^m}{p} \times \overline{1 - \frac{mcz^n}{p+1.x} + \frac{m.m-1\, c^2 z^{2n}}{p+1.p+2.x^2} - \frac{m.m-1.m-2.c^3 z^{3n}}{p+1.p+2.p+3.x^3}}$, &c. and therefore $\frac{dz^{pn}}{n} \times \overline{\frac{a^m}{p} + \frac{m a^{m-1} c z^n}{p+1}}$, &c. $= \frac{dx^m z^{pn}}{pn} \times \overline{1 - \frac{m}{p+1} \times \frac{cz^n}{x} + \frac{m.m-1}{p+1.p+2} \times \frac{c^2 z^{2n}}{x^2}}$, &c.

Q. E. D.

EXAMPLE I.

Let it be required to find the Fluent of $\overline{a \mp z}|^{\frac{1}{2}} \times \frac{2bz^{\frac{1}{2}}\dot{z}}{a}$, or the Area of the Conic-Section, whose Transverse is a, Conjugate b, and Abscissa z. By comparing this with $\overline{a+cz^n}|^m \times dz^{pn-1}\dot{z}$, the foregoing general Expression, we have $c = \mp 1$, $n = 1$, $m = \frac{1}{2}$, $p = \frac{3}{2}$; and therefore $\frac{4bx^{\frac{1}{2}}z^{\frac{3}{2}}}{3a} \times \overline{1 \pm \frac{z}{5x} - \frac{z^2}{5\cdot7x^2} \pm \frac{3z^3}{5\cdot7\cdot9x^3}}$, &c. or $\frac{4bx^{\frac{1}{2}}z^{\frac{3}{2}}}{a} \times \overline{\frac{1}{1\cdot3} \pm \frac{z}{1\cdot3\cdot5x} - \frac{z^2}{3\cdot5\cdot7x^2} \pm \frac{z^3}{5\cdot7\cdot9x^3} - \frac{z^4}{7\cdot9\cdot11x^4}}$, &c. equal to the Value required; where x is $= a \mp z$, that is, equal to $a - z$, in the Circle and Ellipsis, and equal to $a + z$ in the Hyperbola.

EXAMPLE II.

Where it is proposed to find the Fluent of $\overline{1+z^2}|^{-\frac{1}{2}} \times z^8\dot{z}$. Here we have $a = 1$, $c = 1$, $n = 2$, $m = -\frac{1}{2}$, $d = 1$ $(2p - 1 = 8)$

$= 8)\ p = \frac{9}{2},\ 1 + z^2 = x$, and $\frac{x^{-\frac{1}{2}} z^9}{9} \times \overline{1 + \frac{z^2}{11x} + \frac{3z^4}{11 \cdot 13 x^2} + \frac{3 \cdot 5 z^6}{11 \cdot 13 \cdot 15 x^3} + \frac{3 \cdot 5 \cdot 7 z^8}{11 \cdot 13 \cdot 15 \cdot 17 x^4}}$, &c. equal to the required Fluent in this Case; which Series is more simple and converges much faster, than that resulting from the common Method.

PROPOSITION II.

The Fluent (Q) of $\overline{a + cz^n}|^m \times dz^{pn-1} \dot{z}$, or the Area of the Curve, whose Abscissa is z, and Ordinate $\overline{a + cz^n}|^m \times dz^{pn-1}$ being given; to find (R) the exact Area of the Curve, whose Abscissa is also z, and Ordinate $\overline{a + cz^n}|^m \times dz^{pn+vn-1} \times \overline{fz^{rn} + gz^{rn-n} + hz^{rn-2n}}$, &c. continued to $r + 1$ Terms; r and v being any whole positive Numbers.

Let $\overline{a + cz^n}|^{m+1} \times \overline{Az^{qn} + Bz^{qn-n} + Cz^{qn-2n}}$, &c. $+ wQ$ be assumed $=$ (R) the Area sought. Then, by putting the Equation in Fluxions, we have

$$\left.\begin{array}{l} nc\dot{z}z^{n-1} \times \overline{m+1} \times \overline{a+cz^n}|^m \times \overline{Az^{qn} + Bz^{qn-n} + Cz^{qn-2n}}, \text{\&c.} \\ \overline{a+cz^n}|^{m+1} \times \overline{qn\dot{z}Az^{qn-1} + \overline{qn-n} \times \dot{z}Bz^{qn-n-1}}, \text{\&c.} + w\dot{Q} \end{array}\right\} = \dot{R};$$

and therefore by writing $\overline{a + cz^n}|^m \times dz^{pn-1} \dot{z}$, and $\overline{a + cz^n}|^m \times dz^{pn+vn-1} \dot{z} \times \overline{fz^{rn} + gz^{rn-n} + hz^{rn-2n}}$, &c. instead of their respective Equals Q and R, and dividing the whole by $\dot{z} z^{n-1} \times \overline{a + cz^n}|^m$, &c. there will come out

$m+$

$$\left.\begin{array}{l} \overline{m+1\times cn}\times\overline{Az^{qn}+Bz^{qn-n}+Cz^{qn-2n}+Dz^{qn-3n},\ \&c.} \\ \overline{a+cz^n}\times\overline{qnAz^{qn-n}+\overline{q-1}\times nBz^{qn-2n}+\overline{q-2}\times nCz^{qn-3n},\ \&c.} \\ -dz^{pn+vn-n}\times\overline{fz^{rn}+gz^{rn-n}+hz^{rn-2n}+iz^{rn-3n},\ \&c.} \\ \qquad\qquad\qquad\qquad +dwz^{pn-n} \end{array}\right\}=0$$

Which reduced into ſimple Terms, is.

$$\left.\begin{array}{r} \overline{m+q+1}\times cnAz^{qn}+\overline{m+q}\times cnBz^{qn-n}+\overline{m+q-1}\times cnCz^{qn-2n},\ \&c. \\ +qaAz^{qn-n}+\overline{q-1}\times naBz^{qn-2n},\ \&c. \\ -dfz^{pn+vn+rn-n}-dgz^{pn+vn+rn-2n},\ \&c.+dwz^{pn-n} \end{array}\right\}=0$$

Hence, by making qn and $pn+vn+rn-n$, the two greateſt Exponents, equal to each other, putting $p+v+r+m=t$, and comparing the homologous Terms, we have $q=p+v+r-1$, $A=\frac{df}{nct}$, $B=\frac{dg-qaAn}{\overline{t-1}\times cn}$, $C=\frac{dh-\overline{q-1}\times aBn}{\overline{t-2}\times cn}$, $D=\frac{di-\overline{q-2}\times aCn}{\overline{t-3}\times cn}$, &c. Where, becauſe the Index of the firſt Term in the above Equation is $pn+vn+rn-n$, and that of the laſt Term $pn-n$, the Number of Coefficients to be thus taken, excluſive of w, will, it is manifeſt, be $r+v$; w being $=$ the laſt of theſe Coefficients multiply'd by $-\frac{npa}{d}$. Q. E. I.

Note. When p as well as v is a whole poſitive Number, the Fluent or Area ſought will be had in finite Terms, independant of Q, by taking $A=\frac{df}{nct}$, $B=\frac{dg-qaAn}{\overline{t-1}\times cn}$, &c. (as above) and continuing the Operation till ſome one of the Coefficients B, C, &c. becomes $=0$, or the Series breaks off; as is manifeſt from the Nature of the foregoing Proceſs.

CO-

COROLLARY I.

If f be taken $=1$, and $r, g, h,$ &c. each $=0$, and e be put $=p+v$; then will t become $=e+m$, $A=\frac{d}{nct}$, $B=-\frac{qaA}{\overline{t-1}\times c}$, &c. and therefore (R) the Fluent of $\overline{a+cz^n}|^m\times dz^{en-1}\dot{z}$ is equal to $\frac{\overline{a+cz^n}|^{m+1}\times dz^{en-n}}{\overline{e+m}\times cn}\times$ $\overline{1-\frac{e-1}{e+m-1}\times\frac{a}{cz^n}+\frac{e-1}{e+m-1}\times\frac{e-2}{e+m-2}\times\frac{a^2}{c^2z^{2n}}}\,(v)\pm\frac{Qa^v}{c^v}\times$ $\overline{\frac{p}{p+m+1}\times\frac{p+1}{p+m+2}\times\frac{p+2}{p+m+3}}\,(v)$, or $\frac{\overline{a+cz^n}|^{m+1}\times dz^{en-n}}{tcn}\times\overline{1-\frac{e-1}{t-1}\times\frac{a}{cz^n}+\frac{\overline{e-1}\times\overline{e-2}}{\overline{t-1}\times\overline{t-2}}\times\frac{a^2}{c^2z^{2n}}}\,(v)\pm\frac{Qa^v}{c^v}\times\overline{\frac{p}{p+m+1}\times\frac{p+1}{p+m+2}\times\frac{p+2}{p+m+3}}$ (v); where the Sign $+$ or $-$, before $\frac{Qa^v}{c^v}$, obtains according as v is an even or an odd Number, and where $1-\frac{e-1}{t-1}\times\frac{a}{cz^n}$ $+\frac{\overline{e-1}\times\overline{e-2}}{\overline{t-1}\times\overline{t-2}}\times\frac{a^2}{c^2z^{2n}}$ (v) ſignifies the ſame thing as $1-\frac{e-1}{t-1}\times$ $\frac{a}{cz^n}$, &c. continued to v Terms, and $\frac{p}{p+m+1}\times\frac{p+1}{p+m+2}\times\frac{p+2}{p+m+3}$ (v) the ſame as $\frac{p}{p+m+1}\times\frac{p+1}{p+m+2}$, &c. continued to v Factors; which Method of Notation is to be underſtood in what follows.

COROLLARY II.

Hence may the Fluent of $\overline{a+cz^n}|^m\times dz^{pn-vn-1}\dot{z}$, be eaſily derived; for let $-v$ be written inſtead of v, in the laſt Corollary, and we ſhall have $e=p-v$, $t=e+m$, and $\frac{\overline{a+cz^n}|^{m+1}\times dz^{en-n}}{tcn}\times\overline{1-\frac{e-1}{t-1}\times\frac{a}{cz^n}+\frac{\overline{e-1}\times\overline{e-2}}{\overline{t-1}\times\overline{t-2}}\times\frac{a^2}{c^2z^{2n}}}\,(-v)$

&c. for the Value sought. But $1 - \frac{e-1}{t-1} \times \frac{a}{cz^n}$, &c. continued to $-v$ Terms, signifies the same thing as the v first Terms, from Unity, of the Series continued downwards or the contrary Way by the same Law; and the like holds good with regard to $\frac{p}{p+m+1} \times \frac{p+1}{p+m+2}\,(-v)$: Therefore the Fluent of $\overline{a+cz^n}|^{m+1} \times dz^{en-1}\dot{z}$, or the true Value required, will, be

$$\frac{\overline{a+cz^n}|^{m+1} \times dz^{en}}{ena} \times \overline{1 - \frac{t+1}{e+1} \times \frac{cz^n}{a} + \frac{t+1 \times t+2}{e+1 \times e+2} \times \frac{c^2z^{2n}}{a^2} - \frac{t+1 \times t+2 \times t+3}{e+1 \times e+2 \times e+3} \times \frac{c^3z^{3n}}{a^3}}\,(v) \pm \frac{Qc^v}{a^v} \times \overline{\frac{p+m}{p-1} \times \frac{p+m-1}{p-2} \times \frac{p+m-2}{p-3}}\,(v).$$

COROLLARY III.

But if the Fluent of $\overline{a+cz^n}|^m \times dz^{en-1}\dot{z}$ be required, according to any assigned Value of e, independant of Q; let either of the above found Series

$$\frac{\overline{a+cz^n}|^{m+1} \times dz^{en-n}}{tcn} \times \overline{1 - \frac{e-1}{t-1} \times \frac{a}{cz^n} + \frac{e-1}{t-1} \times \frac{e-2}{t-2} \times \frac{a^2}{c^2z^{2n}}}, \text{ \&c. or}$$

$$\frac{\overline{a+cz^n}|^{m+1} \times dz^{en}}{ena} \times \overline{1 - \frac{t+1}{e+1} \times \frac{cz^n}{a} + \frac{t+1}{e+1} \times \frac{t+2}{e+2} \times \frac{c^2z^{2n}}{a^2}}, \text{ \&c.}$$

be continued *in infinitum*, or till it terminates, and it will be equal to the true Value sought. But the former of these Series will always terminate when e is a whole positive Number, and the latter when t, or its equal $e+m$, is a whole negative Number; therefore in these two Cases the Fluent may be exactly had in finite Terms.

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COROLLARY IV.

Moreover the Fluent of $\overline{a+cz^n}|^{m+r} \times dz^{en-1}\dot{z}$, or of $\overline{a+cz^n}|^{m+r} \times dz^{pn+vn-1}\dot{z}$, will from hence be given, in Terms of Q and algebraic Quantities, when both r and v are whole positive Numbers; for $\overline{a+cz^n}|^{m+r}$ $(= \overline{a+cz}|^m \times \overline{cz^n+a}|^r)$ being $= \overline{a+cz^n}|^m \times \overline{c^r z^{rn} + rc^{r-1}az^{rn-n} + r \times \frac{r-1}{2} c^{r-2} a^2 z^{rn-2n}}$, &c. if, in the general Proposition, there be taken $f = c^r$, $g = rc^{r-1}a$, $h = r \times \frac{r-1}{2} \times c^{r-2} a^2$, &c. and these Values be substituted in those of A, B, C, D, E, &c. as there found, A will come out $= \frac{dc^{r-1}}{nt}$, $B = \overline{\frac{r}{t-1} - \frac{q}{t \times t-1}} \times \frac{dc^{r-2}}{n}$, &c. and therefore the Fluent sought $= \frac{dc^{r-1}}{n} \times \overline{a+cz^n}|^{m+1}$ into $\frac{z^{qn}}{t} + \overline{\frac{r}{t-1} - \frac{q}{t \times t-1}} \times \frac{az^{qn-n}}{c} + \overline{\frac{r \times r-1}{2 \times t-2} - \frac{r \times q-1}{t-1 \times t-2} + \frac{q \times q-1}{t \times t-1 \times t-2}} \times \frac{a^2 z^{qn-2n}}{cc} + \overline{\frac{r \times r-1 \times r-2}{2 \times 3 \times t-3} - \frac{r \times r-1 \times q-2}{2 \times t-2 \times t-3} + \frac{r \times q-1 \times q-2}{t-1 \times t-2 \times t-3} - \frac{q \times q-1 \times q-2}{t \times t-1 \times t-2 \times t-3}} \times \frac{a^3 z^{qn-3n}}{c^3}$, &c. where the Law of Continuation is manifest. But the same thing may be had in a more commodious Form, by help of the first Corollary: For, by writing $x = a+cz^n$, we shall get $z^{en} = \frac{\overline{x-a}|^e}{c^e}$, and therefore $z^{en-1}\dot{z} = \frac{\dot{x} \times \overline{x-a}|^{e-1}}{nc^e}$, consequently $\overline{x-a}|^{e-1} \times \frac{dx^m\dot{x}}{nc^e} = \overline{a+cz^n}|^m \times dz^{en-1}$

and

and $\overline{x-a}|^{e-1} \times \frac{dx^{m+r}\dot{x}}{nc^e} = \overline{a+cz^n}|^{m+r} \times dz^{en-1}\dot{z}$. Now, if the Fluent of $\overline{x-a}|^{e-1} \times \frac{dx^m\dot{x}}{nc^e}$ be repreſented by K, and that of $\overline{x-a}|^{e-1} \times \frac{dx^{m+r}\dot{x}}{nc^e}$ by L, and for $a, c, z, n, d, m, p, e, v$, Q, and R, in the foreſaid Corollary, — a, 1, x, 1, $\frac{d}{nc^e}$, $e-1$, $m+1$, $m+1+r$, r, K and L be reſpectively ſubſtituted, we ſhall have $\frac{\overline{x-a}|^e \times dx^{r+m}}{e+m+r \times nc^e}$ into $\overline{1 + \frac{r+m}{r+m+e-1} \times \frac{a}{x} + \frac{r+m \times r+m-1}{r+m+e-1 \times r+m+e-2} \times \frac{a^2}{x^2}\,(r)} + a^r K \times \overline{\frac{m+1}{e+m+1} \times \frac{m+2}{e+m+2} \times \frac{m+3}{e+m+3}\,(r)} = \mathrm{L}$.

But, ſince $\overline{x-a}|^{e-1} \times \frac{dx^m\dot{x}}{nc^e}$ is $= \overline{a+cz^n}|^m \times dz^{en-1}\,\dot{z}$, the Fluents of theſe two Expreſſions, muſt conſequently be equal, that is, $\mathrm{K} = \frac{\overline{a+cz^n}|^{m+1} \times dz^{en-n}}{e+m \times cn} \times \overline{1 - \frac{e-1}{e+m-1} \times \frac{a}{cz^n}}$, &c. (by the ſame Corollary). Therefore, if f be now put $= m+r$, $g = m+e$, $h = e+f$, $\mathrm{F} = \frac{m+1}{g+1} \times \frac{m+2}{g+2} \times \frac{m+3}{g+3}\,(r)$ and $\mathrm{G} = \frac{p}{p+m+1} \times \frac{p+1}{p+m+2}\,(v)$ and theſe Values with that of K, be ſubſtituted in the foregoing Equation, &c. we ſhall have $\frac{dz^{en}x^f}{hn} \times \overline{1 + \frac{fa}{h-1 \times x} + \frac{f \times f-1 \times a^2}{h-1 \times h-2 \times x^2} + \frac{f \times f-1 \times f-2 \times a^3}{h-1 \times h-2 \times h-3 \times x^3}\,(r)} + \frac{d\mathrm{F}a^r x^{m+1} z^{en-n}}{cng} \times \overline{1 - \frac{e-1 \times a}{g-1 \times cz^n} + \frac{e-1 \times e-2 \times a^2}{g-1 \times g-2 \times c^2 z^{2n}}\,(v)}$ $\pm \frac{\mathrm{FGQ} \times a^{v+r}}{c^v}$, equal to the Fluent of $\overline{a+cz^n}|^{m+r} \times dz^{pn+vn-1}\,\dot{z}$, or the true Value required; where e ſtands for

for $p+v$, x for $a+cz^n$, and Q for the Fluent of $\overline{a+cz^n}|^m \times dz^{pn-1}\dot{z}$, and where the laſt Term $\frac{FGQ \times a^{v+r}}{c^v}$ is to be taken with the Sign + or —, according as v is an even or odd Number.

COROLLARY V.

Hence may the Fluent of $\overline{a+cz^n}|^{m+r} \times dz^{pn-vn-1}\dot{z}$ be eaſily derived; for let $-v$ be ſubſtituted inſtead of v, in the laſt Corollary, and we ſhall have $e=p-v$, $f=m+r$, $g=m+e$, $h=e+f$, $\frac{m+1}{g+1} \times \frac{m+2}{g+2}\,(r)=F$, $\frac{p}{p+m+1} \times \frac{p+1}{p+m+2}$ $(-v)=G$, and the Fluent ſought $= \frac{dz^{en}x^f}{hn} \times \overline{1+\frac{fa}{h-a \times x}\,(r)}$ $+\frac{dFa^r x^{m+1} z^{en-n}}{eng} \times \overline{1-\frac{\overline{e-1} \times a}{\overline{g-1} \times cz^n}\,(-v)} \pm \frac{FGQ \times a^{r-v}}{c^{-v}}$.

But ſince $\frac{p}{p+m+1} \times \frac{p+1}{p+m+2}\,(-v)$, or $\frac{p}{p+m+1} \times \frac{p+2}{p+m+2} \times \frac{p+2}{p+m+3}$, &c. continued to $-v$ Factors (as has been before obſerved) ſignifies nothing more than $\frac{p}{p+m+1} \times \frac{p+1}{p+m+2}$, &c. continued the contrary way, by the ſame Law, to v Factors, G will here be $=\frac{p+m}{p-1} \times \frac{p+m-1}{p-2} \times \frac{p+m-2}{p-3}\,(v)$; and, for the like Reaſon, $1-\frac{\overline{e-1} \times a}{\overline{g-1} \times cz^n}+\frac{\overline{e-1} \times \overline{e-2} \times a^2}{\overline{g-1} \times \overline{g-2} \times c^2 z^{2n}}\,(-v)$ will be $\doteq \frac{gcz^n}{ea}-\frac{g \times \overline{g+1} \times c^2 z^{2n}}{e \times \overline{e+1} \times a^2}\,(v)$: And therefore $\frac{dz^{en}x^f}{hn} \times$ $\overline{1+\frac{fa}{\overline{h-1} \times x}+\frac{f \times \overline{f-1} \times a^2}{\overline{h-1} \times \overline{h-2} \times x^2}\,(r)} + \frac{dFa^{r-1} z^{en} x^{m+1}}{en} \times$

$\overline{1 - \frac{g+1 \times cz^n}{e+1 \times a} + \frac{g+1 \times g+2 \times c^2 z^{2n}}{e+1 \times e+2 \times a^2}}\,(v) \pm FGQc^v a^{r-v}$ is the true Fluent in this Caſe.

COROLLARY VI.

In like Manner the Fluent of $\overline{a+cz^n}|^{m-r} \times dz^{pn+vn-1}\dot{z}$, may be determined; for let $-r$ be ſubſtituted inſtead of r, &c. then will $e = p + v$, $f = m - r$, $g = m + e$, $h = e + f$, $F = \frac{g}{m} \times \frac{g-1}{m-1} \times \frac{g-2}{m-2}\,(r)$, $G = \frac{p}{p+m+1} \times \frac{p+1}{p+m+2}\,(v)$, and $\frac{dz^{en}x^{f+1}}{f+1 \times na} \times \overline{1 - \frac{h+1 \times x}{f+2 \times a} - \frac{h+1 \times h+2 \times x^2}{f+2 \times f+3 \times a^2}}\,(r) + \frac{dFx^{m+1}z^{en-n}}{cnga^r} \times \overline{1 - \frac{e-1 \times a}{g-1 \times cz^n} + \frac{e-1 \times e-2 \times a^2}{g-1 \times g-2 \times c^2 z^{2n}}}\,(v) \pm \frac{FGQa^{v-r}}{c^v}$ equal to the Fluent ſought.

COROLLARY VII.

Likewiſe, by proceeding in the ſame Manner, the Fluent of $\overline{a+cz^n}|^{m-r} \times dz^{pn-vn-1}\dot{z}$, will be found, and is equal to $\frac{dz^{en}x^{f+1}}{f+1 \times na} \times \overline{1 - \frac{h+1 \times x}{f+2 \times a} - \frac{h+1 \times h+2 \times x^2}{f+2 \times f+3 \times a^2}}\,(r) + \frac{dFz^{en}x^{m+1}}{ena^{r+1}} \times \overline{1 - \frac{g+1 \times cz^n}{e+1 \times a} + \frac{g+1 \times g+2 \times c^2 z^{2n}}{e+1 \times e+2 \times a^2}}\,(v) \pm \frac{FGQc^v}{a^{v+r}}$; where $e = p - v$, $f = m - r$, $g = m + e$, $h = f + g$, $F = \frac{g}{m} \times \frac{g-1}{m-1} \times \frac{g-2}{m-2}\,(r)$, $G = \frac{p+m}{p-1} \times \frac{p+m-1}{p-2} \times \frac{p+m-2}{p-3}\,(v)$ and all the reſt as in the preceding Corollaries.

COROLLARY VIII.

If c be negative and p, $p + v$, $m + 1$ and $m + r + 1$ affirmative, or c, p, and $p + v$ affirmative, and $m + p$ and

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$m+p+r+v$ negative, and z be ſuppoſed to flow till $a+cz^n$ becomes nothing or infinite, or till Q becomes the Area of the whole Curve, whoſe Ordinate is $\overline{a+cz^n}|^m \times dz^{pn-1}$, it is evident, from Corollary IV. that the Area of the whole Curve, whoſe Ordinate is $\overline{a+cz^n}|^{m+r} \times dz^{pn+vn-1}$ will be truly defined by $\frac{p}{s} \times \frac{p+1}{s+1} \times \frac{p+2}{s+2}\,(v) \times \frac{m+1}{g+1} \times \frac{m+2}{g+2} \times \frac{m+3}{g+3}\,(r) \times \pm \frac{a^{v+r}Q}{c^v}$; where s is $=p+m+1$, and $g=p+m+v$, and where v and r may repreſent any whole Numbers poſitive or negative, under the forementioned Limitations. Therefore if r be taken $=0$, and $c=-b$, then the Area of the whole Curve, whoſe Ordinate is $\overline{a-bz^n}|^m \times dz^{pn+vn-1}$, when the Values of b, p, and $m+1$ are all poſitive, will be equal to $\frac{p}{s} \times \frac{p+1}{s+1} \times \frac{p+2}{s+2}\,(v) \times \frac{Qa^v}{b^v}$: From whence it appears, that the Area of the whole Curve, whoſe Abſciſſa is z, and Ordinate $\overline{a-bz^n}|^m \times dz^{pn-1} \times \overline{A+Bz^n+Cz^{2n}+Dz^{3n}}$, &c. will be truly repreſented by $Q \times \overline{A + \frac{pBa}{sb} + \frac{p.\overline{p+1}.Ca^2}{s.\overline{s+1}.b^2} + \frac{p.\overline{p+1}.\overline{p+2}.Da^3}{s.\overline{s+1}.\overline{s+2}.b^3}}$, &c. where A, B, C, &c. ſtand for any determinate Quantities.

COROLLARY IX.

Hence if t be put to denote any Number at pleaſure, and $A+Bz^n+Cz^{2n}+Dz^{3n}$, &c. be taken $= 1-tlz + \frac{t}{1} \times \frac{t+1}{2} \times l^2z^{2n} - \frac{t}{1} \times \frac{t+1}{2} \times \frac{t+2}{3} \times l^3z^{3n}$, &c. $= \overline{1+lz^n}|^{-t}$, we ſhall then have $A=1$, $B=-tl$, &c. and therefore the Area of the whole Curve, whoſe Ordinate is $\overline{a-bz^n}|^m \times dz^{pn-1} \times \overline{1+lz^n}|^{-t}$, or $\frac{\overline{a-bz^n}|^m \times dz^{pn-1}}{\overline{1+lz^n}|^t}$, will be

$Q \times \overline{1 - \frac{t.p}{1.s} \times \frac{al}{b} + \frac{t.\overline{t+1}.p.\overline{p+1}}{1.2.s.\overline{s+1}} \times \frac{a^2l^2}{b^2} - \frac{t.\overline{t+1}.\overline{t+2}.p.\overline{p+1}.\overline{p+2}}{1.2.3.s.\overline{s+1}.\overline{s+2}} \times \frac{a^3l^3}{b^3}}$, &c. which Value, if w be put $t - s$, will be truly expressed by $\frac{b^p Q}{\overline{b+al}|^p} \times \overline{1 - \frac{w.p}{1.s} \times \frac{al}{b+al} + \frac{w.\overline{w-1}.p.\overline{p+1}}{1.2.s.\overline{s+1}} \times \frac{a^2l^2}{\overline{b+al}|^2} - \frac{w.\overline{w-1}.\overline{w-2}.p.\overline{p+1}.\overline{p+2}}{1.2.3.s.\overline{s+1}.\overline{s+2}} \times \frac{a^3l^3}{\overline{b+al}|^3}}$, &c. as appears from Proposition VI. of the Summation of Series; and therefore in all Cases, where w or $t - s$ is a whole positive Number, the Area from hence may be exactly obtained: And hence also may the Area of the whole Curve, whose Ordinate is $\frac{\overline{a-bz^n}|^m \times dz^{pn-1}}{\overline{k+lz^n}|^t}$, be easily derived; for, since $\overline{k+lz^n}|^t$ may be reduced to $k^t \times \overline{1 + \frac{lz^n}{k}}|^t$, let $\frac{k}{l}$ be substituted instead of l, in the foresaid general Expression, and the whole be divided by k^t, and then we shall have $\frac{b^p k^{p-t} Q}{\overline{bk+al}|^p}$ into $\overline{1 - \frac{w.p}{1.s} \times \frac{al}{bk+al} + \frac{w.\overline{w-1}.p.\overline{p+1}}{1.2.s.\overline{s+1}} \times \frac{a^2l^2}{\overline{bk+al}|^2} - \frac{w.\overline{w-1}.\overline{w-2}.p.\overline{p+1}.\overline{p+2}}{1.2.3.s.\overline{s+1}.\overline{s+2}} \times \frac{a^3l^3}{\overline{bk+al}|^3}}$, &c. for the true Area in this Case; s being (as before specified) $= p + m + 1$.

COROLLARY X.

Also from hence the exact Area of the whole Curve, whose Ordinate is $\frac{\overline{a-bz^n}|^m \times z^{pn-1}}{\overline{1+lz^n}|^t}$, may be deduced, when s and t are both whole positive Numbers, tho' the latter should not happen to be the greater, as is required in the last Corollary: For the Series $Q \times \overline{1 - \frac{t.p}{1.s} \times \frac{al}{b}}$, &c. (universally expressing that Area) may in all such Cases be summed, and by by Corol. I. Prop. VII. of the Summation of Series, is equal to

$$\frac{PQb^p\times\overline{b+al}|^m}{\overline{al}|^{s-1}}\times\overline{1-\frac{m.t-1}{1.s-2}\times\frac{al}{b+al}+\frac{m.m-1.t-1.t-2}{1.2.s-2.s-3}\times\frac{a^2l^2}{\overline{b+al}|^2}}, \text{\&c.} - \frac{PQb^{s-1}}{\overline{al}|^{s-1}}\times\overline{1+\frac{m.w-1}{1.s-2}\times\frac{al}{b}+\frac{m.m-1.w-1.w-2}{1.2.s-2.s-3}\times\frac{a^2l^2}{b^2}},$$

&c. w and P being as hereunder ſpecified: From which Area that of the Curve, whoſe Ordinate is $\frac{\overline{a-bz^n}|^m\times dz^{pn-1}}{\overline{k+lz^n}|^t}$ may be eaſily obtained, and, by proceeding as in the laſt Corollary, will come out as follows,

$$\frac{PQb^p\times\overline{bk+al}|^m}{k^{t-p}\times\overline{al}|^{s-1}}\times\overline{1-\frac{m.t-1}{1.s-2}\times\frac{al}{bk+al}+\frac{m.m-1.t-1.t-2}{1.2.s-2.s-3}\times\frac{a^2l^2}{\overline{bk+al}|^2}-\frac{m.m-1.m-2.t-1.t-2.t-3}{1.2.3.s-2.s-3.s-4}\times\frac{a^3l^3}{\overline{bk+al}|^3}}, \text{\&c.} - \frac{PQ}{k^t}\times$$

$$\overline{\frac{bk}{al}}|^{s-1}\times\overline{1+\frac{m.w-1}{1.s-2}\times\frac{al}{bk}+\frac{m.m-1.w-1.w-2}{1.2.s-2.s-3}\times\frac{a^2l^2}{b^2k^2}+\frac{m.m-1.m-2.w-1.w-2.w-3}{1.2.3.s-2.s-3.s-4}\times\frac{a^3l^3}{b^3k^3}},$$

&c. where both Series are to be continued till they break off, and where w is put $=s-t$, and P $=\pm\frac{t.t+1.t+2\,(w)\times w.w+1.w+2\,(t-1)}{m.m-1.m-2.m-3.m-4\,(s-1)}$; in which laſt Value the prefixed Sign + or — obtains, according as t is an odd or an even Number.

Note. In theſe two laſt and the ſucceeding Corollaries, k and l may denote any Quantities at Pleaſure, provided the Quantities $b+al$, $bk+al$, and $bk\mp al$, wherever they occur, be poſitive.

COROLLARY XII.

Moreover the Fluent of $\overline{a-bz}|^{m} \times dz^{un-1}\dot{z} \times R$, when $a-bz^{n}$ is $=0$, or the Area of the whole Curve, whose Abscissa is z, and Ordinate $\overline{a-bz^{n}}|^{m} \times dz^{un-1} \times R$, supposing $\dot{R} = \overline{k\mp lz^{n}}|^{-t} \times z^{qn-1}\dot{z}$, and u and q any positive Numbers, may from hence be determined; for $\overline{k\mp lz^{n}}|^{-t} \times z^{qn-1}\dot{z}$ being converted to a Series, and the Fluent taken, we have $R = \frac{z^{qn}}{nk^{t}} \times \overline{\frac{1}{q} \pm \frac{t}{q+1} \times \frac{lz^{n}}{k} + \frac{t.t+1}{2.q+2} \times \frac{l^{2}z^{2n}}{k^{2}}}$, &c. and therefore $\overline{a-bz^{n}}|^{m} \times dz^{un-1} \times R$, if p be put $=q+u$, will be equal to $\frac{\overline{a-bz^{n}}|^{m} \times dz^{pn-1}}{nk^{t}} \times \overline{\frac{1}{q} \pm \frac{t}{q+1} \times \frac{lz^{n}}{k} + \frac{t.t+1}{2.q+2} \times \frac{l^{2}z^{2n}}{k^{2}}}$, &c. Wherefore if (as above) s be put $=p+m+1$ and Q be taken to denote the Area of the whole Curve, whose Ordinate is $\overline{a-bz^{n}}|^{m} \times dz^{pn-1}$, and $\frac{1}{q}$, $\frac{t.l}{q+1.k}$, $\frac{t.t+1.l^{2}}{2.q+2.k^{2}}$, &c. be respectively substituted for A, B, C, &c. in the general Expression at the end of Corol. VIII. we shall have $\frac{Q}{nk^{t}} \times \overline{\frac{1}{q} \pm \frac{t.p}{s.q+1} \times \frac{al}{bk} + \frac{t.t+1.p.p+1}{s.s+1.2.q+2} \times \frac{a^{2}l^{2}}{b^{2}k^{2}} \pm \frac{t.t+1.t+2.p.p+1.p+2}{s.s+1.s+2.2.3.q+3} \times \frac{a^{3}l^{3}}{b^{3}k^{3}}}$, &c. for the Area proposed to be found.

COROLLARY XII.

Therefore, if this Area be denoted by S, and $\frac{al}{bk}$ be put $=x$, and $t-s=w$, we shall have $S = \frac{Q}{nk^{t}} \times \overline{\frac{1}{q} \pm \frac{t.p.x}{s.q+1} + \frac{t.t+1.p.p+1.x^{2}}{s.s+1.2.q+2}}$, &c. But it appears, from Prop. VI. of the

Summation of Series, that $\overline{1 \pm \frac{t.px}{s} + \frac{t.\overline{t+1}.p.\overline{p+1}.x^2}{2.s.\overline{s+1}}}$, &c. is universally equal to $\frac{1}{\overline{1 \mp x}|^p} \times \overline{1 \pm \frac{w.px}{1.s.\overline{1 \mp x}} + \frac{w.\overline{w-1}.p.\overline{p+1}.x^2}{1.2.s.\overline{s+1}.\overline{1 \pm x}|^2} \pm \frac{w.\overline{w-1}.\overline{w-2}.p.\overline{p+1}.\overline{p+2}.x^3}{1.2.3.s.\overline{s+1}.\overline{s+2}.\overline{1 \mp x}|^3}}$, &c. be the Value of x, &c. what it will; therefore if x be considered as variable, and both Sides of the Equation be multiply'd by $x^{q-1}\dot{x}$, we shall have $x^{q-1}\dot{x} \times \overline{1 \pm \frac{t.p.x}{s}}$, &c. $= \frac{x^{q-1}\dot{x}}{\overline{1 \mp x}|^p} \times \overline{1 \pm \frac{w.px}{1.s.\overline{1 \mp x}}}$, &c. and consequently $x^q \times \overline{\frac{1}{q} \pm \frac{t.p.x}{s.\overline{q+1}} + \frac{t.\overline{t+1}.p.\overline{p+1}.x^2}{s.\overline{s+1}.2.\overline{q+2}}}$, &c. equal to the Fluent of $\frac{x^{q-1}\dot{x}}{\overline{1 \mp x}|^p} \times \overline{1 \pm \frac{w.p.x}{s.\overline{1 \mp x}} + \frac{w.\overline{w-1}.p.\overline{p+1}.x^2}{2.s.\overline{s+1}.\overline{1 \pm x}|^2}}$, &c. Therefore, if H be taken to represent the Fluent of $\frac{x^{q-1}\dot{x}}{\overline{1 \mp x}|^p} \times \overline{1 \pm \frac{w.p.x}{s.\overline{1 + x}}}$, &c. when x is $= \frac{al}{bk}$, it is evident, that S will be exactly equal to $\frac{b^q QH}{nk^{t-q} \times \overline{al}|^q}$. But, in order for the more ready determining the Value of H, upon which that of S depends, let $\frac{x}{1 \mp x}$ be put $= y$, then will $\frac{x^{q-1}\dot{x}}{\overline{1 \mp x}|^p} \times \overline{1 \pm \frac{w.p.x}{s.\overline{1 + x}}}$, &c. be changed to $\overline{1 \pm y}|^{u-1} \times y^{q-1}\dot{y} \times \overline{1 \pm \frac{wpy}{s} + \frac{w.\overline{w-1}.p.\overline{p+1}.y^2}{2.s.\overline{s+1}} \pm \frac{w.\overline{w-1}.\overline{w-2}.p.\overline{p+1}.\overline{p+2}.y^3}{2.3.s.\overline{s+1}.\overline{s+2}}}$, &c. therefore the Fluent of this last Expression, generated while y from nothing, becomes $= \frac{al}{bk \mp al}$, will give the true Value of H; which Fluent may, in many Cases, be had in finite Terms, and by the Quadrature of the Conic-Sections, if w or $t - s$ be either nothing, or a whole positive Number.

COROLLARY XIII.

But if $t-s$ be a whole negative Number, put $w=s-t$, $\delta=t-q$, and $\beta=\frac{al}{bk}$, and let both Sides of the Equation $S=\frac{Q}{nk^t}\times\overline{\frac{1}{q}\mp\frac{t.p.x}{q+1.s}+\frac{t.t+1.p.p+1.x^2}{2.q+2.s.s+1}}$, &c. as (found above) be divided by $t.t+1.t+2\ldots\ldots s-1$, and there will be $\frac{S}{t.t+1.t+2\ldots s-1}=\frac{Q}{nk^t}$ into $\frac{1}{q.t.t+1.t+2\ldots s-1}\mp\frac{px}{q+1.t+1.t+2.t+3\ldots s}$ $\frac{\frac{p}{1}\times\frac{p+1}{2}\times x^2}{q+2.t+2.t+3.t+4\ldots s+1}$, &c. or to $\frac{Q}{nk^t}$ multiply'd by

$$\frac{1}{q.t.t+1.t+2\,(w+1)}\mp\frac{px}{q+1.t+1.t+2\,(w+1)}+\frac{\frac{p}{1}\times\frac{p+1}{2}\times x^2}{q+2.t+2.t+3\,(w)},\ \&c.$$

which Series (*by Prop.* V. *of the Summation of Series*) is equal to

$$+\frac{1}{\delta.\delta+1.\delta+2\,(w)}\times\overline{\frac{1}{q}\mp\frac{px}{q+1}+\frac{p.p+1.x^2}{2.q+2}},\ \&c.$$
$$-\frac{1}{\delta.1.2.3.4\,(w)}\times\overline{\frac{1}{t}\mp\frac{px}{t+1}+\frac{p.p+1.x^2}{2.t+2}},\ \&c.$$
$$+\frac{w-1}{\delta+1.1.2.3.4\,(w)}\times\overline{\frac{1}{t+1}\mp\frac{px}{t+2}+\frac{p.p+1.x^2}{2.t+3}},\ \&c.$$

&c. &c.

But $\frac{1}{q}\mp\frac{px}{q+1}+\frac{p.p+1.x^2}{2.q+2}$, &c. if x be considered as variable, will, it is manifest, be equal to the Fluent of $\frac{\overline{1\mp x}|^{-p}\times x^{q-1}\dot{x}}{\beta^q}$ when x is equal to β or $\frac{al}{bk}$; and the like may be observed with regard to the Series $\frac{1}{t}\mp\frac{px}{t+1}+\frac{p.p+1.x^2}{2.t+2}$ &c. &c. From whence it will appear, that the true Value of S, S, will, in this Case, be $= \overline{t.t+1.t+2.t+3\,(w)} \times \frac{Q}{zk^t}$ multiply'd by the Fluent of $\frac{\overline{1\mp x}|^{-p} \times x^{q-1}\dot{x}}{\delta.\delta+1.\delta+2\,(w)\beta^q} - \frac{\overline{1\mp x}|^{-p} \times x^{t-1}\dot{x}}{1.2.3.4\,(w-1)\beta^t}$ $\times \overline{\frac{1}{\delta} - \frac{w-1.x}{\delta+1.\beta} + \frac{w-1.w-2.x^2}{2.\delta+2.\beta^2} - \frac{w-1.w-2.w-3.x^3}{2.3.\delta+3.\beta^3}}$, &c. taken in the forementioned Circumstance, when x from nothing, is become $= \frac{al}{bk}$.

EXAMPLE I.

The Fluent (Q) of $\overline{b^2+z^2}|^{\frac{1}{2}} \times z\dot{z}$ being given, 'tis required to find the Fluent of $\overline{b^2+z^2}|^{\frac{1}{2}} \times z^5\dot{z}$.

Here, by comparing $\overline{b^2+z^2}|^{\frac{1}{2}} \times z^5\dot{z}$, with $\overline{a+cz^n}|^m \times dz^{en-1}\dot{z}$ (Vid. Corol. I.) we have $a=b^2$, $c=1$, $n=2$, $m=\frac{1}{2}$, $d=1$, and $en-1$ or $2e-1=5$, whence $e=3$, $v=2$, $p=1$, and $t\,(e+m)=\frac{7}{2}$, and consequently $\frac{\overline{a+cz}|^{m+1} \times dz^{en-n}}{tcn} \times \overline{1-\frac{e-1}{t-1} \times \frac{a}{cz^n}}$, &c. $= \frac{\overline{b^2+z^2}|^{\frac{3}{2}} \times z^4}{7} \times \overline{1-\frac{4b^2}{5z^2}} + Qb^4 \times \frac{2}{5} \times \frac{4}{7}$ equal to the Value sought. But since p is here a whole positive Number, the true Value may be had independant of Q, being (by Corol. III.) equal to $\frac{\overline{a+cz^n}|^{m+1} \times dz^{en-n}}{tcn} \times \overline{1-\frac{e-1}{t-1} \times \frac{a}{cz^n}}$, &c. continued till it terminates; that is, equal to $\frac{\overline{b^2+z^2}|^{\frac{3}{2}} \times z^4}{7} \times \overline{1-\frac{4b^2}{5z^2}+\frac{4.2b^4}{5.3z^4}}$, or to $\frac{\overline{b^2+z^2}|^{\frac{3}{2}} \times \overline{15z^4-12z^2b^2+8b^4}}{105}$.

EX-

EXAMPLE II.

Where it is propoſed to find the Fluent of $\overline{b^3-x^3}|^{-\frac{1}{2}}x^{-\frac{5}{2}}\dot{x}$.

In this Caſe we have $a=b^3$, $c=-1$, $z=x$, $n=3$, $m=-\frac{1}{2}$, $d=1$, $en-1=-\frac{5}{2}$, whence $e=-\frac{1}{2}$, and $t\ (=e+m)=-1$; which laſt Value being a whole Negative, indicates that the required Fluent may be exactly found in finite Terms. Therefore let theſe ſeveral Values be now ſubſtituted in the ſecond general Expreſſion in Corollary III. and we ſhall have $-\frac{\overline{b^3-x^3}|^{\frac{1}{2}}\times 2x^{-\frac{3}{2}}}{3b^3}$ for the true Value ſought.

EXAMPLE III.

Let (Q) the Fluent of $\overline{g^4+x^4}|^{\frac{1}{4}}\times x^{-3}\,\dot{x}$ be given; to find the Fluent of $\overline{g^4+x^4}|^{\frac{1}{4}}\times x^{-15}\,\dot{x}$.

Theſe Expreſſions being compared with thoſe in Cor. II. &c. we have $a=g^4$, $c=1$, $z=x$, $m=\frac{1}{4}$, $n=4$, $d=1$, $v=3$, $p=-\frac{1}{2}$, $e=-\frac{7}{2}$, $t=-\frac{13}{4}$, and $\frac{\overline{a+cz^n}|^{m+1}\times dz^{en}}{ena}\times\overline{1-\frac{t+1}{e+1}\times\frac{cz^n}{a}}$, &c. $=\frac{\overline{g^4+x^4}|^{\frac{5}{4}}\times x^{-14}}{-14g^4}\times\overline{1-\frac{9x^4}{10g^4}+\frac{9}{10}\times\frac{5}{6}\times\frac{x^8}{g^8}}-\frac{1}{6}\times\frac{5}{10}\times\frac{9}{14}\times\frac{Q}{g^{12}}$ equal to the Value required.

EXAMPLE IV.

Where (Q) the Fluent of $\overline{x^3-b^3}|^{\frac{1}{3}}\times z^4\,\dot{z}$, or the Area of the Curve, whoſe Abſciſſa is z, and Ordinate $\overline{z^3-b^3}|^{\frac{1}{3}}\times$

z^4 being given; 'tis required to find the Fluent of $\overline{z^3-b^3}|^{\frac{13}{3}}\times z^{10}\dot{z}$, or the Area of the Curve whose Abscissa is also z, and Ordinate $\overline{z^3-b^3}|^{\frac{13}{3}}\times z^{10}$.

By comparing these Expressions with those in Corol. IV. there will be $c=1$, $a=-b^3$, $d=1$, $n=3$, $m=\frac{1}{3}$, $p=\frac{5}{3}$, $v=2$, $r=4$, $e=\frac{11}{3}$, $f=\frac{13}{3}$, $g=\frac{12}{3}$, $h=\frac{24}{3}$, $x=z^3-b^3$; and therefore, by writing these Values in the last of the Equations there given, we shall have

$$\frac{z^{11}\times\overline{z^3-b^3}|^{\frac{4}{3}}}{24}\times\overline{\overline{z^3-b^3}|^3-\frac{13b^3\times\overline{z^3-b^3}|^2}{21}+\frac{13\times10b^6\times\overline{z^3-b^3}}{21\times18}-\frac{13\times10\times7b^9}{21\times18\times15}}$$

$$+\frac{13\times10\times7\times4b^{12}\times\overline{z^3-b^3}|^{\frac{4}{3}}}{24\times21\times18\times15\times12}\times\overline{z^8+\frac{8z^5b^3}{9}}+\frac{4\times7\times10\times13\times5\times8b^{18}Q}{9\times12\times15\times18\times21\times24}$$

for the true Fluent or Area sought; which therefore, when $z=b$, is barely $=\frac{4\times7\times10\times13\times5\times8b^{18}Q}{9\times12\times15\times18\times21\times24}$.

EXAMPLE V.

Let there be given (Q) the Fluent of $\overline{1-x}|^{-\frac{1}{2}}x^{-\frac{1}{2}}\dot{x}$, when $1-x$ becomes $=0$, or the Periphery of the Circle, whose Diameter is Unity; to find the exact Fluent of $\overline{1-x}|^{r-\frac{1}{2}}\times x^{v-\frac{1}{2}}\dot{x}$, when $1-x$ becomes $=0$; or the Area of the whole Curve whose Ordinate is $\overline{1-x}|^{r-\frac{1}{2}}\times x^{v-\frac{1}{2}}$.

In this Case $a=1$, $c=-1$, $z=x$, $n=1$, $m=-\frac{1}{2}$, $p=\frac{1}{2}$, $s\,(p+m+1)=1$, $g\,(p+m+v)=v$; and therefore, by Corol. VIII. we shall have

$$\frac{1.3.5.7.9,\ \&c.\ \text{to } v \text{ Factors, into } 1.3.5.7.9,\ \&c.\ \text{to } r \text{ Factors.}}{2.4.6.8.10.12.14.16.18.20.22,\ \&c.\ \text{to } r+v \text{ Factors.}}\times Q,$$ for the Value sought.

EXAMPLE VI.

The Fluent Q of $\overline{b^2+x^2}|^{-\frac{2}{3}} \times \dot{x}$ being given, 'tis proposed to find the Fluent of $\overline{b^2+x^2}|^{-\frac{8}{3}} \times x^4 \dot{x}$, when b^2+x^2 becomes infinite, or the Area of the whole Curve, whose Ordinate is $\overline{b^2+x^2}|^{-\frac{8}{3}} \times x^4$.

Here we have $a=b^2$, $c=1$, $z=x$, $n=2$, $m=-\frac{2}{3}$, $d=1$, $p=\frac{1}{2}$, $r=-2$, $v=2$, $s=\frac{5}{6}$, $g=\frac{11}{6}$; whence, by Cor. VIII. there will be $\frac{p}{s} \times \frac{p+1}{s+1}(2) \times \frac{m+1}{g+1} \times \frac{m+2}{g+2}(-2) \times Q = \frac{p}{s} \times \frac{p+1}{s+1}(2) \times \frac{g}{m} \times \frac{g-1}{m-1}(2) \times Q = \frac{3}{5} \times \frac{9}{11} \times \frac{11}{-4} \times \frac{5}{-10} \times Q = \frac{27Q}{40}$ equal to the exact Area which was to be found.

Note. The Area of this same Curve, or of any Part of it, may also be found by Corollary VI.

EXAMPLE VII.

Where (K) the Area of the whole Curve, whose Ordinate is $\overline{f^3-z^3}|^{-\frac{1}{2}}$ being supposed given; tis required to find the exact Area of the whole Curve, whose Ordinate is $\frac{\overline{f^3-z^3}|^{\frac{1}{2}} \times z^3}{\overline{b^3+z^3}|^{\frac{17}{6}}}$.

First, to determine the Area of the Curve whose Ordinate is $\overline{f^3-z^3}|^{\frac{1}{2}} \times z^3$, which is requisite for finding that required,

quired, let $\overline{a+cz^n}|^m \times dz^{pn-1}$ and $\overline{a+cz^n}|^{m+r} \times dz^{pn+vn-1}$ (as expreſſed in Corol. VIII.) be compared with $\overline{f^3-z^3}|^{-\frac{1}{2}}$ and $\overline{f^3-z^3}|^{\frac{1}{2}} \times z^3$, and there will be $a=f^3$, $c=-1$, $n=3$, $m=-\frac{1}{2}$, $m+r=\frac{1}{2}$, $d=1$, $pn-1=0$, and $pn+vn-1=3$; whence $r=1$, $p=\frac{1}{3}$, $v=1$, $s\,(p+m+1)=\frac{5}{6}$, $g\,(p+m+v)=\frac{5}{6}$, and therefore $\left(\frac{p}{s} \times \frac{p+1}{s+1}(v) \times \frac{m+1}{g+1} \times \frac{m+2}{g+2}\ (r)\right.$ into $\left.\pm \frac{a^{v+r}K}{c^v}\right) \frac{6f^6K}{55}$ is equal to the exact Area of the Curve, whoſe Ordinate is $\overline{f^3-z^3}|^{\frac{1}{2}} \times z^3$. Let therefore Q be put $=\frac{6f^6K}{55}$ and let $\frac{\overline{f^3-z^3}|^{\frac{1}{2}} \times z^3}{\overline{b^3+z^3}|^{\frac{17}{6}}}$ be now compar'd with $\frac{\overline{a-bz^n}|^m \times dz^{pn-1}}{\overline{k+lz^n}|^t}$ (Vid. Corol. IX.) and we ſhall have $a=f^3$, $b=1$, $n=3$, $m=\frac{1}{2}$, $d=1$, $p=\frac{4}{3}$, $k=b^3$, $l=1$, $t=\frac{17}{6}$, $s\,(p+m+1)=\frac{17}{6}$, $w\,(t-s)=0$; and therefore $\left(\frac{b^p k^{p-t}\,Q}{\overline{bk+al}|^p} \times 1 - \frac{w.p}{1.s} \times \frac{al}{bk+al},\right.$ &c.$\left.\right) \frac{6Kf^6}{55b^{\frac{9}{2}} \times \overline{b^3+f^3}|^{\frac{4}{3}}}$ is the true Value required.

EXAMPLE VIII.

The ſame being given as in the preceding Example, let it be required to find the exact Area of the whole Curve, whoſe Ordinate is $\frac{\overline{f^3-z^3}|^{\frac{3}{2}} \times z^9}{\overline{b^3-z^3}|^{\frac{41}{6}}}$. Here, by proceeding as above, we ſhall firſt get $\frac{2 \times 8 \times 14 \times 3 \times 9}{5 \times 11 \times 17 \times 23 \times 29} \times Kf^{15}$, for the exact Area of the whole Curve, whoſe Ordinate is $\overline{f^3-z^3}|^{\frac{3}{2}} \times z^9$, which let be denoted by Q, and then, by comparing $\frac{\overline{f^3-z^3}|^{\frac{3}{2}} \times z^9}{\overline{b^3-z^3}|^{\frac{41}{6}}}$ with

with $\frac{\overline{a-bz^n}|^m \times dz^{pn-1}}{\overline{k+lz^n}|^t}$, we ſhall have $a=f^3$, $b=1$, $n=3$, $m=\frac{3}{2}$, $d=1$, $p=\frac{10}{3}$, $k=h^3$, $l=-1$, $t=\frac{41}{6}$, $s=\frac{35}{6}$, and w $(t-s)=1$, and conſequently $\frac{h^{-\frac{21}{2}} \times Q}{\overline{h^3-f^3}|^{\frac{10}{3}}} \times \overline{1+\frac{4f^3}{7\times \overline{h^3-f^3}}}$ or $\frac{864\, K f^{15} \times \overline{7h^3-3f^3}}{623645\, h^{\frac{21}{2}} \times \overline{h^3-f^3}|^{\frac{13}{3}}}$, equal to the required Area in this Caſe.

EXAMPLE IX.

Where the Area of (Q) of the whole Curve, whoſe Ordinate is $\overline{h^n-z^n}|^{\frac{1}{2}} \times z^{\frac{3}{2}n-1}$ being given; 'tis required to find that of the whole Curve, whoſe Ordinate is $\frac{\overline{h^n-z^n}|^{\frac{1}{2}} \times z^{\frac{3}{2}n-1}}{g^n+z^n}$. In this Caſe we have $a=h^n$, $b=1$, $m=\frac{1}{2}$, $d=1$, $p=\frac{3}{2}$, $k=g^n$, $l=1$, $t=1$, and $s\,(p+m+1)=3$: Therefore, s being here a whole Number greater than t, let theſe ſeveral Values be written in Corol. X. and we ſhall have $w=2$, $P=\left(\frac{1\times 2}{\frac{1}{2}\times -\frac{1}{2}}\right)-8$, and $-\frac{8Qg^{\frac{1}{2}n}\times\overline{g^n+h^n}|^{\frac{1}{2}}}{h^{2n}}+\frac{8Q}{g^n}\times\frac{\overline{g^n}|^2}{h^n}\times\overline{1+\frac{\frac{1}{2}\times 1}{1\times 1}\times\frac{h^n}{g^n}}$ or $Q\times\frac{8g^n+4h^n-8\times\overline{g^{2n}+g^nh^n}|^{\frac{1}{2}}}{h^{2n}}$ equal to the exact Area required.

EXAMPLE X.

Let it be required to find the Area of the whole Curve, whoſe Ordinate is $\overline{1-z^2}|^{-\frac{1}{2}} \times R$, ſuppoſing $\dot{R}=\overline{1-z^2}|^{-\frac{1}{2}} \times \dot{z}$.

Theſe

These Expressions being compared with those in Corollary XI. we have $a=1$, $b=1$, $n=2$, $m=-\frac{1}{2}$, $d=1$, $u=\frac{1}{2}$, $k=1$, $l=1$, $t=\frac{1}{2}$, $q=\frac{1}{2}$, $p=1$, and $s=\frac{3}{2}$; therefore Q (the Area of the whole Curve whose Ordinate is $\overline{a-bz^n|}^m \times dz^{pn-1}$) will here be $=1$, and consequently $\frac{Q}{nk^t} \times \overline{\frac{1}{q} + \frac{t.p}{s.p+1} \times \frac{al}{bk}}$, &c. $= 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81}$, &c. equal to the true Value sought. But the Area of the Curve, whose Ordinate is $\overline{1-z^2|}^{-\frac{1}{2}} \times R$, is also equal to the Fluent of $\overline{1-z^2|}^{-\frac{1}{2}}\dot{z} \times R$, or $\dot{R}R$, that is $= \frac{R^2}{2}$; where R, in the proposed Circumstance, is equal to $\frac{1}{4}$ Part of the Periphery of the Circle, whose Radius is Unity. Hence it appears, that the Sum of the Series $1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49}$, &c. is equal to $\frac{1}{8}$ Part of the Square of the Periphery of the Circle, whose Diameter is Unity. But the Sum of the Series $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36}$, &c. is just $\frac{1}{6}$ Part of the Square of that Periphery, because $\frac{1}{4} + \frac{1}{16} + \frac{1}{36}$, &c. the Sum of the Squares of the Reciprocals of the even Numbers, is $= \frac{1}{4} \times \overline{1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}}$, &c. that is $= \frac{1}{4}$ of the whole Series.

EXAMPLE XI.

Where $\dot{R}$ being $= \overline{c^2+z^2|}^{-\frac{1}{2}} \times \dot{z}$, 'tis proposed to find the Area (S) of the whole Curve, whose Ordinate is $\overline{b^2-z^2|}^{-\frac{1}{2}} \times ccR$.

 Here

Here, by proceeding as in the laſt Example, we ſhall have $a=b^2$, $b=1$, $n=2$, $m=-\frac{1}{2}$, $u=\frac{1}{2}$, $d=c^2$, $k=c^2$, $l=1$, $t=\frac{3}{2}$, $q=\frac{1}{2}$, $p=1$, $s=\frac{3}{2}$, and $Q=c^2b$; whence, according to Corol. XII. $w\,(t-s)=0$, and $S\left(\frac{b^t QH}{n k^{t-q}\times\overline{al}|^q}\right)=\frac{H}{2}$; H being the Fluent of $\overline{1-y}|^{-\frac{1}{2}}y^{-\frac{1}{2}}\dot{y}$, when y is $=\frac{b^2}{c^2+b^2}$, or twice the Arch of the Circle, whoſe Diameter is Unity, and verſed Sine $\frac{b^2}{c^2+b^2}$. Therefore the Value here ſought, will be exactly equal to the Meaſure of the ſaid Arch.

SCHOLIUM.

Tho' the chief Deſign of the preceding Propoſitions is to exhibit the Relation of ſuch Fluents, as can be expreſſed in Terms of each other, and algebraic Quantities; yet from thence a Method may be derived for finding Fluents originally, by Infinite Series, much preferable to that commonly made uſe of. Let the general Expreſſion $\overline{a+cz^n}|^m\times dz^{en-1}\dot{z}$ be propoſed, in order to find the Fluent thereof. Take v equal to any whole poſitive Number, and let $p=e+v$, $t=e+m$, $x=a+cz^n$, and $Q=$ the Fluent of $\overline{a+cz^n}|^m\times dz^{pn-1}\dot{z}$, which, according to Prop. I. is $=\frac{dx^m z^{pn}}{pn}\times\overline{1-\frac{m}{p+1}\times\frac{cz^n}{x}+\frac{m}{p+1}\times\frac{m-1}{p+2}\times\frac{c^2z^{2n}}{x^2}}$, &c. Then, by Prop. II. Corol. II. the required Fluent will be truly defined by $\frac{dz^{en}x^{m+1}}{ena}\times\overline{1-\frac{t+1}{e+1}\times\frac{cz^n}{a}+\frac{t+1}{e+1}\times\frac{t+2}{e+2}\times\frac{c^2z^{2n}}{a^2}}\,(v)\pm\frac{Qc^v}{a^v}\times\overline{\frac{p+m}{p-1}\times\frac{p+m-1}{p-2}\times\frac{p+m-2}{p-3}}\,(v)$. Hence, to find the Fluent of

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$\overline{a+cz^n}|^m \times dz^{en-1}\dot{z}$, let there be taken $A = \frac{dz^{en}x^{m+1}}{ena}$, $B = -A \times \frac{t+1}{e+1} \times \frac{cz^n}{a}$, $C = -B \times \frac{t+2}{e+2} \times \frac{cz^n}{a}$, $D = -C \times \frac{t+3}{e+3} \times \frac{cz^n}{a}$, and so on to any Number of Terms (v) and let the last of those Terms be denoted by Q; then take $R = -Q \times \frac{p+m}{p} \times \frac{cz^n}{x}$, $S = -R \times \frac{m}{p+1} \times \frac{cz^n}{x}$, $T = -S \times \frac{m-1}{p+2} \times \frac{cz^n}{x}$, $V = -T \times \frac{m-2}{p+3} \times \frac{cz^n}{x}$, &c. then will $A+B+C+D \ldots\ldots +Q+R+S+T+V+W$, &c. be the true Value sought. Now the chief Advantage of this Method consists in this, that, as v may be any Number at Pleasure, the Value thereof may be so assigned, or such a Number of Terms, A, B, C, &c. of the first Series may be taken, as to make the second Series R+S+T, &c. converge exceeding fast, when the Series resulting from the common Method diverges, or converges so slow, as to be intirely useless. But this will appear better by an Example or two. Let it be required to find the Fluent of $\overline{1+z^2}|^{-1} \times \dot{z}$, when $z=1$ (expressing the length of $\frac{1}{8}$ of the Periphery of the Circle, whose Radius is Unity). Here $\overline{1+z^2}|^{-1} \times \dot{z}$, being compared with $\overline{a+cz^n}|^m \times dz^{en-1}\dot{z}$, there will be $a=1$, $c=1$, $n=2$, $m=-1$, $d=1$, $z=1$, $en-1$, or $2e-1=0$; whence, if v be taken $=6$, we shall have $p\ (e+v) = \frac{13}{2}$, $t = -\frac{1}{2}$, $x=2$, $A=1$, $B = -\frac{1}{3}$, $C = \frac{1}{5}$, $D = -\frac{1}{7}$, $E = \frac{1}{9}$, F (or Q) $= -\frac{1}{11}$, $R = \frac{1}{26}$, $S = \frac{R}{15}$, $T = \frac{2S}{17}$, $V = \frac{3T}{19}$, $W = \frac{4T}{21}$, &c. and consequently $A+B+C \ldots\ldots Q+R+S$, &c. $= 0.785398$; which Number, found by taking only 8 Terms of the Series R+S+T, &c. is right in the last Place, and would have required, at least,

leaſt, 1000000 Terms of the common Series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$, &c. Again, let it be propoſed to find the Fluent of $\overline{1+z^4}|^{\frac{1}{2}} \times \dot{z}$. In this Caſe, taking $v=4$, we have $a=1$, $c=1$, $n=4$, $m=\frac{1}{2}$, $d=1$, $e=\frac{1}{4}$, $p=\frac{17}{4}$, $t=\frac{3}{4}$, $A=zx^{\frac{1}{2}}$, $B=-\frac{7Az^4}{5}$, $C=-\frac{11Bz^4}{9}$, D (or Q) $=-\frac{15Cz^4}{13}$, $R=-\frac{19Qz^4}{17x}$, $S=-\frac{2Rz^4}{21x}$, $T=\frac{2Sz^4}{25x}$, $V=\frac{6Tz^4}{29x}$, $W=\frac{10Vz^4}{33x}$ &c. and $A+B+C+Q+R+S$, &c. equal to the Fluent ſought; which if z be taken $=1$, will become 1.08942, in finding whereof, no more than 6 Terms of the Series $R+S+T$, &c. are requiſite; and if z had been taken ſmaller, the Concluſion would have been ſtill more exact.—Beſides theſe Uſes of the foregoing Method, another Conſequence may be derived therefrom, not altogether inconſiderable. We have proved that the Fluent of $\overline{a+cz^n}|^m \times dz^{pn-1}\dot{z}$, is univerſally equal to $\frac{dx^m z^{pn}}{pn} \times \overline{1 - \frac{m}{p+1} \times \frac{cz^n}{x} + \frac{m}{p+1} \times \frac{m-1}{p+2} \times \frac{c^2z^{2n}}{x^2}}$, &c. Therefore if a be taken $=0$, $c=1$, $d=1$, then x being $=cz^n$, we ſhall have $\frac{z^{pn+mn}}{pn} \times \overline{1 - \frac{m}{p+1} + \frac{m}{p+1} \times \frac{m-1}{p+2}}$, &c. equal to the Fluent of $\overline{z^n}|^m \times z^{pn-1}\dot{z}$, that is $=\frac{z^{mn+pn}}{mn+pn}$; whence $-\frac{m}{p+1} + \frac{m}{p+1} \times \frac{m-1}{p+2}$, &c. $=\frac{-m}{p+m}$: From which the Sum of any Infinite Series as $\frac{m}{p} + \frac{m}{p} \times \frac{m+r}{p+r} + \frac{m}{p} \times \frac{m+r}{p+r} \times \frac{m+2r}{p+2r}$ &c. where m, p, and r denote any Quantities at Pleaſure, is eaſily deduced; for, ſince this Series may be changed to $\frac{\frac{m}{r}}{\frac{p}{r}} + \frac{\frac{m}{r}}{\frac{p}{r}} \times \frac{\frac{m}{r}+1}{\frac{p}{r}+1}$, &c. let $\frac{p}{r}-1$, and $-\frac{m}{r}$ be ſubſtituted

for

for p and m, in the laſt Equation, we ſhall have $\frac{m}{p} + \frac{m}{p} \times \frac{m+r}{p+r} + \frac{m}{p} \times \frac{m+r}{p+r} \times \frac{m+2r}{p+2r} + \frac{m}{p} \times \frac{m+r}{p+r} \times \frac{m+2r}{p+3r} \times \frac{m+3r}{p+3r}$, &c. $= \frac{m}{p-r-m}$. Hence we alſo gather, that if the Terms A, B, C, D, &c. of any Infinite Series, be ſo related that $B = \frac{m}{p} \times A$, $C = \frac{m+r}{p+r} \times B$, $D = \frac{m+2r}{p+2r} \times C$, &c. the Value of that Series, will be truly defined by $\frac{p-r}{p-r-m} \times A$; which therefore is finite or infinite, according as p is greater or leſs than $m+r$. Example: Let it be required to find the Sum of the Infinite Series $\frac{1}{4} + \frac{1}{4} \times \frac{3}{6} + \frac{1}{4} \times \frac{3}{6} \times \frac{5}{8} + \frac{1}{4} \times \frac{3}{6} \times \frac{5}{8} \times \frac{7}{10}$, &c. Then $p=4$, $m=1$, $r=2$, and $\frac{m}{p-r-m} = 1 =$ the Value ſought. Again, let the Series propoſed be $\frac{1}{1.2} + \frac{1}{2\ 3} + \frac{1}{3.4}$, &c. then will $A = \frac{1}{2}$, $B = \frac{1}{3} A$, $C = \frac{2}{4} B$, &c. $m=1$, $p=3$, $r=1$, and $\frac{1}{1.2} + \frac{1}{2.3}$, &c. $(= \frac{p-r}{p-r-m} \times A) = 1$. Or, more univerſally, let the Series propounded be $\frac{1}{m.\overline{m+r}.\overline{m+2r}\,(n)}$ $\frac{1}{\overline{m+r}.\overline{m+2r}.\overline{m+3r}\,(n)} + \frac{1}{\overline{m+2r}.\overline{m+3r}.\overline{m+4r}\,(n)}$, &c. and the Value thereof will come out $\frac{1}{m.\overline{m+r}.\overline{m+2r}\,(n-1) \times rn}$; which laſt Expreſſion is the ſame with that in Page 92 of my Eſſays, but found in a different Manner.

PROPOSITION III.

If x and y be two variable Quantities any how related to each other, and the Fluent of $y x^{f-1} \dot{x}$ *be taken, and multiply'd by* $x^{g-1} \dot{x}$, *and the Fluent of the Product be again taken and multiply'd by* $x^{h-1} \dot{x}$, *and the Fluent of this last Product be also taken, and so on continually; and there be made* $f = r$, $r + g = s$, $s + h = t$, $t + i = v$, *&c.* $A = \frac{1}{\overline{s-r}.\overline{t-r}.\overline{v-r}\,(n)}$, $B = \frac{1}{\overline{r-s}.\overline{t-s}.\overline{v-s}\,(n)}$, $C = \frac{1}{\overline{r-t}.\overline{s-t}.\overline{v-t}\,(n)}$, $D = \frac{1}{\overline{r-v}.\overline{s-v}.\overline{t-v}\,(n)}$, *&c. and the Sum of all the Indices* $f + g + h + i$, *&c. be denoted by* p; *I say, that Fluent whose Place in the Progression is denoted by* $n + 1$, *will, when* x *becomes equal to any given Quantity* a, *be equal to the Fluent of*

$$\frac{a^p y \dot{x}}{x} \times \overline{\frac{A x^r}{a^r} + \frac{B x^s}{a^s} + \frac{C x^t}{a^t} + \frac{D x^v}{a^v} + \frac{E x^w}{a^w}\,(n+1)}.$$

For let $P x^q + Q x^{q+m} + R x^{q+2m} + S x^{q+3m}$, &c. (which is a general Expression for any Quantity whatever) be assumed $= y$; then will $y x^{r-1} \dot{x}$, or $y x^{f-1} \dot{x}$ be $= P x^{q+r-1} \dot{x} + Q x^{q+r+m-1} \dot{x} + R x^{q+r+2m-1} \dot{x}$, &c. and therefore its Fluent $= \frac{P x^{q+r}}{q+r} + \frac{Q x^{q+r+m}}{q+r+m} + \frac{R x^{q+r+2m}}{q+r+2m}$, &c. which being multiply'd by $x^{g-1} \dot{x}$, and the Fluent taken, we have $\frac{P x^{q+s}}{\overline{q+r}.\overline{q+s}} + \frac{Q x^{q+s+m}}{\overline{q+r+m}.\overline{q+s+m}} + \frac{R x^{q+s+2m}}{\overline{q+r+2m}.\overline{q+s+2m}}$, &c. From which Method of Operation it is evident, that the Fluent proposed, or that whose Place in the Progression is denoted by $n+1$, will be be truly defined by $\frac{Px^{p+q}}{q+r.q+s.q+t(n)} + \frac{Qx^{p+q+m}}{q+r+m.q+s+m.q+t+m(n)}$ $+ \frac{Rx^{p+q+2m}}{q+r+2m.q+s+2m.q+t+2m(n)}$, &c. which Value, by Prop. V. of the Summation of Series, is equal to

$$A \times \overline{\frac{Px^{p+q}}{q+r} + \frac{Qx^{p+q+m}}{q+r+m} + \frac{Rx^{p+q+2m}}{q+r+2m}}, \text{ \&c.}$$

$$B \times \overline{\frac{Px^{p+q}}{q+s} + \frac{Qx^{p+q+m}}{q+s+m} + \frac{Rx^{p+q+2m}}{q+s+2m}}, \text{ \&c.}$$

$$C \times \overline{\frac{Px^{p+q}}{q+t} + \frac{Qx^{p+q+m}}{q+t+m} + \frac{Rx^{p+q+2m}}{q+t+2m}}, \text{ \&c.}$$

&c. &c.

But $A \times \overline{\frac{Px^{p+q}}{q+r} + \frac{Qx^{p+q+m}}{q+r+m} + \frac{Rx^{p+q+2m}}{q+r+2m}}$, &c. in the proposed Circumstance, when x is $= a$, will be equal to the Fluent of $A a^{p-r} \dot{x} \times \overline{Px^{q+r-1} + Qx^{q+r+m-1} + Rx^{q+r+2m-1}}$, &c. or of $A a^{p-r} \times y x^{r-1} \dot{x}$ (supposing x and y variable.) And in the same manner it will appear that the second Series $B \times \overline{\frac{Px^{p+q}}{q+s} + \frac{Qx^{p+q+m}}{q+s+m} + \frac{Rx^{p+q+2m}}{q+s+2m}}$, &c. is equal to the Fluent of $Ba^{p-s} \times y x^{s-1} \dot{x}$, &c. &c. Therefore the Sum of all these Series will be equal to the Fluent of $Aa^{p-r} \times y x^{r-1} \dot{x}$ $+ B a^{t-1} \times y x^{s-1} \dot{x} + Ca^{t-1} \times y x^{t-1} \dot{x}$, &c. or of $\frac{a^p y \dot{x}}{x} \times$ $\overline{\frac{Ax^r}{a^r} + \frac{Bx^s}{a^s} + \frac{Cx^t}{a^t}}^{(n+1)}$.

Q. E. D.

Note. That all the Fluents abovementioned are supposed to be contemporaneous, or generated in the same Time.

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COROLLARY I.

If the Fluent propofed be reprefented by K, then, fince the Fluent of $\frac{a^p y \dot{x}}{x} \times \overline{\frac{Ax^r}{a^r} + \frac{Bx^s}{a^s}, \&c.}$ is $= a^p y \times \overline{\frac{Ax^r}{ra^r} + \frac{Bx^s}{sa^s} + \frac{Cx^t}{ta^t}, \&c.}$ — the Fluent of $a^p \dot{y} \times \overline{\frac{Ax^r}{ra^r} + \frac{Bx^s}{sa^s} + \frac{Cx^t}{ta^t}, \&c.}$ this laft Value, on the Right-hand-fide of the Equation, will alfo be equal to K; and will be found more commodious than the former, when the Relation of x and y cannot be exhibited, but by the Meafures of Angles and Ratios, &c.

COROLLARY II.

If all the Indices f, g, h, &c. be equal to each other, then will $s = 2r$, $t = 3r$, $v = 4r$, $p = \overline{n+1} \times r$, A $\left(\frac{1}{r.2r.3r(n)}\right) = \frac{1}{1.2.3.4\ldots nr^n}$, $B = -nA$, $C = \frac{n}{1} \times \frac{n-1}{2} A$, $D = -\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} A$, &c. and confequently K in this Cafe, equal to the Fluent of $\frac{a^{rn+r} y \dot{x}}{1.2.3.4\ldots n \times r^n x} \times \overline{\frac{x^r}{a^r} - \frac{nx^{2r}}{a^{2r}} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{x^{3r}}{a^{3r}}}$ $(n+1)$ or to that of $\frac{\overline{a^r - x^r}|^n \times y x^{r-1} \dot{x}}{1.2.3.4\ldots n \times r^n}$; which Equation, if r be taken $= 1$, will be the fame with that delivered by Sir *Ifaac Newton*, in the eleventh Propofition of his Book of Quadratures.

EXAMPLE.

Let $y = \overline{a^2 - x^2}|^{\frac{1}{2}}$, and f, g, h, &c. each $= 2$, and let it be required to find the third Fluent of the Progreffion, generated while x, from nothing, is increaſed to a. Here, according to Cor. II. we have $\frac{\overline{a^r - x^r}|^n \times y x^{r-1} \dot{x}}{1 . 2 . 3 \ldots n \times r^n} = \frac{\overline{a^2 - x^2}|^{\frac{5}{2}} \times x \dot{x}}{8}$; whoſe Fluent, univerſally expreſſed, is $-\frac{\overline{a^2 - x^2}|^{\frac{7}{2}}}{56} + \frac{a^7}{56}$; in which taking $x = a$, according to the foregoing Preſcript, we have $\frac{a^7}{56}$ for the true Value ſought.

Here follow ſome uſeful Theorems, extracted out of the foregoing Propoſitions.

THEOREM I.

$$\dot{S} = \overline{a + cz^n}|^{r - \frac{1}{2}} \times d z^{vn + \frac{1}{2}v - 1} \dot{z}.$$

Let $b = r + v$, $F = \frac{1}{2v+2} \times \frac{3}{2v+4} \times \frac{5}{2v+6} (r)$, $G = \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} (v)$, $x = a + c z^n$, and Q equal to $\frac{2d}{n c^{\frac{1}{2}}} \times$ Hyp. Log. $\sqrt{1 + \frac{cz^n}{a}} + \sqrt{\frac{cz^n}{a}}$, or to $\frac{2d}{n\sqrt{-c}} \times$ Arch, whoſe Radius is 1, and Sine $\sqrt{\frac{-cz^n}{a}}$, according as the Value of c is poſitive or negative.

Then $S = \frac{d z^{vn} x^r}{bn} \times \sqrt{\frac{z^n}{x}} \times \overline{1 + \frac{2r-1 . a}{2b-2 . x} + \frac{2r-1 . 2r-3 . a^2}{2b-2 . 2b-4 . x^2} + \frac{2r-1 . 2r-3 . 2r-5 . a^3}{2b-2 . 2b-4 . 2b-6 . x^3} (r)} + \frac{d F a^r z^{vn}}{vnc} \times \sqrt{\frac{x}{z^n}} \times \overline{1 - \frac{2v-1 . a}{2v-2 . cz^n} + \frac{2v-1 . 2v-3 . a^2}{2v-2 . 2v-4 . c^2 z^{2n}} (v)} \pm \frac{FGQa^{v+r}}{c^v}$.

Note. That in this and the following Theorems, r and v may denote any whole poſitive Numbers; and that the laſt Term, when there are two Signs prefix'd, is to be taken $+$ or $-$, according as v is even or odd.

THEOREM II.

$$\dot{S}=\overline{a+cz^n}|^{-r-\frac{1}{2}}\times dz^{\frac{1}{2}n+vn-1}\,\dot{z}.$$

Let $b=r-v$, $F=-\frac{2v}{1}\times-\frac{2v-2}{3}\times-\frac{2v-4}{5}\ (r)$ and the rest as in the preceding Theorem.

Then $S=\frac{2dz^{vn}}{\overline{2r-1}\times nax^r}\times\sqrt{xz^n}\times\overline{1+\frac{\overline{2b-2}.x}{\overline{2r-3}.a}+\frac{\overline{2b-2}.\overline{2b-4}.x^2}{\overline{2r-3}.\overline{2r-5}.a^2}\ (r)}+\frac{dFz^{vn-n}}{nvca^r}\times\sqrt{xz^n}\times\overline{1-\frac{\overline{2v-1}.a}{\overline{2v-2}.cz^n}+\frac{\overline{2v-1}.\overline{2v-3}.a^2}{\overline{2v-2}.\overline{2v-4}.c^2z^{2n}}\ (v)}\pm\frac{FGQa^{v-r}}{c^v}$: But when r is greater than v, F will be $=0$, and therefore, in that Case, S is barely $=\frac{2dz^{vn}}{\overline{2r-1}\times nax^r}\times\sqrt{xz^n}\times\overline{1+\frac{\overline{2b-2}.x}{\overline{2r-3}.a}+\frac{\overline{2b-2}.\overline{2b-4}.x^2}{\overline{2r-3}.\overline{2r-5}.a^2}}$, &c. continued till it terminates.

THEOREM III.

$$\dot{S}=\overline{a+cz^n}|^{r-\frac{1}{2}}\times dz^{\frac{1}{2}n-vn-1}\,\dot{z}.$$

Let $b=v-r$, $F=\frac{1}{2}\times\frac{3}{4}\times\frac{5}{6}\ (r)$, $G=\frac{2r}{1}\times\frac{2r-2}{3}\times\frac{2r-4}{5}\ (v)$, and x and Q as above.

Then $S=-\frac{2dx^r}{\overline{2v-1}\times naz^{vn}}\times\sqrt{xz^n}\times\overline{1-\frac{\overline{2b-2}.cz^n}{\overline{2v-3}.a}+\frac{\overline{2b-2}.\overline{2b-4}.c^2z^{2n}}{\overline{2v-3}.\overline{2v-5}.a^2}\ (v)}+\frac{dGc^vx^{r-1}}{rna^v}\times\sqrt{xz^n}\times\overline{1+\frac{\overline{2r-1}.a}{\overline{2r-2}.x}+\frac{\overline{2r-1}.\overline{2r-3}.a^2}{\overline{2r-2}.\overline{2r-4}.x^2}\ (r)}+FGQa^{r-v}\times c^v$: which, when v is greater than r, will be barely $=-\frac{2dx^r}{\overline{2v-1}\times naz^{vn}}\times\sqrt{xz^n}\times$

$\overline{1 - \frac{2b-2.cz^n}{2v-3.a}}$, &c. continued till it terminates, because then $G = 0$.

THEOREM. IV.

$$\dot{S} = \overline{a+cz^n}|^m \times dz^{-mn-vn-1}\dot{z}.$$

Put $v+m=e$, and $a+cz^n=z$.

Then $S = -\frac{dx^{m+1}}{neaz^{en}} \times \overline{1 - \frac{v-1.cz^n}{e-1.a} + \frac{v-1.v-2.c^2z^{2n}}{e-1.e-2.a^2}}$, &c. continued till it terminates.

THEOREM V.

$$\dot{S} = \overline{a+cz^n}|^m \times dz^{vn-1}\dot{z}.$$

Let $v+m=e$, and $a+cz^n=x$.

Then $S = \frac{dx^{m+1} \times z^{vn-n}}{enc} \times \overline{1 - \frac{v-1.a}{e-1.cz^n} + \frac{v-1.v-2.a^2}{e-1.e-2.c^2z^{2n}}}$, &c. continued till it terminates.

Note. In the two last Theorems m may represent any Number whatever, whole or broken, positive or negative.

THEOREM VI.

$$\dot{S} = \overline{a+cz^n}|^{r-\frac{1}{2}} \times dz^{-vn-1}\dot{z}.$$

Put $b = r-v-\frac{1}{2}$, $F = -\frac{1}{2v-1} \times -\frac{3}{2v-3} \times -\frac{5}{2v-5}$ (r), $G = \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6}$ (v), $x = a+cz^n$, and Q equal to $\frac{d}{n\sqrt{a}} \times$ Hyp. Log. $\frac{\overline{a+cz^n}|^{\frac{1}{2}} - a^{\frac{1}{2}}}{\overline{a+cz^n}|^{\frac{1}{2}} + a^{\frac{1}{2}}}$, or to $\frac{2d}{n\sqrt{-a}} \times$ Arch, whose Radius 1, and

and Secant $\sqrt{\frac{cz^n}{-a}}$, according as the Value of a is positive or negative.

Then $S = \frac{dx^{r-\frac{1}{2}}}{bnz^{vn}} \times \overline{1 + \frac{2r-1.a}{2b-2.x} + \frac{2r-1.2r-3.a^2}{2b-2.2b-4.x^2}}\ (r)$ $- \frac{dFa^{r-1}x^{\frac{1}{2}}}{vnz^{vn}} \times \overline{1 - \frac{2v-1.cz^n}{2v-2.a} + \frac{2v-1.2v-3.c^2z^{2n}}{2v-2.2v-4.a^2}}\ (v)$ $\pm FGQc^v a^{r-v}$.

THEOREM VII.

$$\dot{S} = \overline{a+cz^n}|^{-r-\frac{1}{2}} \times dz^{-vn-1}\dot{z}.$$

Let $b = r+v+\frac{1}{2}$, $F = \frac{2v+1}{1} \times \frac{2v+3}{3} \times \frac{2v+5}{5}\ (r)$, $G = \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6}\ (v)$, and x and Q is in the precedent.

Then $S = \frac{2dx^{\frac{1}{2}}}{2r-1 \times nax^r z^{vn}} \times \overline{1 + \frac{2b-2.x}{2r-3.a} + \frac{2b-2.2b-4.x^2}{2r-3.2r-5.a^2}}$ $(r) - \frac{dFx^{\frac{1}{2}}}{vna^{r+1} \times z^{vn}} \times \overline{1 - \frac{2v-1.cz^n}{2v-2.a} + \frac{2v-1.2v-3.c^2z^{2n}}{2v-2.2v-4.a^2}}$ $(v) \pm \frac{FGQc^v}{a^{v+r}}$.

THEOREM VIII.

$$\dot{S} = \overline{a+cz^n}|^{-r} \times dz^{pn-1}\dot{z}.$$

Let $b = r-p$, $x = a+cz^n$, and $Q =$ the Fluent of $\frac{dz^{pn-1}\dot{z}}{a+cz^n}$ (which may be always had by the Measures of Angles and Ratios.)

Then

Then $S = \frac{dz^{pn}}{\overline{r-1} \times nax^{r-1}} \times \overline{1 + \frac{b-1.x}{r-2.a} + \frac{b-1.b-2.x^2}{r-2.r-3.a^2}}$ $+ \frac{b-1.b-2.b-3.x^2}{r-2.r-3.r-4.a^3} (r-1) + \frac{Q}{a^{r-1}} \times \overline{\frac{1-p}{1} \times \frac{2-p}{2} \times \frac{3-p}{3} (r-1)}$ Which, therefore, when p is a whole poſitive Number, leſs than r, will be barely $= \frac{dz^{pn}}{\overline{r-1} \times nax^{r-1}} \times \overline{1 + \frac{b-1.x}{r-2.a}}$, &c. continued till it terminates.

THEOREM IX.

$\dot{S} = \frac{\overline{a-bz^n}|^{r-\frac{1}{2}} \times dz^{vn+\frac{1}{2}n-1}\dot{z}}{\overline{k \mp lz^n}|^t}$; t being any Integer not leſs than $r+v+1$.

Let $p = v + \frac{1}{2}$, $s = r+v+1$, $w = t-s$, $H = \frac{1.3.5(v) \times 1.3.5(r)}{2.4.6.8(r+v)}$, and $Q =$ Periphery of the Circle, whoſe Diameter is Unity.

Then the required Fluent, when $a-bz^n = 0$, or the Value of S generated while bz^n, from Nothing, becomes $= a$, will be accurately $= \frac{da^{r+v} \times b^{\frac{1}{2}} HQ}{nk^{t-p} \times \overline{bk \mp al}|^p} \times \overline{1 \pm \frac{w.p.al}{1.s.bk \mp al}}$ $\overline{+ \frac{w.w-1.p.p+1.a^2l^2}{1.2.s.s+1.\overline{bk \mp al}|^2} \pm \frac{w.w-1.w-2.p.p+1.p+2.a^3l^3}{1.2.3.s.s+1.s+2.\overline{bk \mp al}|^3}}$, &c. which Series will always terminate in $w+1$ Terms.

THEOREM X.

$\dot{S} = \frac{\overline{a-bz^n}|^{r-\frac{1}{2}} \times dz^{vn+\frac{1}{2}n-1}\dot{z}}{\overline{k \mp lz^n}|^t}$; t being any whole poſitive Number, not exceeding $r+v$.

Let p, s, H and Q be as in the precedent, and let $m = r - \frac{1}{2}$, $w = s-t$ and $P = \pm \frac{t.t+1.t+2(w) \times w.w+1.w+2(t-1)}{m.m-1.m-2.m-3.m-4(s-1)}$; in which

 laſt

laſt Value, let the prefixed Sign $+$ or $-$ obtain, according as t is an odd or an even Number. Then the Value of S, in the abovementioned Circumſtance, will be $= \dfrac{\overline{bk \mp al}|^{m} \times db^{\frac{1}{2}} HPQ}{nk^{t-p} \times \overline{\mp l}|^{s-1}} \times$

$$\overline{1 \pm \frac{m . t-1 . al}{1 . s-2 . bk \mp al} + \frac{m . m-1 . t-1 . t-2 . a^2 l^2}{1 . 2 . s-2 . s-3 . \overline{bk \mp al}|^2}}$$

$$\overline{\pm \frac{m . m-1 . m-2 . t-1 . t-2 . t-3 . a^3 l^3}{1 . 2 . 3 . s-2 . s-3 . s-4 . \overline{bk \mp al}|^3}}, \text{ \&c.} - \frac{db^r \times k^{w-1} \times HPQ}{n \times \overline{\mp l}|^{s-1}}$$

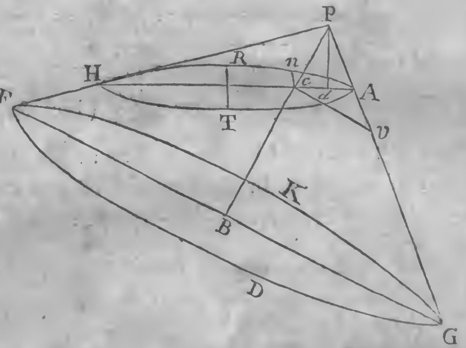
$$\overline{\times 1 \mp \frac{m . w-1 . al}{1 . s-2 . bk} + \frac{m . m-1 . w-1 . w-2 . a^2 l^2}{1 . 2 . s-2 . s-3 . b^2 k^2}}, \text{ \&c.}$$

Note. In the two laſt Theorems, the Value of $bk \mp al$ muſt be poſitive.

An

An INVESTIGATION *of the Curve described by the Shadow of an Object on the Plane of the Horizon, according to any given Declination of the Sun, and Elevation of the Pole.*

Let *c*ARHT be the Plane of the Horizon, P*d* the perpendicular Height of the proposed Object, and A*n*RHTA the required Curve, described by the Shadow of its Summit; supposing P*c*B parallel to the Earth's Axis, and the Angles BPF, BPG, each equal to the Complement of the given Declina-



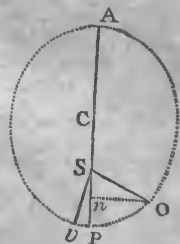
tion: Then, since all the Rays intercepted at P, during one whole apparent, diurnal Revolution of the Sun (supposing the Declination to continue invariable) make Angles with the Earth's Axis or PB, equal to (BPF) the said Complement of Declination, it is plain, that if those Rays had proceeded on without Interruption, they would have formed a conical Surface PFKGD; and therefore, as the required Curve is made by the Intersection of this Surface and the Plane of the Horizon ARHTA,

ARHTA, it muſt conſequently be a Conic-Section. Let, therefore, cn be now made parallel to RT the conjugate Axis of the ſaid Section, and cv to F G; putting P$d=h$, the Sine of the given Latitude (AcP) to the Radius 1, $=b$, its Coſine $=c$, the Sine of Declination $=d$, and the Sine of its Complement (cPH or cPA) $=p$; then cHF being equal to the Sum, and PAc to the Difference of the Angles HcP and cPH, we ſhall have $bd+cp=$ the Sine of cHF, and $bd-cp$ = Sine of PAc, by the Elements of Trigonometry. Therefore ſeeing the Sine of PcA, is to Radius, as Pd to Pc $(=\frac{h}{b})$ we ſhall have (by plain Trigonometry) as $bd+cp : p :: \frac{h}{b} : Hc = \frac{hp}{b \times \overline{bd+cp}}$, and as $bd-cp : p :: \frac{h}{b} : Ac = \frac{hp}{b \times \overline{bd-cp}}$; whence $AH = \frac{hp}{b} \times \overline{\frac{1}{bd+cp} + \frac{1}{bd-cp}} = \frac{2hdp}{b^2d^2-c^2p^2}$ $= \frac{2hdp}{b^2d^2-c^2 \times \overline{1-dd}} = \frac{2hdp}{\overline{bb+cc} \times \overline{dd-cc}} = \frac{2hpd}{dd-cc}$ (becauſe $p^2 = 1 - d^2$, and $b^2+c^2=1$). Moreover as $d : p :: \frac{h}{b}$ (Pc) : $\frac{ph}{db}$ $= vc$ or cn; wherefore, by the Property of the Curve, it will be $\frac{p^2h^2}{b^2 \times \overline{d^2-c^2}}$ $(Hc \times Ac) : \frac{p^2h^2}{d^2b^2}$ $(cn^2) :: \frac{4h^2p^2d^2}{\overline{d^2-c^2}|^2}$ (AH^2) $: \frac{4h^2p^2}{d^2-c^2} = RT^2$. Hence it appears, that if d be greater than c, or the Declination greater than the Complement of Latitude, the Curve deſcribed will be an Ellipſis, whoſe tranſverſe and conjugate Axes are $\frac{2pdh}{d^2-c^2}$ and $\frac{2ph}{\overline{d^2-c^2}|^{\frac{1}{2}}}$ reſpectively, and its Parameter equal to $\frac{2ph}{d}$: Therefore when the Declination is equal to the Complement of Latitude, then $\frac{2hpd}{d^2-c^2}$ becoming infinite, the Ellipſis will degenerate to a Parabola, whoſe Parameter is $\frac{2bh}{c}$; but if the Declination be leſs than the Com-

plement

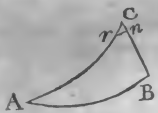
plement of Latitude, then $\frac{2hpd}{d^2-c^2}$ being negative, the Curve will become an Hyperbola, whofe Tranfverfe and Conjugate Axes are $\frac{2hpd}{c^2-d^2}$ and $\frac{2ph}{\sqrt{c^2-d^2}}$, and its Parameter $=\frac{2ph}{d}$; except when $d=0$, or the Sun is in the Equinox, in which Cafe, the Parameter $\frac{2ph}{d}$ becoming infinite, the Hyperbola degenerates to a Right-line.

A Determination of the Time of the Year when Days lengthen the fafteft, according to apparent Time, and to any affigned Excentricity of the Earth's Orbit.

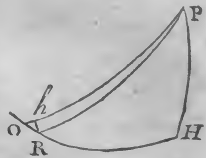


Let AvPOA be the Orbit of the Earth, AP its principal Axis, C the Centre, and S the Sun, in one of the Foci; and, v being the Place of the Earth at the Winter Solftice, let O be its Place at the Time required: Draw Sv and SO, and alfo On, perpendicular to AP; puting PC$=a$, SC$=e$, SO$=z$, the Sine of vSP, to the Radius $1, = m$, its Cofine $=n$, and the Cofine of vSO, or the Sine of the Sun's Diftance from the Equinoctial Point at the required Time $=x$. Therefore, the Sine of vSO being $=\sqrt{1-xx}$, and the Angle PSO equal to the Difference of the two Angles vSO and vSP, the Cofine of PSO will be $=nx+m\sqrt{1-xx}$, by the Elements of Trigonometry; wherefore it will be, as 1 (Radius) : $nx+m\sqrt{1-xx}$: : z (SO) : Sn $=nxz+mz\sqrt{1-xx}$. But, by the Property of the Curve, $e : a : : a-z :$ C$n=\frac{aa-az}{e}$; whence (Sn)$=\frac{a^2-az}{e}-e=nxz$

$+ m z \sqrt{1-xx}$, consequently $z = \frac{a^2-e^2}{a+nex+me\sqrt{1-xx}}$; and therefore $\frac{\overline{a^2-e^2}|^2}{z^2} = \overline{a+nex+me\sqrt{1-xx}}|^2$; which last Expression, since the Alteration of Longitude, or of the Angle PSO, in a given Particle of Time, is inversely as the Square of Radius SO, will, it is manifest, be also as the Alteration of Longitude, in a given Particle of Time, or in one whole Day very nearly.



This being now obtained, let AC denote the Ecliptic, AB the Equinoctial, CB a Meridian, and C and r those two Points of the Ecliptic, wherein the Sun is at rising on the two required Days, when the Difference of the Hour and Minute of his rising is the greatest possible, and let Cn be the Difference of Declination in those Points; putting Cr=y, and the Sine of CAB, the Sun's greatest Declination $= d$: Then as 1 (Radius : d : : x (Sine of AC) : $dx =$ Sine BC; therefore its Cosine $= \sqrt{1-d^2x^2}$; Again, as $\sqrt{1-xx}$ (Cosine of AC) : 1 (Radius) : : $\frac{\sqrt{1-dd}}{d}$ (Co-Tang. of A) : $\frac{\sqrt{1-dd}}{d\sqrt{1-xx}}$ = Tangent of ACB, or rCn, therefore its Secant $= \frac{\sqrt{1-d^2x^2}}{d\sqrt{1-xx}}$: Wherefore, because the Triangle Crn, by Reason of its smallness, may be consider'd as rectilineal, it will be as $\frac{\sqrt{1-d^2x^2}}{d\sqrt{1-xx}}$: 1 : : y (Cr) : C$n = \frac{dy\sqrt{1-xx}}{\sqrt{1-d^2x^2}}$ equal to the Alteration of Declination, from Sun-rising to Sun-rising, on the said two Days very nearly. Let this Alteration, therefore, be now represented by Ob, supposing HRO to be



be the Horizon, PH the Meridian, PR and PO Complements of Declination, at the Times abovementioned; and let the Cosine of the Latitude (PH) be denoted by b: Then it will be as $\sqrt{1-d^2x^2}$ (Sine of PO) : 1 (Radius) : : $\sqrt{1-bb}$ (Sine of the Latitude) : $\frac{\sqrt{1-bb}}{\sqrt{1-d^2x^2}}$ = Sine of POH; therefore its Cosine, or the Sine of O R b is $=\frac{\sqrt{b^2-d^2x^2}}{\sqrt{1-d^2x^2}}$, therefore it will be as $\frac{\sqrt{b^2-d^2x^2}}{\sqrt{1-d^2x^2}} : \frac{\sqrt{1-bb}}{\sqrt{1-d^2x^2}}$ $:: \frac{dy\sqrt{1-xx}}{\sqrt{1-d^2x^2}}$ (O b) : $\frac{dy\sqrt{1-xx}\times\sqrt{1-bb}}{\sqrt{1-d^2x^2}\times\sqrt{b^2-d^2x^2}}$ = R b; but as $\sqrt{1-d^2x^2} : 1 :: \frac{dy\sqrt{1-xx}\times\sqrt{1-bb}}{\sqrt{1-d^2x^2}\times\sqrt{b^2-d^2x^2}} : \frac{dy\sqrt{1-xx}\times\sqrt{1-bb}}{\overline{1-d^2x^2}\times\sqrt{b^2-d^2x^2}}$ = the Arch of the Equator, measuring the Angle RPO, or the Difference of the Semi-diurnal Arches of the Sun on the two Days above specified. This Difference therefore, since y, by the former Part of the Problem, is found to be as $\overline{a+nex+me\sqrt{1-xx}}|^2$, will be as $\frac{\sqrt{1-xx}\times\overline{a+nex+me\sqrt{1-xx}}|^2}{\overline{1-d^2x^2}\times\sqrt{b^2-d^2x^2}}$;

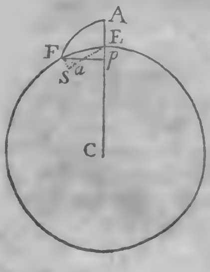
where, if the Fluxion be taken, and made equal to Nothing, the required Value of x may, it is manifest, be determined, let (e) the Excentricity of the Orbit and the Latitude of the Place be what they will. But the Excentricity, as given from Observation being small, the greatest lengthening of Days, if the proposed Place be not very near the Frigid Zone, must be near the Time of the vernal Equinox, and the Value of x but small; therefore, if the foresaid Expression be converted into a Series, and all the Terms wherein more than two Dimensions of e and x are concerned, be neglected as inconsiderable, it will be be rėduced to $aa + 2ame - \frac{a^2x^2}{2} \times \overline{1 - 2d^2 - \frac{d^2}{b^2}} + 2aenx$, where, by taking the Fluxion, *&c.* x comes out $= \frac{2ne}{a \times \overline{1 - 2d^2 - \frac{d^2}{b^2}}}$.

Note. From the Equation foregoing, the greateſt lengthening of Days at *London*, will be found to be about 7 Days before the vernal Equinox.

A DETERMINATION *how far a heavy Body, freely deſcending from Reſt, falls from perpendicular, by Means of the Earth's Rotation.*

PROPOSITION I.

SUppoſing the Earth to be perfectly Spherical, and that a heavy Body deſcends from a given Point above its Surface in any given Latitude; to find how far it will impinge from a perpendicular, let fall from that Point to the Surface, thro' the Cauſe above ſpecified.

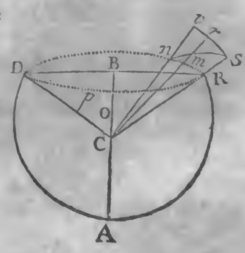


Let the Axis of the Earth be conſidered as abſolutely at Reſt, and let EA be the perpendicular Height from whence the Body is let fall, and by the Force of Gravity and the Motion acquired by the Earth's Rotation, begins to deſcribe the elliptical Area ACFA, in the Plane of the great Circle EFC, about C the Centre of Force, while the Point E is carry'd by the Rotation of the Earth, in in its Parallel of Latitude EaS from E to-

towards *a*; let F be the Place where the Ball falls, and FS the Diſtance of that Place from the ſaid Parallel; and let the Point *a* be the Poſition of E, at the Time when the Body impinges on the Surface at F. Therefore, ſince the Velocity acquired by the Rotation of the Earth, and the Attraction at the Point A are both given, the Ellipſis AF will be given both in Magnitude and Species (by Page 23 of my Eſſays) whence EF and E*a* will be given, and conſequently the required Diſtance F*a*. But when the Height AE is ſuppoſed ſmall in reſpect to the Earth's Radius EC, as in the Caſe propoſed, the Solution may be, otherwiſe, more eaſily inveſtigated: For then S*a* being ſmall in reſpect of FS, the latter of theſe may, without ſenſible Error, be taken for F*a*; but FS is to the verſed Sine E*p*, of the Arch FE, as the Tangent of the given Latitude to Radius nearly. But FE is given from the Time of Deſcent, whence FS will be given alſo. Q. E. I.

PROPOSITION II.

To determine the ſame as in the laſt Propoſition; ſuppoſing all Bodies gravitate perpendicularly to the Surface of the Earth.



Let ACDR, &c. repreſent the Earth (whether under a ſpherical or an oval Figure) AB, &c. its Axis conſidered as abſolutely at reſt, and R*p*D*n*R the given Parallel of Latitude; let CRS be perpendicular to the Surface at R, RS the given Height, or Diſtance deſcended, and SO the Direction in which the Body would fall was it not for the Earth's Rotation. Then, as the Attraction, exerted at S, acts in the Direction SO, the Body, upon its

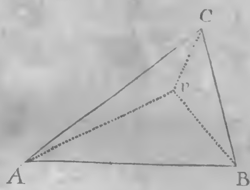
 leaving

leaving S, will begin, thro' that Attraction and the Motion received from the Earth's Rotation, to move in a Curve Line S*n*, that may, without ſenſible Error, be conſidered as Part of an Ellipſis, formed by the Interſection of the Conical Surface CR*n*D*p* produced, and a Plane paſſing through S and O; and will continue to deſcribe the ſame Areas, in equal Times, about the Point O or C, as it did before its leaving S (ſetting aſide what ariſes from the Alteration of the Centre of Attraction, *&c.* which is too minute here, to require a particular Conſideration.) Hence if the Point *m* be ſo taken in the given Parallel of Latitude, that the Area of the Sector C S *rm* C, may be equal to the Area CS*n*C, then will the Point *m*, it is evident, be the Poſition of the Place R, at the time when the Body impinges on the Surface at *n*. Now the Height R S being ſmall, when compared with the Diameter of the Earth, the Curve S*n* may be taken as a Semi-Parabola, whoſe Vertex is S, and R*n* as a Right-line; whence the Area *n*SR*n* is found $= SR \times \frac{2}{3} Rn$, and therefore $SnvS = SR \times \frac{1}{3} Rn$; which is alſo the Area of the Sector *r*C*nv*, becauſe CS*r*C being equal to CS*n*C, let each of theſe be taken from CS*v*C, and there remains *r* C *n v* equal to S *n v* S: Therefore will *n m* be $= \frac{Sn \times 2RS}{3RC}$ nearly, and ſo much will the Body fall eaſterly of the Perpendicular.

N. B. The two foregoing Propoſitions might be of Service in proving the Motion of the Earth, by the Deſcent of heavy Bodies, provided the Experiment could be made with ſufficient Accuracy.

A DEMONSTRATION *of the Law of Motion that a Body deflected by two Forces tending to two fixed Points, will describe equal Solids in equal Times about the Right-line joining those Points.*

Let A and B be the two propofed Points, and C any Place of the Body, and let the Direction of its Motion, at that Place, make any given Angle with the Plane A B C, or with any Right-line drawn in that Plane; and fuppofe the Body, upon its leaving C, to be impelled by any Forces whatever, tending either to the Points A and B, or to any Parts of the Line AB, and let C *v* be the Right-line, which afterwards, by its compound Motion, it will proceed to defcribe, and let the motive Force, before the Impulfe at C, be refolved into two others, one in the Direction of a Right-line lying in the Plane ACB, and the other perpendicular thereto. Then, fince the laft of thefe is not at all affected by the Impulfe, acting in the Plane, the perpendicular Diftance of the Body, from the Plane at the end of a given Time, will, it is manifeft, be the fame, let the greatnefs of the Impulfe be what it will, and therefore in different Times, directly as thofe Times. But A C B *v*, the Solid defcribed about the Line AB, being an oblique Pyramid, is known to be as the faid perpendicular Diftance, and therefore muft likewife be as the Time: Hence it appears, that whether the Body be, or be not impelled at the Point C, the Magnitude or Content of the Solid defcribed about A B, will be the fame, and proportional to the Time in which it is defcribed: Therefore, feeing no fingle Impulfe, however great, can affect the equable Defcription of Solids about AB, it is evident, that no



Num-

Number of ſuch Impulſes can, nor any Forces tending continually to the Points A and B. Q. E. D.

N. B. The Propoſition would have been equally true, and the Demonſtration the very ſame, had there been ſuppoſed ever ſo many Forces tending to the fixed Line AB; and if inſtead of the Solids deſcribed about that whole Line, thoſe deſcribed about any given Part of it had been taken.

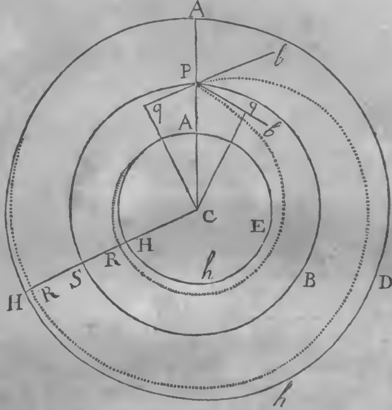
A DETERMINATION *in what Caſes a Body, acted on by a centripetal Force, may continually deſcend in a ſpiral Line towards the Centre, and yet never ſo far as to approach it within a certain Diſtance; and alſo in what Caſes it may continually aſcend, yet never riſe to a certain aſſignable Altitude.*

Mr. *Mac-Laurin*, at the End of his Treatiſe of Fluxions, has found, that if the centripetal Force be as the 5th Power of the Diſtance inverſely, a Body may continually deſcend towards the Centre, and yet never ſo low as to come within a certain Circle, or may recede for ever from the Centre, yet never riſe to a certain Height; which remarkable Circumſtance had not been taken Notice of by any preceding Authors. But the ſame Thing will alſo happen in an Infinity of other Caſes.

For let C be the Centre of Force, and let the Body proceed from P in any given Direction Pq, with a Velocity, which is to the Velocity whereby it might deſcribe the Circle PBS, in the Ratio of p to 1; let R be any Point in the Trajectory; and make Cq perpendicular to Pq; putting $cP=1$, $Cq=s$, CR $=x$, and PBS$=$A. Then, if the centripetal Force be ſuppoſed as any Power (n) of the Diſtance we ſhall have $\dot{A} = \frac{sp\dot{x}}{x\sqrt{p^2+\frac{2}{n+1}\times \overline{x^2-p^2s^2-\frac{2x^{n+3}}{n+1}}}}$; and the Velocity of the Body at R, will be to the Velocity whereby it might deſcribe a Circle

at

at the Diſtance CR, in the Ratio of $\sqrt{\overline{p^2+\frac{2}{n+1}}\times\frac{1}{x^{n+1}}-\frac{2}{n+1}}$ to 1, as is proved in Page 31 of my Eſſays. This being pre-



miſed, let $x\sqrt{\overline{p^2+\frac{2}{n+1}}\times x^2-p^2s^2-\frac{2x^{n+3}}{n+1}}$ be now taken $=0$, and $\sqrt{\overline{p^2+\frac{2}{n+1}}\times\frac{1}{x^{n+1}}-\frac{2}{n+1}}=1$; and then, the Equations being duly ordered, we ſhall have $x=\overline{\frac{2+\overline{n+1}\times p^2}{n+3}}\Big|^{\frac{1}{n+1}}$, and p^2s^2 $=\overline{\frac{2+\overline{n+1}\times p^2}{n+3}}\Big|^{\frac{n+3}{n+1}}$ $(=x^{n+3})$. Wherefore, with this Value of x, as a Radius, conceive the Circle AbH to be deſcribed, and let the Velocity at P be ſuch, that p^2s^2 (when poſſible) may

may be $= \overline{\frac{2+\overline{n+1}\times p^2}{n+3}}\Big|^{\frac{n+3}{n+1}}$; then if AC be greater than CP, and the Body upon its leaving P begins to aſcend, it will continue to aſcend *ad infinitum*, and yet never riſe ſo high as the Circle AD*b*H: For it cannot begin to deſcend before it arrives at its higher A*pſe*, which (if it can properly be ſaid to have any) will be in that Circle, becauſe AC will be the Value of x, when $\sqrt{p^2+\overline{\frac{2}{n+1}}\times x^2-p^2s^2-\frac{2x^{n+3}}{n+1}}$ is equal to Nothing: Nor can it ever riſe ſo high as the Circle A*b*H; for if it ſhould, its Velocity there being juſt ſufficient to retain it in that Circle, it would continue to move therein, and not deſcend again in the ſame manner it aſcended, which is abſurd. By a parity of Reaſoning it will appear, that if AC be leſs than CP, and the Body upon leaving P begins to deſcend, it will continue to deſcend for ever, but never ſo low as to enter within the Circle AER. It therefore now only remains to find in what Caſes the forementioned Equation, $p^2s^2 = \overline{\frac{2+\overline{n+1}\times p^2}{n+3}}\Big|^{\frac{n+3}{n+1}}$ is poſſible, and in what Caſes it is not.

In order to which, let the Fluxion of $\overline{\frac{2+\overline{n+1}\times p^2}{n+3}}\Big|^{\frac{n+3}{n+1}}$ $\times\frac{1}{p^2}$ $(=s^2)$ be taken, conſidering p as variable, and you will have $\frac{p^2-1}{n+3}\times\frac{4\dot{p}}{p^3}\times\overline{\frac{2+\overline{n+1}\times p^2}{n+3}}\Big|^{\frac{2}{n+1}}$; which Fluxion will be poſitive or negative, according as $\frac{p^2-1}{n+3}$ is poſitive or negative; becauſe $\overline{\frac{2+\overline{n+1}\times p^2}{n+3}}\Big|^{\frac{2}{n+1}}$ muſt be poſitive, elſe s^2 cannot be ſo.

But

But when p is $=1$, $\frac{p^2-1}{n+3}$ will be $=0$, and $\frac{\overline{2+\overline{n+1}\times p^2}\big|^{\frac{n+3}{n+1}}}{n+3}$ $\times\frac{1}{p^2}=1$; which laſt is manifeſtly the greateſt or leaſt Value, poſſible, of that Expreſſion; that is, the greateſt when $n+3$ is negative, becauſe then the Fluxion, while p is leſs than 1, will be poſitive, and afterwards negative; but the leaſt when $n+3$ is poſitive, ſince then the Fluxion of $\frac{\overline{2+\overline{n+1}\times p^2}\big|^{\frac{n+3}{n+1}}}{n+3}$ $\times\frac{1}{p^2}$ is firſt negative and then poſitive: In the former of which Caſes only the Equation $\frac{\overline{2+\overline{n+1}\times p^2}\big|^{\frac{n+3}{n+1}}}{n+3}\times\frac{1}{p^2}=s^2$, can be poſſible, ſeeing s, by the Nature of the Problem, muſt be leſs than 1, or Cq than CP. Therefore ſince it appears that the forementioned Circumſtances can only take Place, when the Value of $n+3$ is negative, or the Law of centripetal Force more than the Cube of the Diſtance inverſely, let $-m-3$ be ſubſtituted inſtead of n, in order to reduce the Equation to a Form more commodious for this Caſe; then we ſhall have $\frac{\overline{\overline{2+m}\times p^2-2}\big|^{\frac{-1}{m+2}}}{m}=AC$, and $\frac{\overline{\overline{2+m}\times p^2-2}\big|^{\frac{m}{m+2}}}{m}=p^2s^2$; where it is evident, from what has been ſaid above, that the Root p, let s be what it will, has two poſitive Values, one of them leſs than Unity, the other greater; whereof the former (which gives AC greater than AP) muſt be taken when the Body aſcends, but the latter when it deſcends. Q. E. I.

An

An easy and general Way of Investigating the common Theorems relating to Compound-Interest and Annuities, without being obliged to sum up the Terms of a geometrical Progression.

Let R be the amount of one Pound in one Year, *viz.* Principal and Interest, P any Sum put out at Interest for any Number n of Years, a its Amount, A any Annuity forborn n Years, m its Amount, and v its worth in present Money, for the same Time.

Therefore, since one Pound put out at Interest, in the first Year is increased to R, it will be as 1 to R, so is R, the Sum forborn the second Year, to R^2, the amount of one Pound in two Years; and therefore as 1 to R, so is R^2, the Sum forborn the third Year, to R^3, the amount in three Years: Whence it appears that R^n, or R, raised to the Power, whose Exponent is the Number of Years, will be the amount of one Pound in those Years: But as 1 *l.* to its amount R^n, so is P to (a) its amount in the same Time; whence we have $P \times R^n = a$. Moreover, because the amount of one Pound in n Years is R^n, its Increase in that Time will be $R^n - 1$; but its Interest for one single Year, or the Annuity answering to that Increase, is $R - 1$; therefore as $R - 1$ to $R^n - 1$, so is A to m. Hence we get $\frac{A \times \overline{R^n - 1}}{R - 1} = m$. Furthermore, since it appears that one Pound ready Money, is equivalent to R^n, to be received at the Expiration of n Years, we have as R^n to 1, so is $\frac{A \times \overline{R^n - 1}}{R - 1}$ (the Sum in Arrear) to v, its worth in Ready-Money; whence $\frac{A \times \overline{1 - \frac{1}{R^n}}}{R - 1} = v$. From which three Theorems, or Equations, the various Questions relating to Compound-Interest, Annuities in Arrear, and purchasing of Annuities, are, respectively, resolved.

FINIS.