Numerical Quadrature and Hyperinterpolation over Spherical Triangles/Polygons by the dCATCH Package.

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The main purpose of the talk is to show a novel strategy for determining by Caratheodory-Tchakaloff compression implemented in the d-CATCH package

- the computation of nodes and weights of a low-cardinality positive quadrature formula, nearly exact for polynomials of a given degree,
- hyperinterpolation of mild degree,

over spherical polygons.

The Matlab software used in this talk is available at the authors' homepage.

A great circle is the intersection of the unit-sphere $\mathbb{S}^2$ with a plane passing through the origin.

Let $P_1, \ldots, P_L$ be distinct points of $\mathbb{S}^2$ and set $P_0 = P_L$.

A spherical polygon is the region $\mathcal{P} \subset \mathbb{S}^2$ whose boundary $\delta\mathcal{P}$

- is determined by the geodesic arcs $\{\gamma_k\}_{k=0,\ldots,L}$, where each γ_k is the portion of the great circles joining P_k with P_{k+1}.
- is oriented counterclockwise.

In this talk we suppose that $\mathcal{P}$

- is contained in a cap Ω whose *polar angle* is strictly inferior than π;
- $\mathcal{P}$ is simple, i.e. it has no self intersections.

Under these assumptions,

- if C is the center of such a cap Ω containing $\mathcal{P}$ then determine the stereographic projection $\mathcal{P}'$ of $\mathcal{P}$ on the plane π_C tangent in C to the unit-sphere;
- compute a triangulation over the planar polygon $\mathcal{P}'$ (e.g. by Matlab built-in environment `polyshape`), i.e. $\mathcal{P}' = \cup_{i=1}^{m} T_i'$ where $T_i' \subset \pi_C$ are planar triangles whose interior do not overlap, i.e. if $j \neq k$ then $\mathrm{int}(T_j') \cap \mathrm{int}(T_k') = \emptyset$;
- map back to the sphere, by means of the inverse of the stereographic projection, each planar triangle T_k' into a spherical triangle T_k.

The spherical triangles $\{T_k\}_{k=1,\ldots,M}$ determine a spherical triangulation of $\mathcal{P}$.

The purpose of this section is to determine a cubature rule over the spherical polygon $\mathcal{P}$, that has internal nodes $\{Q_k\}_{k=1,\ldots,N}$ and positive weights $\{w_k\}_{k=1,\ldots,N}$ so that if f is a continuous function then

$$\int_{\mathcal{P}} f(x,y,z)d\sigma \approx \sum_{k=1}^{N} w_k f(x_k, y_k, z_k).$$

Since $\{T_k\}_{k=1,\ldots,M}$ is a spherical triangulation of $\mathcal{P}$, if we determine such a cubature rule over each spherical triangle T_k then by the addivity of integration we have such a rule on $\mathcal{P}$ either.

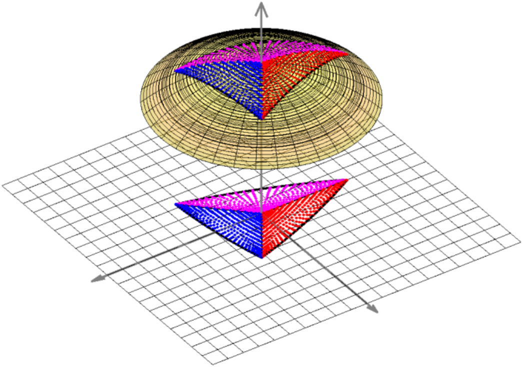

Figure: What we will get: quadrature nodes (and weights) on a spherical triangle lifted from the projected elliptical triangle.

With no loss of generality, up to a suitable rotation, we concentrate on spherical triangles $\mathcal{T} = ABC$ with centroid $(A+B+C)/\|A+B+C\|_2$ at the north pole, that are completely contained in the northern-hemisphere, and do not touch the equator.

Then, if $f \in C(\mathcal{T})$, $g(x,y) = \sqrt{1-x^2-y^2}$,

$$I_{\mathcal{T}} := \int_{\mathcal{T}} f(x,y,z)d\sigma = \int_{\mathcal{T}^{\perp}} f(x,y,g(x,y))\frac{1}{g(x,y)}dxdy,$$

where $\mathcal{T}^{\perp}$ is the projection of $\mathcal{T}$ onto the xy-plane, that is the curvilinear triangle whose vertices, say $\hat{A}$, $\hat{B}$, $\hat{C}$, are the xy-coordinates of A, B, C, respectively.

The sides of $\mathcal{T}^{\perp}$ are arcs of ellipses centered at the origin, being the projections of great circle arcs. Then we can split the planar integral into the sum of the integrals on three elliptical sectors S_i with $i = 1, 2, 3$, obtained by joining the origin with the vertices $\hat{A}$, $\hat{B}$, $\hat{C}$, namely

$$I_{\mathcal{T}} = \int_{\mathcal{T}^{\perp}} f(x, y, g(x, y)) \frac{1}{g(x, y)} dxdy = \sum_{i=1}^{3} \int_{S_i} f(x, y, g(x, y)) \frac{1}{g(x, y)} dxdy$$

If the purpose is to compute an algebraic rule over T, with degree of precision n, positive weights and internal nodes (i.e. rules of PI-type) we face the problem that while f is a polynomial of total degree n then being $g(x, y) = \sqrt{1 - x^2 - y^2}$, we have that in general $f(x, y, g(x, y))\frac{1}{g(x,y)}$ may not be a polynomial.

To see this properly let

- $f(x, y, z) = x^\alpha y^\beta z^\gamma, \quad 0 \leq \alpha + \beta + \gamma \leq n, \ \alpha, \beta, \gamma \in \mathbb{N}$;
- $g(x, y) = \sqrt{1 - x^2 - y^2}$.

Thus

$$f(x, y, g(x, y)) \frac{1}{g(x, y)} = x^\alpha y^\beta (1 - x^2 - y^2)^{(\gamma-1)/2}.$$

Thus if γ is

- odd then $f(x, y, g(x, y)) \frac{1}{g(x,y)}$ is a polynomial of degree at most n,
- even then $f(x, y, g(x, y)) \frac{1}{g(x,y)}$ is $1/g$ multiplied for a polynomial of degree at most n.

Key points:

- Approximate $1/g$ by a polynomial p_ϵ of degree $m = m_\epsilon$ such that $|p_\epsilon - 1/g| \leq \epsilon \cdot 1/|g|$.
- Since $f/g \approx f \cdot p_\epsilon \in \mathbb{P}_{n+m}$, we integrate $f \cdot p_\epsilon$ instead of f/g on the elliptical sectors S_i, $i = 1, 2, 3$.

By determining a rule of algebraic degree of precision $n + m$ over each elliptical sector S_i, $i = 1, 2, 3$, with internal nodes, and positive weights then we have a rule on $\mathcal{T}^\perp := \cup_{i=1}^3 S_i$ with nodes $(x_k, y_k)_{k=1,\ldots,N_{n+m}}$, weights $w_{k=1,\ldots,N_{n+m}}$ of PI-type.

Mapping back the nodes on the sphere, we have a rule over the spherical triangle that is near algebraic with ADE n since

$$\int_{\mathcal{T}} f(x, y, z) d\sigma \approx \sum_{j=1}^{N_{n+m}} \frac{w_i}{\sqrt{1 - x_j^2 - y_j^2}} f(x_j, y_j, \sqrt{1 - x_j^2 - y_j^2}). \quad (1)$$

Some details:

- Rule over elliptical sector S_i: since each S_i is an affine transformation of a circular sector of the unit-disk S_i^*, we determine a formula on S_i^* and map it to S_i (some care on the weights that must be multiplied by absolute value of the transformation matrix determinant);
- Computation of $m = m_\epsilon$: it is sufficient to find the degree of a (near) optimal univariate polynomial approximation (up to ϵ) to $1/\sqrt{1-t}$ where $t \in [0, \max\{\|\hat{A}\|_2^2, \|\hat{B}\|_2^2, \|\hat{C}\|_2^2\}]$.

  Thus $m = m_\epsilon$ can be estimated by Chebfun (even stored in tables!).
- Caratheodory-Tchakaloff rule compression: from the nodes $\{P_k\}$ and weights $\{w_k\}$ of the PI-type rule on $\mathcal{T}$, we extract one with cardinality at most $(n+1)^2$, by Lawson-Hanson algorithm (see dCATCH suite implementation).

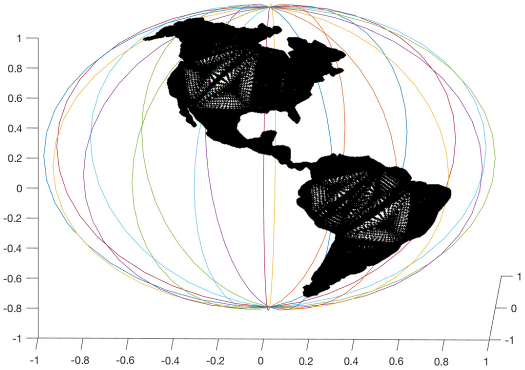

Figure: Quadrature nodes on a spherical polygon of a rule of PI-type, with ADE=8, 380544 points, before Caratheodory-Tchakaloff compression. Cputime: about 10 seconds.

Quadrature on spherical polygons: example

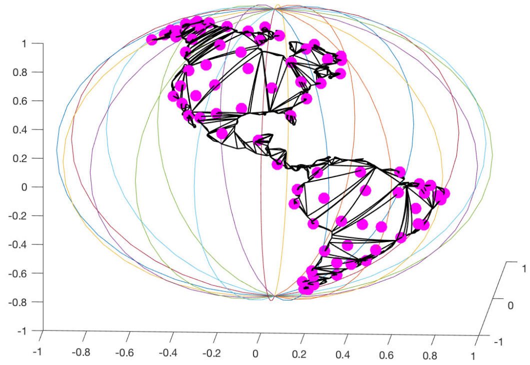

Figure: Triangulation of a spherical polygon (967 spherical triangles) and quadrature rule of PI-type with ADE=8, 81 points, after Caratheodory-Tchakaloff compression (magenta). Cputime: 3.5 seconds.

As introduced by I.H. Sloan in 1995, given

- an orthonormal basis of $\mathbb{P}_n^d(\Omega)$ (the subspace of d-variate polynomials of total-degree not exceeding n, restricted to a compact set or manifold $\Omega \subset R^d$) w.r.t. a given measure $d\mu$ on Ω), say $\{p_j\}$, $1 \leq j \leq N_n = \dim(\mathbb{P}_d^n(\Omega))$,
- a quadrature formula exact for $\mathbb{P}^{2n}{}_d(\Omega)$ with nodes $X = \{x_i\} \subset \Omega$ and positive weights $\mathbf{w} = \{w_i\}$, $1 \leq i \leq M$ with $M \geq N_n$,

the discretized orthogonal projection (hyperinterpolation) of $f \in C(\Omega)$ is

$$(\mathcal{L}_n f)(x) \;=\; \sum_{j=1}^{M} \langle f, p_j \rangle_{l_{2,\mathbf{w}}(X)} p_j(x) = \sum_{i=1}^{M} w_i f(x_i) \sum_{j=1}^{N_n} p_j(x_i) p_j(x).$$

Letting Ω be the spherical polygon,

- by means of the routines in our Matlab package `dCATCH` we determine the required orthonormal basis $\{p_j\}$,
- we compute the quadrature rule of degree $2n$ on Ω,
- we finally get the hyperinterpolant of degree n.

As previously mentioned, the Matlab software implementing this approach is available at the authors' homepage.

As example, we take into account a course map of Australia (without taking into account its smaller islands).

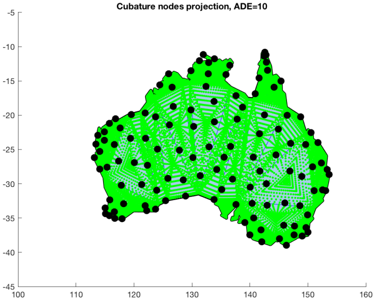

Figure: Quadrature nodes of PI-type on a coarse approximation of Australia, useful for hyperinterpolation. ADE is 10, there are 167 sph. triangles, the full rule has 81711 points, the compressed one 121, with moments error of $\approx 5 \cdot 10^{-15}$. The whole process took about 3 seconds.

Setting $(x_0, y_0, z_0) \approx (-6.325e-01, 6.668e-01, -3.908e-01)$ as an approximation of australian centroid, and

$$h(x,y,z) = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2$$

we define

1. $f_1(x,y,z) = 1 + x + y^2 + x^2 \cdot y + x^4 + y^5 + x^2 \cdot y^2 \cdot z^2$ (polynomial, total degree 6);
2. $f_2(x,y,z) = \cos(10 \cdot (x+y+z))$ (oscillations);
3. $f_3(x,y,z) = \sin(-h(x,y,z))$ (regular);
4. $f_4(x,y,z) = \exp(-h(x,y,z))$ (regular);
5. $f_5(x,y,z) = ((x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2)^{3/2}$ (hard);
6. $f_6(x,y,z) = ((x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2)^{5/2}$ (medium);

Quadrature on spherical polygons: hyperinterpolation test

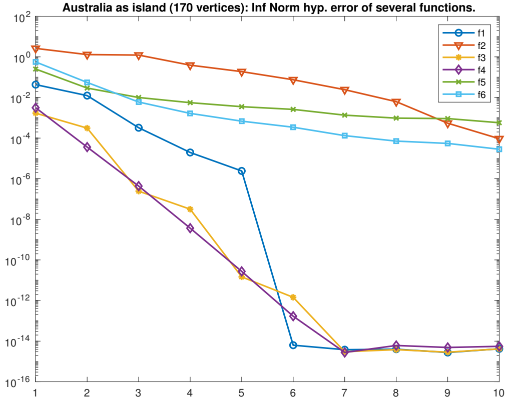

Figure: Inf-Norm hyperinterpolation error.

Quadrature on spherical polygons: hyperinterpolation test

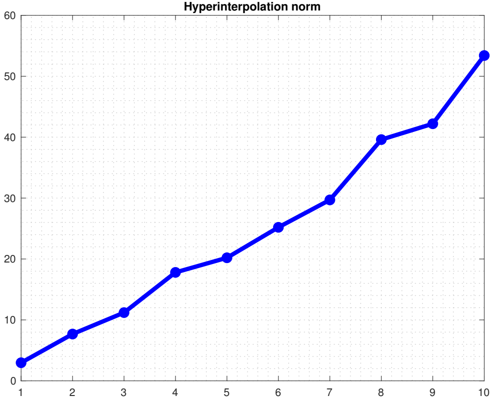

Figure: Hyperinterpolation operator Inf-Norm.

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