1
The Equations of Motion

In this chapter we develop the basic equations of fluid mechanics. These equations are derived from the conservation laws of mass, momentum, and energy. We begin with the simplest assumptions, leading to Euler's equations for a perfect fluid. These assumptions are relaxed in the third section to allow for viscous effects that arise from the molecular transport of momentum. Throughout the book we emphasize the intuitive and mathematical aspects of vorticity; this job is begun in the second section of this chapter.

1.1 Euler's Equations

Let D be a region in two- or three-dimensional space filled with a fluid. Our object is to describe the motion of such a fluid. Let $\mathbf{x} \in D$ be a point in D and consider the particle of fluid moving through $\mathbf{x}$ at time t. Relative to standard Euclidean coordinates in space, we write $\mathbf{x} = (x, y, z)$. Imagine a particle (think of a particle of dust suspended) in the fluid; this particle traverses a well-defined trajectory. Let $\mathbf{u}(\mathbf{x}, t)$ denote the velocity of the particle of fluid that is moving through $\mathbf{x}$ at time t. Thus, for each fixed time, $\mathbf{u}$ is a vector field on D, as in Figure 1.1.1. We call $\mathbf{u}$ the (***spatial***) ***velocity field of the fluid***.

For each time t, assume that the fluid has a well-defined ***mass density*** $\rho(\mathbf{x}, t)$. Thus, if W is any subregion of D, the mass of fluid in W at time t

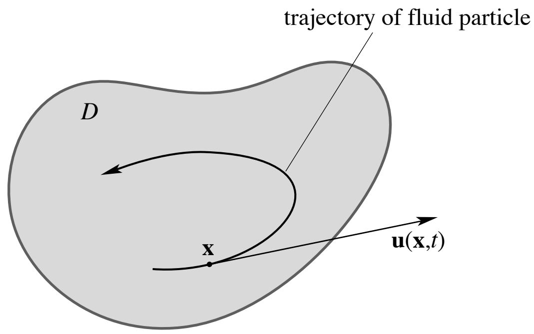

FIGURE 1.1.1. Fluid particles flowing in a region D.

is given by

$$m(W, t) = \int_W \rho(\mathbf{x}, t)\, dV,$$

where dV is the volume element in the plane or in space.

In what follows we shall assume that the functions $\mathbf{u}$ and ρ (and others to be introduced later) are smooth enough so that the standard operations of calculus may be performed on them. This assumption is open to criticism and indeed we shall come back and analyze it in detail later.

The assumption that ρ exists is a ***continuum assumption***. Clearly, it does not hold if the molecular structure of matter is taken into account. For most macroscopic phenomena occurring in nature, it is believed that this assumption is extremely accurate.

Our derivation of the equations is based on three basic principles:

i *mass is neither created nor destroyed*;

ii *the rate of change of momentum of a portion of the fluid equals the force applied to it* (***Newton's second law***);

iii *energy is neither created nor destroyed.*

Let us treat these three principles in turn.

i Conservation of Mass

Let W be a fixed subregion of D (W does *not* change with time). The rate of change of mass in W is

$$\frac{d}{dt} m(W, t) = \frac{d}{dt} \int_W \rho(\mathbf{x}, t)\, dV = \int_W \frac{\partial \rho}{\partial t}(\mathbf{x}, t)\, dV.$$

Let ∂W denote the boundary of W, assumed to be smooth; let $\mathbf{n}$ denote the unit outward normal defined at points of ∂W; and let dA denote the area element on ∂W. The volume flow rate across ∂W per unit area is $\mathbf{u} \cdot n$ and the mass flow rate per unit area is $\rho\mathbf{u} \cdot n$ (see Figure 1.1.2).

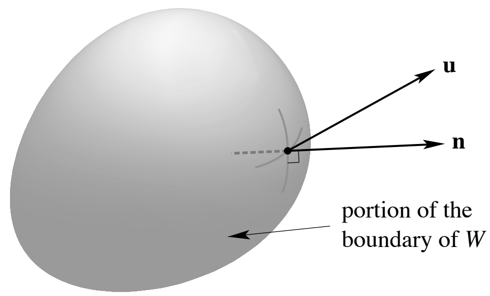

FIGURE 1.1.2. The mass crossing the boundary ∂W per unit time equals the surface integral of $\rho\mathbf{u} \cdot \mathbf{n}$ over ∂W.

The principle of conservation of mass can be more precisely stated as follows: The rate of increase of mass in W equals the rate at which mass is crossing ∂W in the *inward* direction; *i.e.,*

$$\boxed{\frac{d}{dt}\int_W \rho\, dV = -\int_{\partial W} \rho\mathbf{u} \cdot \mathbf{n}\, dA.}$$

This is the ***integral form of the law of conservation of mass.*** By the divergence theorem, this statement is equivalent to

$$\int_W \left[\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\mathbf{u})\right] dV = 0.$$

Because this is to hold for all W, it is equivalent to

$$\boxed{\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\mathbf{u}) = 0.}$$

The last equation is the ***differential form of the law of conservation of mass***, also known as the ***continuity equation.***

If ρ and $\mathbf{u}$ are not smooth enough to justify the steps that lead to the differential form of the law of conservation of mass, then the integral form is the one to use.

ii Balance of Momentum

Let $\mathbf{x}(t) = (x(t), y(t), z(t))$ be the path followed by a fluid particle, so that the velocity field is given by

$$\mathbf{u}(x(t), y(t), z(t), t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t)),$$

that is,

$$\mathbf{u}(\mathbf{x}(t), t) = \frac{d\mathbf{x}}{dt}(t).$$

This and the calculation following explicitly use standard Euclidean coordinates in space (delete z for plane flow).[1]

The acceleration of a fluid particle is given by

$$\mathbf{a}(t) = \frac{d^2}{dt^2}\mathbf{x}(t) = \frac{d}{dt}\mathbf{u}(x(t), y(t), z(t), t).$$

By the chain rule, this becomes

$$\mathbf{a}(t) = \frac{\partial \mathbf{u}}{\partial x}\dot{x} + \frac{\partial \mathbf{u}}{\partial y}\dot{y} + \frac{\partial \mathbf{u}}{\partial z}\dot{z} + \frac{\partial \mathbf{u}}{\partial t}.$$

Using the notation

$$\mathbf{u}_x = \frac{\partial \mathbf{u}}{\partial x}, \quad \mathbf{u}_t = \frac{\partial \mathbf{u}}{\partial t}, \quad \text{etc.},$$

and

$$\mathbf{u}(x, y, z, t) = (u(x, y, z, t), v(x, y, z, t), w(x, y, z, t)),$$

we obtain

$$\mathbf{a}(t) = u\mathbf{u}_x + v\mathbf{u}_y + w\mathbf{u}_z + \mathbf{u}_t,$$

which we also write as

$$\mathbf{a}(t) = \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u},$$

where

$$\partial_t \mathbf{u} = \frac{\partial \mathbf{u}}{\partial t} \quad \text{and} \quad \mathbf{u} \cdot \nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}.$$

[1] Care must be used if other coordinate systems (such as spherical or cylindrical) are employed. Other coordinate systems can be handled in two ways: first, one can proceed more intrinsically by developing intrinsic (i.e., coordinate free) formulas that are valid in any coordinate system, or, second, one can do all the derivations in Euclidean coordinates and transform final results to other coordinate systems at the end by using the chain rule. The second approach is clearly faster, although intellectually less satisfying. See Abraham, Marsden and Ratiu [1988] (listed in the front matter) for information on the former approach. For reasons of economy we shall do most of our calculations in standard Euclidean coordinates.

We call

$$\frac{D}{Dt} = \partial_t + \mathbf{u} \cdot \nabla$$

the ***material derivative***; it takes into account the fact that the fluid is moving and that the positions of fluid particles change with time. Indeed, if $f(x, y, z, t)$ is any function of position and time (scalar or vector), then by the chain rule,

$$\boxed{\frac{d}{dt} f(x(t), y(t), z(t), t) = \partial_t f + \mathbf{u} \cdot \nabla f = \frac{Df}{Dt}(x(t), y(t), z(t), t).}$$

For any continuum, forces acting on a piece of material are of two types. First, there are forces of *stress*, whereby the piece of material is acted on by forces across its surface by the rest of the continuum. Second, there are external, or body, forces such as gravity or a magnetic field, which exert a force per unit volume on the continuum. The clear isolation of surface forces of stress in a continuum is usually attributed to Cauchy.

Later, we shall examine stresses more generally, but for now let us define an ***ideal fluid*** as one with the following property: *For any motion of the fluid there is a function $p(\mathbf{x}, t)$ called the **pressure** such that if S is a surface in the fluid with a chosen unit normal $\mathbf{n}$, the force of stress exerted across the surface S per unit area at $\mathbf{x} \in S$ at time t is $p(\mathbf{x}, t)\mathbf{n}$; i.e.,*

$$\text{force across } S \text{ per unit area} = p(\mathbf{x}, t)\mathbf{n}.$$

Note that the force is in the direction $\mathbf{n}$ and that the force acts orthogonally to the surface S; that is, there are no tangential forces (see Figure 1.1.3).

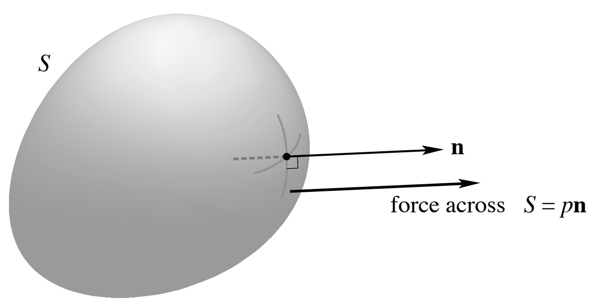

FIGURE 1.1.3. Pressure forces across a surface S.

Of course, the concept of an ideal fluid as a mathematical definition is not subject to dispute. However, the physical relevance of the notion (or mathematical theorems we deduce from it) must be checked by experiment. As we shall see later, ideal fluids exclude many interesting real physical phenomena, but nevertheless form a crucial component of a more complete theory.

Intuitively, the absence of tangential forces implies that there is no way for rotation to start in a fluid, nor, if it is there at the beginning, to stop. This idea will be amplified in the next section. However, even here we can detect physical trouble for ideal fluids because of the abundance of rotation in real fluids (near the oars of a rowboat, in tornadoes, etc.).

If W is a region in the fluid at a particular instant of time t, the total force exerted *on* the fluid inside W by means of stress on its boundary is

$$\mathbf{S}_{\partial W} = \{\text{force on } W\} = -\int_{\partial W} p\mathbf{n}\, dA$$

(negative because $\mathbf{n}$ points outward). If $\mathbf{e}$ is any fixed vector in space, the divergence theorem gives

$$\mathbf{e} \cdot \mathbf{S}_{\partial W} = -\int_{\partial W} p\mathbf{e} \cdot \mathbf{n}\, dA = -\int_W \operatorname{div}(p\mathbf{e})\, dV = -\int_W (\operatorname{grad} p) \cdot \mathbf{e}\, dV.$$

Thus,

$$\mathbf{S}_{\partial W} = -\int_W \operatorname{grad} p\, dV.$$

If $\mathbf{b}(\mathbf{x}, t)$ denotes the given body force *per unit mass*, the total body force is

$$\mathbf{B} = \int_W \rho\mathbf{b}\, dV.$$

Thus, on any piece of fluid material,

$$\text{force per unit volume} = -\operatorname{grad} p + \rho\mathbf{b}.$$

By Newton's second law (force = mass × acceleration) we are led to the differential form of the law of ***balance of momentum***:

$$\boxed{\rho \frac{D\mathbf{u}}{Dt} = -\operatorname{grad} p + \rho\mathbf{b}.} \tag{BM1}$$

Next we shall derive an integral form of balance of momentum in two ways. We derive it first as a deduction from the differential form and second from basic principles.

From balance of momentum in differential form, we have

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\rho(\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla p + \rho\mathbf{b}$$

and so, using the equation of continuity,

$$\frac{\partial}{\partial t}(\rho\mathbf{u}) = -\operatorname{div}(\rho\mathbf{u})\mathbf{u} - \rho(\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla p + \rho\mathbf{b}.$$

If $\mathbf{e}$ is any fixed vector in space, one checks that

$$\begin{aligned}\mathbf{e}\cdot\frac{\partial}{\partial t}(\rho\mathbf{u}) &= -\operatorname{div}(\rho\mathbf{u})\mathbf{u}\cdot\mathbf{e} - \rho(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{e} - (\nabla p)\cdot\mathbf{e} + \rho\mathbf{b}\cdot\mathbf{e}\\ &= -\operatorname{div}(p\mathbf{e} + \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{e})) + \rho\mathbf{b}\cdot\mathbf{e}.\end{aligned}$$

Therefore, if W is a *fixed* volume in space, the rate of change of momentum in direction $\mathbf{e}$ in W is

$$\mathbf{e}\cdot\frac{d}{dt}\int_W \rho\mathbf{u}\,dV = -\int_{\partial W}(p\mathbf{e} + \rho\mathbf{u}(\mathbf{e}\cdot\mathbf{u}))\cdot\mathbf{n}\,dA + \int_W \rho\mathbf{b}\cdot\mathbf{e}\,dV$$

by the divergence theorem. Thus, the integral form of balance of momentum becomes:

$$\boxed{\frac{d}{dt}\int_W \rho\mathbf{u}\,dV = -\int_{\partial W}(p\mathbf{n} + \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n}))\,dA + \int_W \rho\mathbf{b}\,dV.} \tag{BM2}$$

The quantity $p\mathbf{n}+\rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})$ is the ***momentum flux per unit area*** crossing ∂W, where $\mathbf{n}$ is the unit outward normal to ∂W.

This derivation of the integral balance law for momentum proceeded via the differential law. With an eye to assuming as little differentiability as possible, it is useful to proceed to the integral law directly and, as with conservation of mass, derive the differential form from it. To do this carefully requires us to introduce some useful notions.

As earlier, let D denote the region in which the fluid is moving. Let $\mathbf{x}\in D$ and let us write $\varphi(\mathbf{x}, t)$ for the trajectory followed by the particle that is at point $\mathbf{x}$ at time $t = 0$. We will assume φ is smooth enough so the following manipulations are legitimate and for fixed t, φ is an invertible mapping. Let φ_t denote the map $\mathbf{x}\mapsto\varphi(\mathbf{x}, t)$; that is, with fixed t, this map advances each fluid particle from its position at time $t = 0$ to its position at time t. Here, of course, the subscript does *not* denote differentiation. We call φ the ***fluid flow map***. If W is a region in D, then $\varphi_t(W) = W_t$ is the volume W *moving with the fluid*. See Figure 1.1.4.

The "primitive" integral form of balance of momentum states that

$$\boxed{\frac{d}{dt}\int_{W_t}\rho\mathbf{u}\,dV = S_{\partial W_t} + \int_{W_t}\rho\mathbf{b}\,dV,} \tag{BM3}$$

that is, the rate of change of momentum of a moving piece of fluid equals the total force (surface stresses plus body forces) acting on it.

These *two forms of balance of momentum* (BM1) and (BM3) *are equivalent*. To prove this, we use the change of variables theorem to write

$$\frac{d}{dt}\int_{W_t}\rho\mathbf{u}\,dV = \frac{d}{dt}\int_W(\rho\mathbf{u})(\varphi(\mathbf{x}, t), t)J(\mathbf{x}, t)\,dV,$$

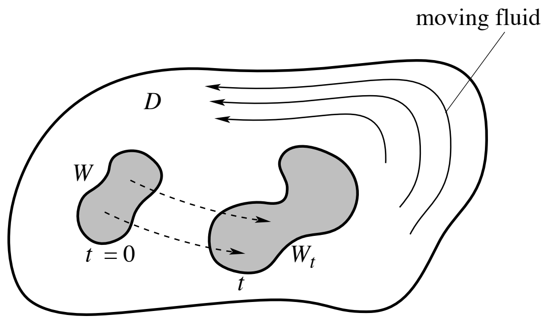

FIGURE 1.1.4. W_t is the image of W as particles of fluid in W flow for time t.

where $J(\mathbf{x}, t)$ is the Jacobian determinant of the map φ_t. Because the volume is fixed at its initial position, we may differentiate under the integral sign. Note that

$$\frac{\partial}{\partial t}(\rho\mathbf{u})(\varphi(\mathbf{x}, t), t) = \left(\frac{D}{Dt}\rho\mathbf{u}\right)(\varphi(\mathbf{x}, t), t)$$

is the material derivative, as was shown earlier. (If you prefer, this equality says that D/Dt is differentiation following the fluid.) Next, we learn how to differentiate $J(\mathbf{x}, t)$.

Lemma

$$\frac{\partial}{\partial t}J(\mathbf{x}, t) = J(\mathbf{x}, t)[\operatorname{div}\mathbf{u}(\varphi(\mathbf{x}, t), t)].$$

Proof Write the components of φ as $\xi(\mathbf{x}, t), \eta(\mathbf{x}, t)$, and $\zeta(\mathbf{x}, t)$. First, observe that

$$\frac{\partial}{\partial t}\varphi(\mathbf{x}, t) = \mathbf{u}(\varphi(\mathbf{x}, t), t),$$

by definition of the velocity field of the fluid.

The determinant J can be differentiated by recalling that the determinant of a matrix is multilinear in the columns (or rows). Thus, holding $\mathbf{x}$

fixed throughout, we have

$$\frac{\partial}{\partial t}J = \begin{bmatrix} \frac{\partial}{\partial t}\frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} & \frac{\partial \zeta}{\partial x} \\ \frac{\partial}{\partial t}\frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} & \frac{\partial \zeta}{\partial y} \\ \frac{\partial}{\partial t}\frac{\partial \xi}{\partial z} & \frac{\partial \eta}{\partial z} & \frac{\partial \zeta}{\partial z} \end{bmatrix} + \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial}{\partial t}\frac{\partial \eta}{\partial x} & \frac{\partial \zeta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial}{\partial t}\frac{\partial \eta}{\partial y} & \frac{\partial \zeta}{\partial y} \\ \frac{\partial \xi}{\partial z} & \frac{\partial}{\partial t}\frac{\partial \eta}{\partial z} & \frac{\partial \zeta}{\partial z} \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} & \frac{\partial}{\partial t}\frac{\partial \zeta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} & \frac{\partial}{\partial t}\frac{\partial \zeta}{\partial y} \\ \frac{\partial \xi}{\partial z} & \frac{\partial \eta}{\partial z} & \frac{\partial}{\partial t}\frac{\partial \zeta}{\partial z} \end{bmatrix}.$$

Now write

$$\begin{aligned} \frac{\partial}{\partial t}\frac{\partial \xi}{\partial x} &= \frac{\partial}{\partial x}\frac{\partial \xi}{\partial t} = \frac{\partial}{\partial x}u(\varphi(\mathbf{x},t),t), \\ \frac{\partial}{\partial t}\frac{\partial \xi}{\partial y} &= \frac{\partial}{\partial y}\frac{\partial \xi}{\partial t} = \frac{\partial}{\partial y}u(\varphi(\mathbf{x},t),t), \\ &\vdots \\ \frac{\partial}{\partial t}\frac{\partial \zeta}{\partial z} &= \frac{\partial}{\partial z}\frac{\partial \zeta}{\partial t} = \frac{\partial}{\partial z}w(\varphi(\mathbf{x},t),t). \end{aligned}$$

The components u, v, and w of $\mathbf{u}$ in this expression are functions of x, y, and z through $\varphi(\mathbf{x}, t)$; therefore,

$$\begin{aligned} \frac{\partial}{\partial x}u(\varphi(\mathbf{x},t),t) &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial x}, \\ &\vdots \\ \frac{\partial}{\partial z}w(\varphi(\mathbf{x},t),t) &= \frac{\partial w}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial w}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial w}{\partial \zeta}\frac{\partial \zeta}{\partial z}. \end{aligned}$$

When these are substituted into the above expression for $\partial J/\partial t$, one gets for the respective terms

$$\frac{\partial u}{\partial x}J + \frac{\partial v}{\partial y}J + \frac{\partial w}{\partial z}J = (\operatorname{div}\mathbf{u})J. \qquad \blacksquare$$

From this lemma, we get

$$\begin{aligned}\frac{d}{dt}\int_{W_t}\rho\mathbf{u}\,dV &= \int_W \left\{\left(\frac{D}{Dt}\rho\mathbf{u}\right)(\varphi(\mathbf{x},t),t) + (\rho\mathbf{u})(\operatorname{div}\mathbf{u})(\varphi(\mathbf{x},t),t)\right\} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times J(\mathbf{x},t)\,dV \\ &= \int_{W_t}\left\{\frac{D}{Dt}(\rho\mathbf{u}) + (\rho\ \operatorname{div}\mathbf{u})\mathbf{u}\right\} dV,\end{aligned}$$

where the change of variables theorem was again used. By conservation of mass,

$$\frac{D}{Dt}\rho + \rho\ \operatorname{div}\mathbf{u} = \frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{u}) = 0,$$

and thus

$$\frac{d}{dt}\int_{W_t}\rho\mathbf{u}\,dV = \int_{W_t}\rho\frac{D\mathbf{u}}{Dt}\,dV.$$

In fact, this argument proves the following theorem.

Transport Theorem *For any function f of $\mathbf{x}$ and t, we have*

$$\boxed{\frac{d}{dt}\int_{W_t}\rho f\,dV = \int_{W_t}\rho\frac{Df}{Dt}\,dV.}$$

In a similar way, one can derive a form of the transport theorem *without a mass density* factor included, namely,

$$\boxed{\frac{d}{dt}\int_{W_t} f\,dV = \int_{W_t}\left(\frac{\partial f}{\partial t} + \operatorname{div}(f\mathbf{u})\right) dV.}$$

If W, and hence, W_t, is arbitrary and the integrands are continuous, we have proved that the "primitive" integral form of balance of momentum is equivalent to the differential form (BM1). Hence, all three forms of balance of momentum—(BM1), (BM2), and (BM3)—are mutually equivalent. As an exercise, the reader should derive the two integral forms of balance of momentum directly from each other.

The lemma $\partial J/\partial t = (\operatorname{div}\mathbf{u})\,J$ is also useful in understanding incompressibility. In terms of the notation introduced earlier, we call a flow ***incompressible*** if for any fluid subregion W,

$$\operatorname{volume}(W_t) = \int_{W_t} dV = \text{constant in } t.$$

Thus, incompressibility is equivalent to

$$0 = \frac{d}{dt} \int_{W_t} dV = \frac{d}{dt} \int_W J\, dV = \int_W (\operatorname{div} \mathbf{u}) J\, dV = \int_{W_t} (\operatorname{div} \mathbf{u})\, dV$$

for all moving regions W_t. Thus, the following are equivalent:

(i) *the fluid is incompressible*;

(ii) $\operatorname{div} \mathbf{u} = 0$;

(iii) $J \equiv 1$.

From the equation of continuity

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0, \quad \text{i.e.,} \quad \frac{D\rho}{Dt} + \rho \operatorname{div} \mathbf{u} = 0,$$

and the fact that $\rho > 0$, we see that a *fluid is incompressible if and only if* $D\rho/Dt = 0$, *that is, the mass density is constant following the fluid*. If the fluid is ***homogeneous***, that is, $\rho =$ constant in space, it also follows that the flow is incompressible if and only if ρ is constant in time. Problems involving inhomogeneous incompressible flow occur, for example, in oceanography.

We shall now "solve" the equation of continuity by expressing ρ in terms of its value at $t = 0$, the flow map $\varphi(\mathbf{x}, t)$, and its Jacobian $J(\mathbf{x}, t)$. Indeed, set $f = 1$ in the transport theorem and conclude the equivalent condition for mass conservation,

$$\frac{d}{dt} \int_{W_t} \rho\, dV = 0$$

and thus,

$$\int_{W_t} \rho(\mathbf{x}, t) dV = \int_{W_0} \rho(\mathbf{x}, 0)\, dV.$$

Changing variables, we obtain

$$\int_{W_0} \rho(\varphi(\mathbf{x}, t), t) J(\mathbf{x}, t)\, dV = \int_{W_0} \rho(\mathbf{x}, 0)\, dV.$$

Because W_0 is arbitrary, we get

$$\boxed{\rho(\varphi(\mathbf{x}, t), t) J(\mathbf{x}, t) = \rho(\mathbf{x}, 0)}$$

as another form of mass conservation. As a corollary, a fluid that is homogeneous at $t = 0$ but is compressible will generally not remain homogeneous. However, the fluid will remain homogeneous if it is incompressible. The example $\varphi((x, y, z), t) = ((1 + t)x, y, z)$ has $J((x, y, z), t) = 1 + t$ so the flow is not incompressible, yet for $\rho((x, y, z), t) = 1/(1 + t)$, one has mass conservation and homogeneity for all time.

iii Conservation of Energy

So far we have developed the equations

$$\rho \frac{D\mathbf{u}}{Dt} = -\operatorname{grad} p + \rho \mathbf{b} \quad \text{(balance of momentum)}$$

and

$$\frac{D\rho}{Dt} + \rho \operatorname{div} \mathbf{u} = 0 \qquad \text{(conservation of mass).}$$

These are four equations if we work in 3-dimensional space (or $n+1$ equations if we work in n-dimensional space), because the equation for $D\mathbf{u}/Dt$ is a vector equation composed of three scalar equations. However, we have *five* functions: $\mathbf{u}$, ρ, and p. Thus, one might suspect that to specify the fluid motion completely, one more equation is needed. This is in fact true, and conservation of energy will supply the necessary equation in fluid mechanics. This situation is more complicated for general continua, and issues of general thermodynamics would need to be discussed for a complete treatment. We shall confine ourselves to two special cases here, and later we shall treat another case for an ideal gas.

For fluid moving in a domain D, with velocity field $\mathbf{u}$, the ***kinetic energy*** contained in a region $W \subset D$ is

$$E_{\text{kinetic}} = \frac{1}{2} \int_W \rho \|\mathbf{u}\|^2 \, dV$$

where $\|\mathbf{u}\|^2 = (u^2 + v^2 + w^2)$ is the square length of the vector function $\mathbf{u}$. We assume that the total energy of the fluid can be written as

$$E_{\text{total}} = E_{\text{kinetic}} + E_{\text{internal}}$$

where E_{internal} is the ***internal energy***, which is energy we cannot "see" on a macroscopic scale, and derives from sources such as intermolecular potentials and internal molecular vibrations. If energy is pumped into the fluid or if we allow the fluid to do work, E_{total} will change.

The rate of change of kinetic energy of a moving portion W_t of fluid is calculated using the transport theorem as follows:

$$\begin{aligned}
\frac{d}{dt} E_{\text{kinetic}} &= \frac{d}{dt}\left[\frac{1}{2}\int_{W_t} \rho \|\mathbf{u}\|^2 \, dV\right] \\
&= \frac{1}{2}\int_{W_t} \rho \frac{D\|\mathbf{u}\|^2}{Dt} \, dV \\
&= \int_{W_t} \rho \left(\mathbf{u} \cdot \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right)\right) dV.
\end{aligned}$$

Here we have used the following Euclidean coordinate calculation

$$\begin{aligned}
\frac{1}{2}\frac{D}{Dt}\|\mathbf{u}\|^2 &= \frac{1}{2}\frac{\partial}{\partial t}(u^2+v^2+w^2) + \frac{1}{2}\left(u\frac{\partial}{\partial x}(u^2+v^2+w^2)\right. \\
&\quad \left. + v\frac{\partial}{\partial y}(u^2+v^2+w^2) + w\frac{\partial}{\partial z}(u^2+v^2+w^2)\right) \\
&= u\frac{\partial u}{\partial t} + v\frac{\partial v}{\partial t} + w\frac{\partial w}{\partial t} + u\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x}\right) \\
&\quad + v\left(u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} + w\frac{\partial w}{\partial y}\right) + w\left(u\frac{\partial u}{\partial z} + v\frac{\partial v}{\partial z} + w\frac{\partial w}{\partial z}\right) \\
&= \mathbf{u}\cdot\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot(\mathbf{u}\cdot\nabla)\mathbf{u}).
\end{aligned}$$

A general discussion of energy conservation requires more thermodynamics than we shall need. We limit ourselves here to two examples of energy conservation; a third will be given in Chapter **3**.

1 Incompressible Flows

Here we assume *all* the energy is kinetic and that *the rate of change of kinetic energy in a portion of fluid equals the rate at which the pressure and body forces do work*:

$$\frac{d}{dt}E_{\text{kinetic}} = -\int_{\partial W_t} p\mathbf{u}\cdot\mathbf{n}\,dA + \int_{W_t} \rho\mathbf{u}\cdot\mathbf{b}\,dV.$$

By the divergence theorem and our previous formulas, this becomes

$$\begin{aligned}
\int_{W_t} \rho\left\{\mathbf{u}\cdot\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right)\right\} dV &= -\int_{W_t} (\operatorname{div}(p\mathbf{u}) - \rho\mathbf{u}\cdot\mathbf{b})\,dV \\
&= -\int_{W_t} (\mathbf{u}\cdot\nabla p - \rho\mathbf{u}\cdot\mathbf{b})\,dV
\end{aligned}$$

because $\operatorname{div}\mathbf{u} = 0$. The preceding equation is also a consequence of balance of momentum. This argument, in addition, shows that *if we assume* $E = E_{\text{kinetic}}$, *then the fluid must be incompressible* (unless $p = 0$). In summary, in this incompressible case, the ***Euler equations*** are:

$$\begin{aligned}
\rho\frac{D\mathbf{u}}{Dt} &= -\operatorname{grad} p + \rho\mathbf{b} \\
\frac{D\rho}{Dt} &= 0 \\
\operatorname{div}\mathbf{u} &= 0
\end{aligned}$$

with the boundary conditions

$$\mathbf{u}\cdot\mathbf{n} = 0 \quad \text{on } \partial D.$$

2 Isentropic Fluids

A compressible flow will be called ***isentropic*** if there is a function w, called the ***enthalpy***, such that

$$\operatorname{grad} w = \frac{1}{\rho} \operatorname{grad} p.$$

This terminology comes from thermodynamics. We shall not need a detailed discussion of thermodynamics concepts in this book, and so it is omitted, except for a brief discussion of entropy in Chapter **3** in the context of ideal gases. For the readers' convenience, we just make a few general comments.

In thermodynamics one has the following basic quantities, each of which is a function of $\mathbf{x}, t$ depending on a given flow:

$p =$ ***pressure***
$\rho =$ ***density***
$T =$ ***temperature***
$s =$ ***entropy***
$w =$ ***enthalpy*** (per unit mass)
$\epsilon = w - (p/\rho) =$ ***internal energy*** (per unit mass).

These quantities are related by the ***First Law of Thermodynamics***, which we accept as a basic principle:[2]

$$dw = T\,ds + \frac{1}{\rho}\,dp \tag{TD1}$$

The first law is a statement of conservation of energy; a statement equivalent to (TD1) is, as is readily verified,

$$d\epsilon = T\,ds + \frac{p}{\rho^2}\,d\rho. \tag{TD2}$$

If the pressure is a function of ρ only, then the flow is clearly isentropic with s as a constant (hence the name ***isentropic***) and

$$w = \int^{\rho} \frac{p'(\lambda)}{\lambda}\,d\lambda,$$

which is the integrated version of $dw = dp/\rho$ (see TD1). As above, the internal energy $\epsilon = w - (p/\rho)$ then satisfies $d\epsilon = (p d\rho)/\rho^2$ (see TD2) or, as a function of ρ,

$$p = \rho^2 \frac{\partial \varepsilon}{\partial p}, \quad \text{or} \quad \epsilon = \int^{\rho} \frac{p(\lambda)}{\lambda^2}\,d\lambda.$$

[2] A. Sommerfeld [1964] *Thermodynamics and Statistical Mechanics,* reprinted by Academic Press, Chapters **1** and **4**.

For isentropic flows with p a function of ρ, the integral form of energy balance reads as follows: *The rate of change of energy in a portion of fluid equals the rate at which work is done on it*:

$$\begin{aligned}\frac{d}{dt}E_{\text{total}} &= \frac{d}{dt}\int_{W_t}\left(\tfrac{1}{2}\rho\|\mathbf{u}\|^2+\rho\epsilon\right)dV \\ &= \int_{W_t}\rho\mathbf{u}\cdot\mathbf{b}\,dV-\int_{\partial W_t}p\mathbf{u}\cdot\mathbf{n}\,dA.\end{aligned} \tag{BE}$$

This follows from balance of momentum using our earlier expression for $(d/dt)E_{\text{kinetic}}$, the transport theorem, and $p=\rho^2\partial\epsilon/\partial\rho$. Alternatively, one can start with the assumption that p is a function of ρ and then (BE) and balance of mass and momentum implies that $p=\rho^2\partial\epsilon/\partial\rho$, which is *equivalent* to $dw=dp/\rho$, as we have seen.[3]

Euler's equations for isentropic flow are thus

$$\begin{aligned}\frac{\partial\mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}&=-\nabla w+\mathbf{b},\\ \frac{\partial\rho}{\partial t}+\operatorname{div}(\rho\mathbf{u})&=0\end{aligned}$$

in D, and

$$\mathbf{u}\cdot\mathbf{n}=0$$

on ∂D (or $\mathbf{u}\cdot\mathbf{n}=\mathbf{V}\cdot\mathbf{n}$ if ∂D is moving with velocity $\mathbf{V}$).

Later, we will see that in general these equations lead to a well-posed initial value problem only if $p'(\rho)>0$. This agrees with the common experience that increasing the surrounding pressure on a volume of fluid causes a decrease in occupied volume and hence an increase in density.

Gases can often be viewed as isentropic, with

$$p=A\rho^\gamma,$$

where A and γ are constants and $\gamma\geq 1$. Here,

$$w=\int^\rho\frac{\gamma A s^{\gamma-1}}{s}\,ds=\frac{\gamma A\rho^{\gamma-1}}{\gamma-1}\quad\text{and}\quad\epsilon=\frac{A\rho^{\gamma-1}}{\gamma-1}.$$

Cases 1 and 2 above are rather opposite. For instance, if $\rho=\rho_0$ is a constant for an incompressible fluid, then clearly p cannot be an invertible function of ρ. However, the case $\rho=$ constant may be regarded as a limiting case $p'(\rho)\to\infty$. In case 2, p is an explicit function of ρ (and therefore

[3] One can carry this even further and use balance of energy and its invariance under Euclidean motions to derive balance of momentum and mass, a result of Green and Naghdi. See Marsden and Hughes [1994] for a proof and extensions of the result that include formulas such as $p=p^2\partial\varepsilon/\partial p$ amongst the consequences as well.

depends on $\mathbf{u}$ through the coupling of ρ and $\mathbf{u}$ in the equation of continuity); in case 1, p is implicitly determined by the condition $\operatorname{div} \mathbf{u} = 0$. We shall discuss these points again later.

Finally, notice that in neither case 1 or 2 is the possibility of a loss of kinetic energy due to friction taken into account. This will be discussed at length in §**1.3**.

Given a fluid flow with velocity field $\mathbf{u}(\mathbf{x}, t)$, a ***streamline*** at a fixed time is an integral curve of $\mathbf{u}$; that is, if $\mathbf{x}(s)$ is a streamline at the instant t, it is a curve parametrized by a variable, say s, that satisfies

$$\frac{d\mathbf{x}}{ds} = \mathbf{u}(\mathbf{x}(s), t), \qquad t \text{ fixed.}$$

We define a fixed ***trajectory*** to be the curve traced out by a particle as time progresses, as explained at the beginning of this section. Thus, a trajectory is a solution of the differential equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}(t), t)$$

with suitable initial conditions. If $\mathbf{u}$ is independent of t (i.e., $\partial_t \mathbf{u} = 0$), streamlines and trajectories coincide. In this case, the flow is called ***stationary***.

Bernoulli's Theorem *In stationary isentropic flows and in the absence of external forces, the quantity*

$$\tfrac{1}{2}\|\mathbf{u}\|^2 + w$$

is constant along streamlines. The same holds for homogeneous ($\rho =$ constant in space $= \rho_0$) incompressible flow with w replaced by p/ρ_0. The conclusions remain true if a force $\mathbf{b}$ is present and is conservative; i.e., $\mathbf{b} = -\nabla\varphi$ for some function φ, with w replaced by $w + \varphi$.

Proof From the table of vector identities at the back of the book, one has

$$\tfrac{1}{2}\nabla(\|\mathbf{u}\|^2) = (\mathbf{u} \cdot \nabla)\mathbf{u} + \mathbf{u} \times (\nabla \times \mathbf{u}).$$

Because the flow is steady, the equations of motion give

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla w$$

and so

$$\nabla\left(\tfrac{1}{2}\|\mathbf{u}\|^2 + w\right) = \mathbf{u} \times (\nabla \times \mathbf{u}).$$

Let $\mathbf{x}(s)$ be a streamline. Then

$$\begin{aligned} \tfrac{1}{2}\left(\|\mathbf{u}\|^2 + w\right)\Big|_{\mathbf{x}(s_1)}^{\mathbf{x}(s_2)} &= \int_{\mathbf{x}(s_1)}^{\mathbf{x}(s_2)} \nabla\left(\tfrac{1}{2}\|\mathbf{u}\|^2 + w\right) \cdot \mathbf{x}'(s)\, ds \\ &= \int_{\mathbf{x}(s_1)}^{\mathbf{x}(s_2)} (\mathbf{u} \times (\nabla \times \mathbf{u})) \cdot \mathbf{x}'(s)\, ds = 0 \end{aligned}$$

because $\mathbf{x}'(s) = \mathbf{u}(\mathbf{x}(s))$ is orthogonal to $\mathbf{u} \times (\nabla \times \mathbf{u})$. ■

See Exercise 1.1-3 at the end of this section for another view of why the combination $\frac{1}{2}\|\mathbf{u}\|^2 + w$ is the correct quantity in Bernoulli's theorem.

We conclude this section with an example that shows the limitations of the assumptions we have made so far.

Example Consider a fluid-filled channel, as in Figure 1.1.5.

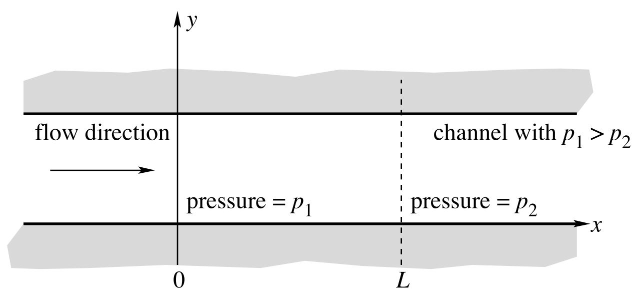

FIGURE 1.1.5. Fluid flow in a channel.

Suppose that the pressure p_1 at $x = 0$ is larger than that at $x = L$ so the fluid is pushed from left to right. We seek a solution of Euler's incompressible homogeneous equations in the form

$$\mathbf{u}(x, y, t) = (u(x, t), 0) \quad \text{and} \quad p(x, y, t) = p(x).$$

Incompressibility implies $\partial_x u = 0$. Thus, Euler's equations become $\rho_0 \partial_t u = -\partial_x p$. This implies that $\partial_x^2 p = 0$, and so

$$p(x) = p_1 - \left(\frac{p_1 - p_2}{L}\right) x.$$

Substitution into $\rho_0 \partial_t u = -\partial_x p$ and integration yields

$$u = \frac{p_1 - p_2}{\rho_0 L} t + \text{constant}.$$

This solution suggests that the velocity in channel flow with a constant pressure gradient increases indefinitely. Of course, this cannot be the case in a real flow; however, in our modeling, we have not yet taken friction into account. The situation will be remedied in §**1.3.** ♦

Exercises

◇ **Exercise 1.1-1** Prove the following properties of the material derivative

(i) $\dfrac{D}{Dt}(f+g) = \dfrac{Df}{Dt} + \dfrac{Dg}{Dt}$,

(ii) $\dfrac{D}{Dt}(f \cdot g) = f\dfrac{Dg}{Dt} + g\dfrac{Df}{Dt}$ (Leibniz or product rule),

(iii) $\dfrac{D}{Dt}(h \circ g) = (h' \circ g)\dfrac{Dg}{Dt}$ (chain rule).

◇ **Exercise 1.1-2** Use the transport theorem to establish the following formula of Reynolds:

$$\frac{d}{dt}\int_{W_t} f(x,t)\,dV = \int_{W_t} \frac{\partial f}{\partial t}(x,t)\,dV + \int_{\partial W_t} f\mathbf{u}\cdot\mathbf{n}\,dA.$$

Interpret the result physically.

◇ **Exercise 1.1-3** Consider isentropic flow without any body force. Show that for a *fixed* volume W in space (*not* moving with the flow).

$$\frac{d}{dt}\int_W \left(\tfrac{1}{2}\rho\|\mathbf{u}\|^2 + \rho\epsilon\right)\,dV = -\int_{\partial W} \rho\left(\tfrac{1}{2}\|\mathbf{u}\|^2 + w\right)\mathbf{u}\cdot\mathbf{n}\,dA.$$

Use this to justify the term ***energy flux vector*** for the vector function $\rho\mathbf{u}\left(\frac{1}{2}\|\mathbf{u}\|^2 + w\right)$ and compare with Bernoulli's theorem.

1.2 Rotation and Vorticity

If the velocity field of a fluid is $\mathbf{u} = (u,v,w)$, then its curl,

$$\boldsymbol{\xi} = \nabla \times \mathbf{u} = (\partial_y w - \partial_z v, \partial_z u - \partial_x w, \partial_x v - \partial_y u)$$

is called the ***vorticity field*** of the flow.

We shall now demonstrate that in a small neighborhood of each point of the fluid, $\mathbf{u}$ *is the sum of a (rigid) translation, a deformation (defined later), and a (rigid) rotation with rotation vector* $\boldsymbol{\xi}/2$. This is in fact a general statement about vector fields $\mathbf{u}$ on $\mathbb{R}^3$; the specific features of fluid mechanics are irrelevant for this discussion. Let $\mathbf{x}$ be a point in $\mathbb{R}^3$, and let $\mathbf{y} = \mathbf{x} + \mathbf{h}$ be a nearby point. What we shall prove is that

$$\mathbf{u}(\mathbf{y}) = \mathbf{u}(\mathbf{x}) + \mathbf{D}(\mathbf{x})\cdot\mathbf{h} + \tfrac{1}{2}\boldsymbol{\xi}(\mathbf{x})\times\mathbf{h} + O(h^2), \tag{1.2.1}$$

where $\mathbf{D}(x)$ is a symmetric 3×3 matrix and $h^2 = \|\mathbf{h}\|^2$ is the squared length of $\mathbf{h}$. We shall discuss the meaning of the several terms later.

Proof of Formula (1.2.1) Let

$$\nabla\mathbf{u} = \begin{bmatrix} \partial_x u & \partial_y u & \partial_z u \\ \partial_x v & \partial_y v & \partial_z v \\ \partial_x w & \partial_y w & \partial_z w \end{bmatrix}$$

denote the Jacobian matrix of $\mathbf{u}$. By Taylor's theorem,

$$\mathbf{u}(\mathbf{y}) = \mathbf{u}(\mathbf{x}) + \nabla\mathbf{u}(\mathbf{x}) \cdot \mathbf{h} + O(h^2), \tag{1.2.2}$$

where $\nabla\mathbf{u}(\mathbf{x}) \cdot \mathbf{h}$ is a matrix multiplication, with $\mathbf{h}$ regarded as a column vector. Let

$$\mathbf{D} = \tfrac{1}{2}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right],$$

where ^{T} denotes the transpose, and

$$\mathbf{S} = \tfrac{1}{2}\left[\nabla\mathbf{u} - (\nabla\mathbf{u})^T\right].$$

Thus,

$$\nabla\mathbf{u} = \mathbf{D} + \mathbf{S}. \tag{1.2.3}$$

It is easy to check that the coordinate expression for $\mathbf{S}$ is

$$\mathbf{S} = \frac{1}{2}\begin{bmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{bmatrix}$$

and that

$$\mathbf{S} \cdot \mathbf{h} = \tfrac{1}{2}\boldsymbol{\xi} \times \mathbf{h}, \tag{1.2.4}$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$. Substitution of (1.2.3) and (1.2.4) into (1.2.2) yields (1.2.1). ■

Because $\mathbf{D}$ is a symmetric matrix,

$$\mathbf{D}(\mathbf{x}) \cdot \mathbf{h} = \operatorname{grad}_h \psi(\mathbf{x}, \mathbf{h}),$$

where ψ is the quadratic form associated with $\mathbf{D}$; *i.e.,*

$$\psi(\mathbf{x}, \mathbf{h}) = \tfrac{1}{2}\langle \mathbf{D}(\mathbf{x}) \cdot \mathbf{h}, \mathbf{h}\rangle,$$

where $\langle\, ,\rangle$ is the inner product of $\mathbb{R}^3$. We call $\mathbf{D}$ the ***deformation tensor***. We now discuss its physical interpretation. Because $\mathbf{D}$ is symmetric, there is, for $\mathbf{x}$ fixed, an orthonormal basis $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3$ in which $\mathbf{D}$ is diagonal:

$$\mathbf{D} = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}.$$

Keep $\mathbf{x}$ fixed and consider the original vector field as a function of y. The motion of the fluid is described by the equations

$$\frac{d\mathbf{y}}{dt} = \mathbf{u}(\mathbf{y}).$$

If we ignore all terms in (1.2.1) except $\mathbf{D} \cdot \mathbf{h}$, we find

$$\frac{d\mathbf{y}}{dt} = \mathbf{D} \cdot \mathbf{h}, \quad \text{i.e.,} \quad \frac{d\mathbf{h}}{dt} = \mathbf{D} \cdot \mathbf{h}.$$

This vector equation is equivalent to three linear differential equations that separate in the basis $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3$:

$$\frac{d\tilde{h}_i}{dt} = d_i \tilde{h}_i, \quad i = 1, 2, 3.$$

The rate of change of a unit length along the $\tilde{\mathbf{e}}_i$ axis at $t = 0$ is thus d_i. The vector field $\mathbf{D} \cdot \mathbf{h}$ is thus merely expanding or contracting along each of the axes $\tilde{\mathbf{e}}_i$—hence, the name "deformation." The rate of change of the volume of a box with sides of length $\tilde{h}_1, \tilde{h}_2, \tilde{h}_3$ parallel to the $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3$ axes is

$$\begin{aligned}\frac{d}{dt}(\tilde{h}_1\tilde{h}_2\tilde{h}_3) &= \left[\frac{d\tilde{h}_1}{dt}\right]\tilde{h}_2\tilde{h}_3 + \tilde{h}_1\left[\frac{d\tilde{h}_2}{dt}\right]\tilde{h}_3 + \tilde{h}_1\tilde{h}_2\left[\frac{d\tilde{h}_3}{dt}\right] \\ &= (d_1 + d_2 + d_3)(\tilde{h}_1\tilde{h}_2\tilde{h}_3).\end{aligned}$$

However, the trace of a matrix is invariant under orthogonal transformations. Hence,

$$d_1 + d_2 + d_3 = \text{trace of } \mathbf{D} = \text{trace of } \tfrac{1}{2}\left((\nabla\mathbf{u}) + (\nabla\mathbf{u})^T\right) = \operatorname{div}\mathbf{u}.$$

This confirms the fact proved in §**1.1** that volume elements change at a rate proportional to $\operatorname{div}\mathbf{u}$. Of course, the constant vector field $\mathbf{u}(\mathbf{x})$ in formula (1.2.1) induces a flow that is merely a translation by $\mathbf{u}(\mathbf{x})$. The other term, $\frac{1}{2}\boldsymbol{\xi}(\mathbf{x}) \times \mathbf{h}$, induces a flow

$$\frac{d\mathbf{h}}{dt} = \tfrac{1}{2}\boldsymbol{\xi}(\mathbf{x}) \times \mathbf{h}, \qquad (\mathbf{x} \text{ fixed}).$$

The solution of this linear differential equation is, by elementary vector calculus,

$$\mathbf{h}(t) = \mathbf{R}(t, \boldsymbol{\xi}(\mathbf{x}))\mathbf{h}(0),$$

where $\mathbf{R}(t, \boldsymbol{\xi}(\mathbf{x}))$ is the matrix that represents a rotation through an angle t about the axis $\boldsymbol{\xi}(\mathbf{x})$ (in the oriented sense). Because rigid motion leaves volumes invariant, the divergence of $\frac{1}{2}\boldsymbol{\xi}(\mathbf{x}) \times \mathbf{h}$ is zero, as may also be checked by noting that $\mathbf{S}$ has zero trace. This completes our derivation and discussion of the decomposition (1.2.1).

We remarked in §**1.1** that our assumptions so far have precluded any tangential forces, and thus any mechanism for starting or stopping rotation. Thus, intuitively, we might expect rotation to be conserved. Because rotation is intimately related to the vorticity as we have just shown, we can expect the vorticity to be involved. We shall now prove that this is so.

Let C be a simple closed contour in the fluid at $t = 0$. Let C_t be the contour carried along by the flow. In other words,

$$C_t = \varphi_t(C),$$

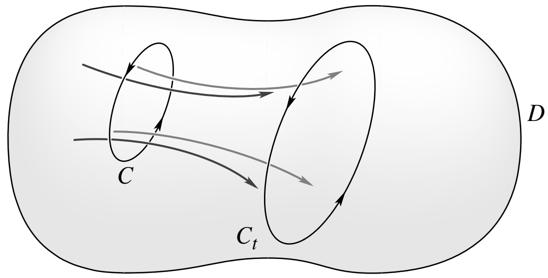

FIGURE 1.2.1. Kelvin's circulation theorem.

where φ_t is the fluid flow map discussed in §**1.1** (see Figure 1.2.1).

The ***circulation*** around C_t is defined to be the line integral

$$\boxed{\Gamma_{C_t} = \oint_{C_t} \mathbf{u} \cdot d\mathbf{s}.}$$

Kelvin's Circulation Theorem *For isentropic flow without external forces, the circulation, Γ_{C_t} is constant in time.*

For example, we note that if the fluid moves in such a way that C_t shrinks in size, then the "angular" velocity around C_t increases. The proof of Kelvin's circulation theorem is based on a version of the transport theorem for curves.

Lemma *Let u be the velocity field of a flow and C a closed loop, with $C_t = \varphi_t(C)$ the loop transported by the flow* (Figure 1.2.1). *Then*

$$\frac{d}{dt}\int_{C_t} \mathbf{u} \cdot\, ds = \int_{C_t} \frac{D\mathbf{u}}{Dt}\, d\mathbf{s}. \tag{1.2.5}$$

Proof Let $\mathbf{x}(s)$ be a parametrization of the loop C, $0 \le s \le 1$. Then a parameterization of C_t is $\varphi(\mathbf{x}(s), t), 0 \le s \le 1$. Thus, by definition of the line integral and the material derivative,

$$\begin{aligned}
\frac{d}{dt}\int_{C_t} \mathbf{u} \cdot d\mathbf{s} =& \frac{d}{dt}\int_0^1 \mathbf{u}(\varphi(\mathbf{x}(s), t), t) \cdot \frac{\partial}{\partial s}\varphi(\mathbf{x}(s), t)\, ds \\
=& \int_0^1 \frac{D\mathbf{u}}{Dt}(\varphi(\mathbf{x}(s), t), t) \cdot \frac{\partial}{\partial s}\varphi(\mathbf{x}(s), t)\, ds \\
&+ \int_0^1 \mathbf{u}(\varphi(\mathbf{x}(s), t), t) \cdot \frac{\partial}{\partial t}\frac{\partial}{\partial s}\varphi(\mathbf{x}(s), t)\, ds.
\end{aligned}$$

Because $\partial\varphi/\partial t = \mathbf{u}$, the second term equals

$$\int_0^1 \mathbf{u}(\varphi(\mathbf{x}(s),t),t) \cdot \frac{\partial}{\partial s}\mathbf{u}(\varphi(\mathbf{x}(s),t),t)\,ds = \frac{1}{2}\int_0^1 \frac{\partial}{\partial s}(\mathbf{u}\cdot\mathbf{u})(\varphi(\mathbf{x}(s),t),t)\,ds = 0$$

(since C_t is closed). The first term equals

$$\int_{C_t} \frac{D\mathbf{u}}{Dt}\,d\mathbf{s},$$

so the lemma is proved. ■

Proof of the Circulation Theorem Using the lemma and the fact that $D\mathbf{u}/Dt = -\nabla w$ (the flow is isentropic and without external forces), we find

$$\begin{aligned}\frac{d}{dt}\Gamma_{C_t} = \frac{d}{dt}\int_{C_t}\mathbf{u}\,d\mathbf{s} &= \int_{C_t}\frac{D\mathbf{u}}{Dt}\,d\mathbf{s} \\ &= -\int_{C_t}\nabla w\cdot d\mathbf{s} = 0 \quad \text{(since } C_t \text{ is closed).}\end{aligned}$$ ■

We now use Stokes' theorem, which will bring in the vorticity. If Σ is a surface whose boundary is an oriented closed oriented contour C, then Stokes' theorem yields (see Figure 1.2.2)

$$\Gamma_C = \int_C \mathbf{u}\cdot d\mathbf{s} = \iint_\Sigma (\nabla\times\mathbf{u})\cdot\mathbf{n}\;dA = \iint_\Sigma \boldsymbol{\xi}\cdot d\mathbf{A}.$$

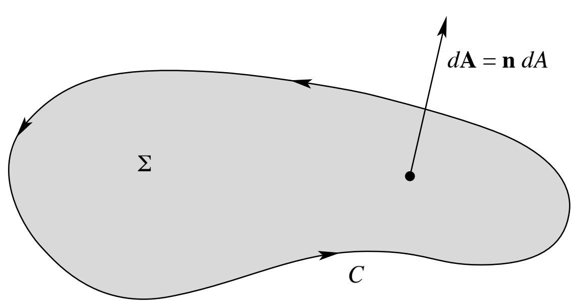

FIGURE 1.2.2. The circulation around C is the integral of the vorticity over Σ.

Thus, as a corollary of the circulation theorem, we can conclude that *the flux of vorticity across a surface moving with the fluid is constant in time.*

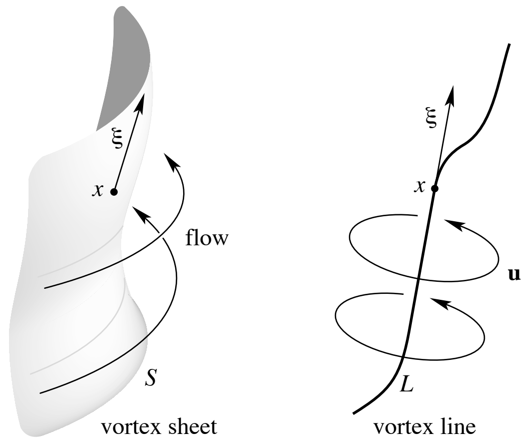

FIGURE 1.2.3. Vortex sheets and lines remain so under the flow.

By definition, a ***vortex sheet*** (or ***vortex line***) is a surface S (or a curve $\mathcal{L}$) that is tangent to the vorticity vector $\boldsymbol{\xi}$ at each of its points (Figure 1.2.3).

Proposition *If a surface (or curve) moves with the flow of an isentropic fluid and is a vortex sheet (or line) at $t = 0$, then it remains so for all time.*

Proof Let $\mathbf{n}$ be the unit normal to S, so that at $t = 0$, $\boldsymbol{\xi} \cdot \mathbf{n} = 0$ by hypothesis. By the circulation theorem, the flux of $\boldsymbol{\xi}$ across any portion $\tilde{S} \subset S$ at a later time is also zero, *i.e.,*

$$\iint_{\tilde{S}_t} \boldsymbol{\xi} \cdot \mathbf{n} \, dA = 0.$$

It follows that $\boldsymbol{\xi} \cdot \mathbf{n} = 0$ identically on S_t, so S remains a vortex sheet.

One can show (using the implicit function theorem) that if $\boldsymbol{\xi}(\mathbf{x}) \neq \mathbf{0}$, then, locally, a vortex line is the intersection of two vortex sheets. ■

Next, we show that the vorticity (per unit mass), that is, $\boldsymbol{\omega} = \boldsymbol{\xi}/\rho$, is *propagated by the flow* (see Figure 1.2.4). This fact can also be used to give another proof of the preceding theorem. We assume we are in three dimensions; the two-dimensional case will be discussed later.

Proposition *For isentropic flow (in the absence of external forces) with $\boldsymbol{\xi} = \nabla \times \mathbf{u}$ and $\boldsymbol{\omega} = \boldsymbol{\xi}/\rho$, we have*

$$\frac{D\boldsymbol{\omega}}{Dt} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} = 0 \tag{1.2.6}$$

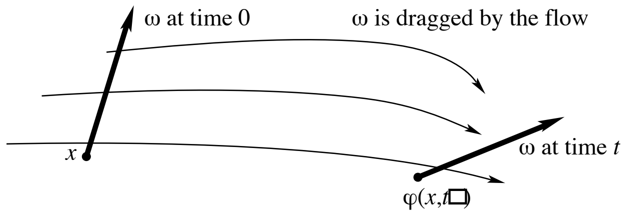

FIGURE 1.2.4. The vorticity is transported by the Jacobian matrix of the flow map.

and

$$\boldsymbol{\omega}(\varphi(\mathbf{x},t),t) = \nabla\varphi_t(\mathbf{x}) \cdot \boldsymbol{\omega}(\mathbf{x},0), \tag{1.2.7}$$

*where φ_t is the flow map (see §**1.1**) and $\nabla\varphi_t$ is its Jacobian matrix.*

Proof Start with the following vector identity (see the table of vector identities at the back of the book)

$$\tfrac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) = \mathbf{u} \times \operatorname{curl}\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}.$$

Substituting this into the equations of motion yields

$$\frac{\partial \mathbf{u}}{\partial t} + \tfrac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) - \mathbf{u}\times\operatorname{curl}\mathbf{u} = -\nabla w.$$

Taking the curl and using the identity $\nabla \times \nabla f = 0$ gives

$$\frac{\partial \boldsymbol{\xi}}{\partial t} - \operatorname{curl}(\mathbf{u}\times\boldsymbol{\xi}) = \mathbf{0}.$$

Using the identity (also from the back of the book)

$$\operatorname{curl}(\mathbf{F}\times\mathbf{G}) = \mathbf{F}\operatorname{div}\mathbf{G} - \mathbf{G}\operatorname{div}\mathbf{F} + (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G}$$

for the curl of a vector product, gives

$$\frac{\partial \boldsymbol{\xi}}{\partial t} - [\,(\mathbf{u}(\nabla\cdot\boldsymbol{\xi}) - \boldsymbol{\xi}(\nabla\cdot\mathbf{u}) + \boldsymbol{\xi}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\boldsymbol{\xi}\,] = \mathbf{0},$$

that is,

$$\frac{D\boldsymbol{\xi}}{Dt} - (\boldsymbol{\xi}\cdot\nabla)\mathbf{u} + \boldsymbol{\xi}(\nabla\cdot\mathbf{u}) = \mathbf{0}, \tag{1.2.8}$$

since $\boldsymbol{\xi}$ is divergence free. Also,

$$\frac{D\boldsymbol{\omega}}{Dt} = \frac{D}{Dt}\left(\frac{\boldsymbol{\xi}}{\rho}\right) = \frac{1}{\rho}\frac{D\boldsymbol{\xi}}{Dt} + \frac{\boldsymbol{\xi}}{\rho}(\nabla\cdot\mathbf{u}) \tag{1.2.9}$$

by the continuity equation. Substitution of (1.2.8) into (1.2.9) yields (1.2.6).

To prove (1.2.7), let

$$\mathbf{F}(\mathbf{x},t) = \boldsymbol{\omega}(\varphi(\mathbf{x},t),t) \quad \text{and} \quad \mathbf{G}(\mathbf{x},t) = \nabla\varphi_t(\mathbf{x}) \cdot \boldsymbol{\omega}(\mathbf{x},0).$$

By (1.2.6), $\partial\mathbf{F}/\partial t = (\mathbf{F}\cdot\nabla)\mathbf{u}$. On the other hand, by the chain rule:

$$\begin{aligned}\frac{\partial\mathbf{G}}{\partial t} &= \nabla\left[\frac{\partial\varphi}{\partial t}(\mathbf{x},t)\right]\cdot\boldsymbol{\omega}(\mathbf{x},0) = \nabla(\mathbf{u}(\varphi(\mathbf{x},t),t))\cdot\boldsymbol{\omega}(\mathbf{x},0)\\ &= (\nabla\mathbf{u})\cdot\nabla\varphi_t(\mathbf{x})\cdot\boldsymbol{\omega}(\mathbf{x},0) = (\mathbf{G}\cdot\nabla)\mathbf{u}\end{aligned}$$

Thus, $\mathbf{F}$ and $\mathbf{G}$ satisfy the same linear first-order differential equation. Because they coincide at $t = 0$ and solutions are unique, they are equal. ■

The reader may wish to compare (1.2.7) with the formula

$$\rho(\mathbf{x},0) = \rho(\varphi(\mathbf{x},t),t)J(\mathbf{x},t) \tag{1.2.10}$$

proved in §**1.1**.

As an exercise, we invite the reader to prove the preservation of vortex sheets and lines by the flow using (1.2.7) and (1.2.10).

For two-dimensional flow, where $\mathbf{u} = (u,v,0)$, $\boldsymbol{\xi}$ has only one component; $\boldsymbol{\xi} = (0,0,\xi)$. The circulation theorem now states that if Σ_t is any region in the plane that is moving with the fluid, then

$$\int_{\Sigma_t} \xi\, dA = \text{constant in time.} \tag{1.2.11}$$

In fact, one can say more using (1.2.7). In two dimensions, (1.2.7) specializes to

$$\frac{\xi}{\rho}(\varphi(\mathbf{x},t),t) = \frac{\xi}{\rho}(\mathbf{x},0), \tag*{(1.2.7)$'$}$$

that is, ξ/ρ is propagated as a scalar by the flow. Employing (1.2.10) and the change of variables theorem gives (1.2.11) as a special case.

In three-dimensional flows, the relation (1.2.7) allows rather complicated behavior. We shall now discuss the three-dimensional geometry a bit further.

A ***vortex tube*** consists of a two-dimensional surface S that is nowhere tangent to $\boldsymbol{\xi}$, with vortex lines drawn through each point of the bounding curve C of S. These vortex lines are integral curves of $\boldsymbol{\xi}$ and are extended as far as possible in each direction. See Figure 1.2.5.

In fluid mechanics it is customary to be sloppy about this definition and make tacit assumptions to the effect that the tube really "looks like" a tube. More precisely, we assume S is diffeomorphic to a disc (i.e., related to a disc by a one-to-one invertible differentiable transformation) and that the resulting tube is diffeomorphic to the product of the disc and the real line. This tacitly assumes that $\boldsymbol{\xi}$ has no zeros (of course, $\boldsymbol{\xi}$ *could* have zeros!).

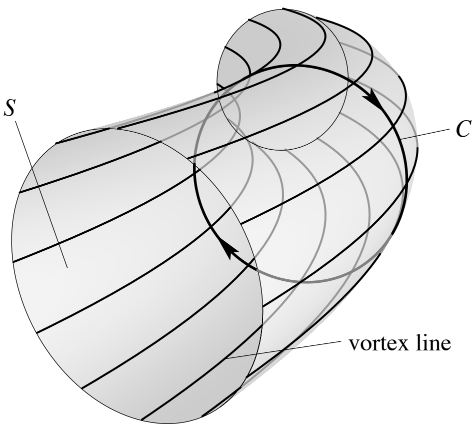

FIGURE 1.2.5. A vortex tube consists of vortex lines drawn through points of C.

Helmholtz's Theorem *Assume the fluid is isentropic. Then*

(*a*) *If C_1 and C_2 are any two curves encircling the vortex tube, then*

$$\int_{C_1} \mathbf{u} \cdot d\mathbf{s} = \int_{C_2} \mathbf{u} \cdot d\mathbf{s}.$$

This common value is called the ***strength*** *of the vortex tube.*

(*b*) *The strength of the vortex tube is constant in time, as the tube moves with the fluid.*

Proof (a) Let C_1 and C_2 be oriented as in Figure 1.2.6.

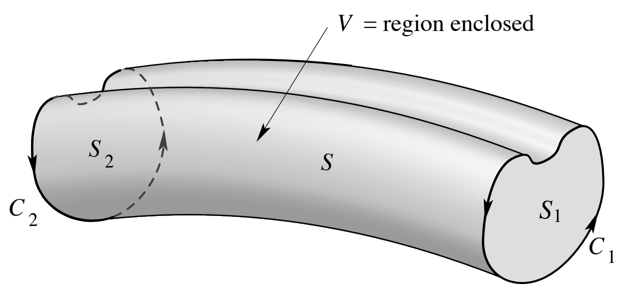

FIGURE 1.2.6. A vortex tube enclosed between two curves, C_1 and C_2.

The lateral surface of the vortex tube enclosed between C_1 and C_2 is denoted by S, and the end faces with boundaries C_1 and C_2 are denoted by S_1 and S_2, respectively. Since $\boldsymbol{\xi}$ is tangent to the lateral surface, S is a vortex sheet. Let V denote the region of the vortex tube between C_1 and C_2 and $\Sigma = S \cup S_1 \cup S_2$ denote the boundary of V. By Gauss' theorem,

$$0 = \int_V \nabla \cdot \boldsymbol{\xi}\, dx = \int_\Sigma \boldsymbol{\xi} \cdot d\mathbf{A} = \int_{S_1 \cup S_2} \boldsymbol{\xi} \cdot d\mathbf{A} + \int_S \boldsymbol{\xi} \cdot d\mathbf{A}.$$

By Stokes' theorem

$$\int_{C_1} \mathbf{u} \cdot d\mathbf{s} = \int_{S_1} \boldsymbol{\xi} \cdot d\mathbf{A} \quad \text{and} \quad \int_{C_2} \mathbf{u} \cdot d\mathbf{s} = -\int_{S_2} \boldsymbol{\xi} \cdot d\mathbf{A},$$

so (a) holds. Part (b) now follows from Kelvin's circulation theorem. ■

Observe that if a vortex tube gets stretched and its cross-sectional area decreases, then the magnitude of $\boldsymbol{\xi}$ must increase. Thus, the stretching of vortex tubes can increase vorticity, but it cannot create it.

A vortex tube with nonzero strength cannot "end" in the interior of the fluid. It either forms a ring (such as the smoke from a cigarette), extends to infinity, or is attached to a solid boundary. The usual argument supporting this statement goes like this: suppose the tube ended at a certain cross section S, inside the fluid. Because the tube cannot be extended, we must have $\boldsymbol{\xi} = \mathbf{0}$ on C_1. Thus, the strength is zero—a contradiction.

This "proof" is hopelessly incomplete. First of all, why should a vortex tube end in a nice regular way on a surface? Why can't it split in two, as in Figure 1.2.7? There is no *a priori* reason why this sort of thing cannot happen unless we merely exclude it by tacit assumption.[4]. In particular, note that the assertion often made that a vortex line cannot end in the fluid is clearly false if we allow $\boldsymbol{\xi}$ to have zeros and probably is false even if $\boldsymbol{\xi}$ has no zeros (an orbit of a vector field can wander around forever without accumulating at an endpoint—as with a line with irrational slope on a torus)

Thus, our assertion about vortex tubes "ending" is correct if we interpret "ending" properly. But the reader is cautioned that this may not be all that can happen, and that this time-honored statement is not at all a proved theorem.

The difference between the two-dimensional and three-dimensional conservation laws for vorticity is very important. The conservation of vorticity $(1.2.7)'$ in two dimensions is a helpful tool in establishing a rigorous theory of existence and uniqueness of the Euler (and later Navier–Stokes) equations. The lack of the same kind of conservation in three dimensions is a major obstacle to the rigorous understanding of crucial properties of the solutions of the equations of fluid dynamics. The main point here is to get existence theorems for *all time*. At the moment, it is known only in two dimension that all time smooth solutions exist.

[4]H. Lamb [1895] *Mathematical Theory of the Motion of Fluids,* Cambridge Univ. Press, p. 149.

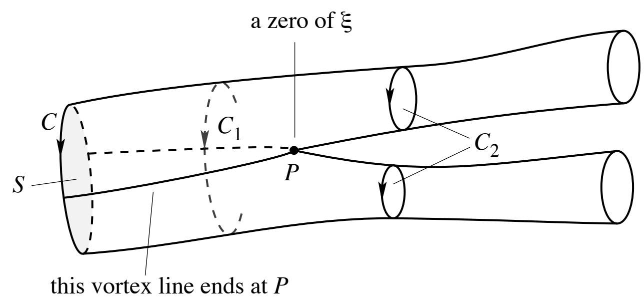

FIGURE 1.2.7. Can this be a vortex tube generated by S? Is the circulation around C_1 equal to that around C_2?

Our last main goal in this section is to develop the vorticity equation somewhat further for the important special case of incompressible flow. For two-dimensional homogeneous incompressible flow, the ***vorticity equation*** is

$$\frac{D\xi}{Dt} = \partial_t \xi + (\mathbf{u} \cdot \nabla)\xi = 0, \tag{1.2.12}$$

where $\xi = \xi(x, y, t) = \partial_x v - \partial_y u$ is the (scalar) vorticity field of the flow and u, v are the components of $\mathbf{u}$. Assume that the flow is contained in some plane domain D with a fixed boundary ∂D, with the boundary condition

$$\mathbf{u} \cdot \mathbf{n} = 0 \qquad \text{on } \partial D, \tag{1.2.13}$$

where $\mathbf{n}$ is the unit outward normal to ∂D. Let us assume D is simply connected (i.e., has no "holes"). Then, by incompressibility, $\partial_x u = -\partial_y v$, and so from vector calculus there is a scalar function $\psi(x, y, t)$ on D unique up to an additive constant such that

$$u = \partial_y \psi \quad \text{and} \quad v = -\partial_x \psi. \tag{1.2.14}$$

The function ψ is the ***stream function*** for fixed t; streamlines lie on level curves of ψ. Indeed, let $(x(s), y(s))$ be a streamline, so $x' = u(x, y)$ and $y' = v(x, y)$. Then

$$\frac{d}{ds}\psi(x(s), y(s), t) = \partial_x \psi \cdot x' + \partial_y \psi \cdot y' = -vu + uv = 0.$$

In particular, by (1.2.13), ∂D lies on a level curve of ψ, and we can adjust the constant so that

$$\psi(x, y, t) = 0 \qquad \text{for } (x, y) \in \partial D.$$

This convention and (1.2.14) determine ψ uniquely. (∂D need not be a whole streamline, but can be composed of streamlines separated by zeros of $\mathbf{u}$, that is, by ***stagnation points***.) The scalar vorticity is now given by

$$\xi = \partial_x v - \partial_y u = -\partial_x^2 \psi - \partial_y^2 \psi = -\Delta\psi,$$

where $\Delta = \partial_x^2 + \partial_y^2$ is the Laplace operator in the plane.

We can summarize the equations for ξ for two-dimensional incompressible flow as follows:

$$\left.\begin{aligned}
&\tfrac{D\xi}{Dt} \equiv \partial_t\xi + (\mathbf{u}\cdot\nabla)\xi = 0,\\
&\Delta\psi = -\xi,\\
&\text{with}\\
&\psi = 0 \quad \text{on} \quad \partial D,\\
&\text{and with}\\
&u = \partial_y\psi \quad \text{and} \quad v = -\partial_x\psi.
\end{aligned}\right\} \tag{1.2.15}$$

These equations completely determine the flow. Note that given ξ, the function ψ is determined by $\Delta\psi = -\xi$ and the boundary conditions, and hence $\mathbf{u}$ by the last equations in (1.2.15). Thus, ξ completely determines $\partial_t\xi$ and hence the evolution of ξ and, through it, ψ and $\mathbf{u}$.

Another remark is useful:

$$\begin{aligned}
(\mathbf{u}\cdot\nabla)\xi = u\partial_x\xi + v\partial_y\xi &= (\partial_y\psi)(\partial_x\xi) - (\partial_x\psi)(\partial_y\xi)\\
&= \det\begin{bmatrix} \partial_x\xi & \partial_y\xi \\ \partial_x\psi & \partial_y\psi \end{bmatrix} = J(\xi,\psi),
\end{aligned}$$

the Jacobian of ξ and ψ. Thus, *the flow is stationary* (*time independent*) *if and only if ξ and ψ are functionally dependent.* (If functional dependence holds at one instant it will hold for all time as a consequence.)

Example Suppose at $t = 0$ the stream function $\psi(x,y)$ is a function only of the radial distance $r = (x^2 + y^2)^{1/2}$. Thus, the streamlines are concentric circles. Write $\psi(x,y) = \psi(r)$ and assume $\psi_r > 0$. The velocity vector is given by

$$u = \partial_y\psi = \partial_r\psi\partial_y r = \frac{y}{r}\partial_r\psi, \tag{1.2.16}$$

$$v = -\partial_x\psi = -\partial_r\psi\partial_x r = -\frac{x}{r}\,\partial_r\psi, \tag{1.2.17}$$

that is, $\mathbf{u}$ is tangent to the circle of radius r with magnitude $|\partial_r\psi|$ and oriented clockwise if $\psi_r > 0$ and counterclockwise if $\psi_r < 0$. Next, observe that

$$\xi = -\Delta\psi = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right),$$

a function of r alone. Because $\psi_r \neq 0, r$ is a function of ψ so ξ is also a function of ψ. Thus, $J(\xi,\psi) = 0$. Hence, motion in concentric circles with $\mathbf{u}$ defined as above is a solution of the two-dimensional *stationary* incompressible equations of ideal flow.

For three-dimensional incompressible ideal flow, the analogue of (1.2.15) is

$$\left.\begin{aligned} \frac{D\boldsymbol{\xi}}{Dt} - (\boldsymbol{\xi}\cdot\nabla)\mathbf{u} = 0, \\ \Delta\mathbf{A} = -\boldsymbol{\xi}, \qquad \operatorname{div}\mathbf{A} = 0, \\ \mathbf{u} = \nabla\times\mathbf{A}. \end{aligned}\right\} \tag{1.2.18}$$

Here we used $\nabla\cdot\mathbf{u} = 0$ to write $\mathbf{u} = \nabla\times\mathbf{A}$, where $\operatorname{div}\mathbf{A} = 0$. (This requires not that D be simply connected, but that it not have any "solid holes" in it; for instance, if D is convex, this will hold.) Then

$$\boldsymbol{\xi} = \operatorname{curl}\mathbf{u} = \operatorname{curl}(\operatorname{curl}\mathbf{A}) = -\Delta A + \nabla(\operatorname{div}\mathbf{A}) = -\Delta\mathbf{A}.$$

One of the troubles with (1.2.18) is that given $\boldsymbol{\xi}$, the vector field $\mathbf{A}$ is not uniquely determined (we cannot impose boundary condition such as $\mathbf{A} = 0$ on ∂D because $\mathbf{A}$ need not be constant on ∂D as was the case with ψ). ♦

Exercises

◇ **Exercise 1.2-1** Derive a formula akin to the transport theorem and Kelvin's circulation theorem for

$$\frac{d}{dt}\int_{S_t}\mathbf{v}\cdot\mathbf{n}\,dA,$$

where S_t is a *moving surface* and $\mathbf{v}$ is a vector field.

◇ **Exercise 1.2-2** ***Couette flow.*** Let Ω be the region between two concentric cylinders of radii R_1 and R_2, where $R_1 < R_2$. Define $\mathbf{v}$ in cylindrical coordinates by

$$v_r = 0, \qquad v_z = 0,$$

and

$$v_\theta = \frac{A}{r} + Br,$$

where

$$A = -\frac{R_1^2R_2^2(\omega_2-\omega_1)}{R_2^2-R_2^1} \quad\text{and}\quad B = -\frac{R_1^2\,\omega_1 - R_2^2\,\omega_2}{R_2^2-R_1^2}.$$

Show that

(i) $\mathbf{v}$ is a stationary solution of Euler's equations with $\rho = 1$;

(ii) $\boldsymbol{\omega} = \nabla\times\mathbf{v} = (0, 0, 2B)$;

(iii) the deformation tensor is

$$D = -\frac{A}{r^2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and discuss its physical meaning;

(iv) the angular velocity of the flow on the two cylinders is ω_1 and ω_2.

1.3 The Navier–Stokes Equations

In §**1.1** we defined an ideal fluid as one in which forces across a surface were normal to that surface. We now consider more general fluids. To understand the need for the generalization beyond the examples already given, consider the situation shown in Figure 1.3.1. Here the velocity field $\mathbf{u}$ is parallel to a surface S but jumps in magnitude either suddenly or rapidly as we cross S. If the forces are all normal to S, there will be no transfer of momentum between the fluid volumes denoted by B and B' in Figure 1.3.1. However, if we remember the kinetic theory of matter, we see that this is actually unreasonable. Faster molecules from above S will diffuse across S and impart momentum to the fluid below, and, likewise, slower molecules from below S will diffuse across S to slow down the fluid above S. For reasonably fast changes in velocity over short distance, this effect is important.[5]

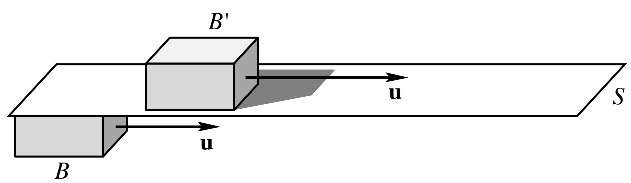

FIGURE 1.3.1. Faster molecules in B' can diffuse across S and impart momentum to B.

We thus change our previous definition. Instead of assuming that

$$\text{force on } S \text{ per unit area} = -p(\mathbf{x}, t)\mathbf{n},$$

where $\mathbf{n}$ is the normal to S, we now assume that

$$\text{force on } S \text{ per unit area} = -p(\mathbf{x}, t)\mathbf{n} + \boldsymbol{\sigma}(\mathbf{x}, t) \cdot \mathbf{n}, \tag{1.3.1}$$

where $\boldsymbol{\sigma}$ is a *matrix* called the ***stress tensor***, about which some assumptions will have to be made. The new feature is that $\boldsymbol{\sigma} \cdot \mathbf{n}$ need not be parallel to $\mathbf{n}$. The separation of the forces into pressure and other forces in (1.3.1) is somewhat ambiguous because $\boldsymbol{\sigma} \cdot \mathbf{n}$ may contain a component parallel to $\mathbf{n}$. This issue will be resolved later when we give a more definite functional form to $\boldsymbol{\sigma}$.

[5] For more information, see J. Jeans [1867] *An Introduction to the Kinetic Theory of Gases*, Cambridge Univ. Press.

As before, Newton's second law states that the rate of change of any moving portion of fluid W_t equals the force acting on it (balance of momentum):

$$\frac{d}{dt}\int_{W_t} \rho \mathbf{u}\, dV = -\int_{\partial W_t} (p \cdot \mathbf{n} - \boldsymbol{\sigma} \cdot \mathbf{n})\, dA$$

(compare (BM3) in §**1.1**). Thus, we see that $\boldsymbol{\sigma}$ modifies the transport of momentum across the boundary of W_t. We will choose $\boldsymbol{\sigma}$ so that it approximates in a reasonable way the transport of momentum by molecular motion.

One can legitimately ask why the force (1.3.1) acting on S should be a *linear* function of $\mathbf{n}$. In fact, if one just assumes the force is a continuous function of $\mathbf{n}$, then, using balance of momentum, one can *prove* it is linear in $\mathbf{n}$. This result is called ***Cauchy's Theorem***.[6]

Our assumptions on $\boldsymbol{\sigma}$ are as follows:

1. $\boldsymbol{\sigma}$ *depends linearly on the velocity gradients* $\nabla \mathbf{u}$ that is, $\boldsymbol{\sigma}$ is related to $\nabla \mathbf{u}$ by some linear transformation at each point.

2. $\boldsymbol{\sigma}$ *is invariant under rigid body rotations*, that is, if $\mathbf{U}$ is an orthogonal matrix,

$$\boldsymbol{\sigma}(\mathbf{U} \cdot \nabla \mathbf{u} \cdot \mathbf{U}^{-1}) = \mathbf{U} \cdot \boldsymbol{\sigma}(\nabla \mathbf{u}) \cdot \mathbf{U}^{-1}.$$

   This is reasonable, because when a fluid undergoes a rigid body rotation, there should be no diffusion of momentum.

3. $\boldsymbol{\sigma}$ *is symmetric*. This property can be deduced as a consequence of balance of angular momentum.[7]

Since $\boldsymbol{\sigma}$ is symmetric, if follows from properties 1 and 2 that $\boldsymbol{\sigma}$ can depend only on the symmetric part of $\nabla \mathbf{u}$; that is, on the deformation $\mathbf{D}$. Because $\boldsymbol{\sigma}$ is a linear function of $\mathbf{D}$, $\boldsymbol{\sigma}$ and $\mathbf{D}$ commute and so can be simultaneously diagonalized. Thus, the eigenvalues of $\boldsymbol{\sigma}$ are linear functions of those of $\mathbf{D}$. By property 2, they must also be symmetric because we can choose $\mathbf{U}$ to permute two eigenvalues of $\mathbf{D}$ (by rotating through an angle $\pi/2$ about an eigenvector), and this must permute the corresponding eigenvalues of $\boldsymbol{\sigma}$. The only linear functions that are symmetric in this sense are of the form

$$\sigma_i = \lambda(d_1 + d_2 + d_3) + 2\mu d_i, \qquad i = 1, 2, 3,$$

where σ_i are the eigenvalues of $\boldsymbol{\sigma}$, and d_i are those of $\mathbf{D}$. This defines the constants λ and μ. Recalling that $d_1 + d_2 + d_3 = \operatorname{div} \mathbf{u}$, we can use property 2 to transform σ_i back to the usual basis and deduce that

$$\boldsymbol{\sigma} = \lambda(\operatorname{div} \mathbf{u})\mathbf{I} + 2\mu\, \mathbf{D}, \tag{1.3.2}$$

[6] For a proof and further references, see, for example, Marsden and Hughes [1994].
[7] Op. cit.

where $\mathbf{I}$ is the identity. We can rewrite this by putting all the trace in one term:

$$\boldsymbol{\sigma} = 2\mu[\mathbf{D} - \tfrac{1}{3}(\operatorname{div}\mathbf{u})\mathbf{I}] + \zeta(\operatorname{div}\mathbf{u})\mathbf{I} \tag{1.3.2$'$}$$

where μ is the ***first coefficient of viscosity***, and $\zeta = \lambda + \frac{2}{3}\mu$ is the ***second coefficient of viscosity***.

If we employ the transport theorem and the divergence theorem, as we did in connection with (BM3), balance of momentum yields the ***Navier–Stokes equations***,

$$\boxed{\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + (\lambda+\mu)\nabla(\operatorname{div}\mathbf{u}) + \mu\Delta\mathbf{u}} \tag{1.3.3}$$

where

$$\Delta\mathbf{u} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\mathbf{u}$$

is the Laplacian of $\mathbf{u}$. Together with the equation of continuity and an energy equation, (1.3.3) completely describes the flow of a compressible viscous fluid.

In the case of incompressible homogeneous flow $\rho = \rho_0 =$ constant, the complete set of equations becomes the ***Navier–Stokes equations for incompressible flow***,

$$\boxed{\begin{gathered}\frac{D\mathbf{u}}{Dt} = -\operatorname{grad} p' + \nu\Delta\mathbf{u}\\ \operatorname{div}\mathbf{u} = 0\end{gathered}} \tag{1.3.4}$$

where $\nu = \mu/\rho_0$ is the coefficient of *kinematic viscosity*, and $p' = p/\rho_0$.

These equations are supplemented by boundary conditions. For Euler's equations for ideal flow we use $\mathbf{u}\cdot\mathbf{n} = 0$, that is, fluid does not cross the boundary but may move tangentially to the boundary. For the Navier–Stokes equations, the extra term $\nu\Delta\mathbf{u}$ raises the number of derivatives of $\mathbf{u}$ involved from one to two. For both experimental and mathematical reasons, this is accompanied by an increase in the number of boundary conditions. For instance, on a solid wall at rest we add the condition that the tangential velocity also be zero (the "no-slip condition"), so the full boundary conditions are simply

$$\mathbf{u} = \mathbf{0} \ \text{ on solid walls at rest.}$$

The mathematical need for extra boundary conditions hinges on their role in proving that the equations are well posed; that is, that a unique solution exists and depends continuously on the initial data. In three dimensions, it is known that smooth solutions to the incompressible equations exist for a short time and depend continuously on the initial data.[8] It is a major open problem in fluid mechanics to prove or disprove that solutions of the incompressible equations exist for all time. In two dimensions, solutions are known to exist for all time, for both viscous and inviscid flow[9]. In any case, adding the tangential boundary condition is crucial for viscous flow.

The physical need for the extra boundary conditions comes from simple experiments involving flow past a solid wall. For example, if dye is injected into flow down a pipe and is carefully watched near the boundary, one sees that the velocity approaches zero at the boundary to a high degree of precision. The no-slip condition is also reasonable if one contemplates the physical mechanism responsible for the viscous terms, namely, molecular diffusion. Our opening example indicates that molecular interaction between the solid wall with zero tangential velocity (or zero average velocity on the molecular level) should impart the same condition to the immediately adjacent fluid.

Another crucial feature of the boundary condition $\mathbf{u} = \mathbf{0}$ is that it provides a mechanism by which a boundary can produce vorticity in the fluid. We shall describe this in some detail in Chapter **2**.

Next, we shall discuss some scaling properties of the Navier–Stokes equations with the aim of introducing a parameter (the Reynolds number) that measures the effect of viscosity on the flow.

For a given problem, let L be a ***characteristic length*** and U a ***characteristic velocity***. These numbers are chosen in a somewhat arbitrary way. For example, if we consider flow past a sphere, L could be either the radius or the diameter of the sphere, and U could be the magnitude of the fluid velocity at infinity. L and U are merely reasonable length and velocity scales typical of the flow at hand. Their choice then determines a time scale by $T = L/U$.

We can measure $\mathbf{x}, \mathbf{u}$, and t as fractions of these scales, by changing variables and introducing the following dimensionless quantities

$$\mathbf{u}' = \frac{\mathbf{u}}{U}, \quad \mathbf{x}' = \frac{\mathbf{x}}{L}, \quad \text{and} \quad t' = \frac{t}{T}.$$

The x component of the (homogeneous) incompressible Navier–Stokes equation is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right].$$

---

[8] For a review of much of what is known, see O. A. Ladyzhenskaya [1969] *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach. See also R. Temam [1977] *Navier–Stokes Equations*, North Holland.

[9] Op. cit. and W. Wolibner, *Math. Zeit.* **37** [1933], 698–726; V. Judovich, *Mat. Sb. N.S.* **64** [1964], 562–588; and T. Kato, *Arch. Rational Mech. Anal.* **25** [1967], 188–200.

The change of variables produces

$$\frac{\partial(u'U)}{\partial t'}\frac{\partial t'}{\partial t} + Uu'\frac{\partial(u'U)}{\partial x'}\frac{\partial x'}{\partial x} + Uv'\frac{\partial(u'U)}{\partial y'}\frac{\partial y'}{\partial y} + Uw'\frac{\partial(u'U)}{\partial z'}\frac{\partial w'}{\partial z}$$
$$= -\frac{1}{\rho_0}\frac{\partial p}{\partial x'}\frac{\partial x'}{\partial x} + \nu\left[\frac{\partial^2(u'U)}{\partial(Lx')^2} + \frac{\partial^2(u'U)}{\partial(Ly')^2} + \frac{\partial^2(u'U)}{\partial(Lz')^2}\right],$$

$$\left[\frac{U^2}{L}\right]\left[\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} + w'\frac{\partial u'}{\partial z'}\right]$$
$$= -\left[\frac{U^2}{L}\right]\frac{\partial(p/(\rho_0 U^2))}{\partial x'} + \left[\frac{U}{L^2}\right]\nu\left[\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2} + \frac{\partial^2 u'}{\partial z'^2}\right].$$

Similar equations hold for the y and z components. If we combine all three components and divide out by U^2/L, we obtain

$$\frac{\partial \mathbf{u}'}{\partial t'} + (\mathbf{u}'\cdot\nabla')\mathbf{u}' = -\operatorname{grad}\mathbf{p}' + \frac{\nu}{LU}\Delta'\mathbf{u}', \tag{1.3.5}$$

where $p' = p/(\rho_0 U^2)$. Incompressibility still reads

$$\operatorname{div}\mathbf{u}' = 0.$$

The equations (1.3.5) are the Navier–Stokes equations in dimensionless variables. We define the ***Reynolds number*** R to be the dimensionless number

$$R = \frac{LU}{\nu}.$$

For example, consider two flows past two spheres centered at the origin but with differing radii, one with a fluid where $U_\infty = 10$ km/hr past a sphere of radius 10 m and the other with the same fluid but with $U_\infty =$ 100 km/hr and radius = 1 m. If we choose L to be the radius and U to be the velocity U_∞ at infinity, then the Reynolds number is the same for each flow. The equations satisfied by the dimensionless variables are thus identical for the two flows.

Two flows with the same geometry and the same Reynolds number are called ***similar***. More precisely, let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two flows on regions D_1 and D_2 that are related by a scale factor λ so that $L_1 = \lambda L_2$. Let choices of U_1 and U_2 be made for each flow, and let the viscosities be ν_1 and ν_2 respectively. If

$$R_1 = R_2, \quad \text{i.e.,} \quad \frac{L_1 U_1}{\nu_1} = \frac{L_2 U_2}{\nu_2},$$

then the dimensionless velocity fields $\mathbf{u}_1'$ and $\mathbf{u}_2'$ satisfy exactly the same equation on the same region. Thus, we can conclude that $\mathbf{u}_1$ may be obtained from a suitably rescaled solution $\mathbf{u}_2$; in other words, $\mathbf{u}_1$ and $\mathbf{u}_2$ are similar.

This idea of the similarity of flows is used in the design of experimental models. For example, suppose we are contemplating a new design for an aircraft wing and we wish to know the behavior of a fluid flow around it. Rather than build the wing itself, it may be faster and more economical to perform the initial tests on a scaled-down version. We design our model so that it has the same geometry as the full-scale wing and we choose values for the undisturbed velocity, coefficient of viscosity, and so on, such that the Reynolds number for the flow in our experiment matches that of the actual flow. We can then expect the results of our experiment to be relevant to the actual flow over the full-scale wing.

We shall be especially interested in cases where R is large. We stress that one cannot say that if ν is small, then viscous effects are unimportant, because such a comment fails to consider the other dimensions of the problem, that is, "ν is small" *is not* a physically meaningful statement unless some scaling is chosen, but "$1/R$ is small" *is* a meaningful statement.

As with incompressible ideal flow, the pressure p in incompressible viscous flow is determined through the equation $\operatorname{div} \mathbf{u} = 0$. We now shall explore the role of the pressure in incompressible flow in more depth. Let D be a region in space (or in the plane) with smooth boundary ∂D.

We shall use the following decomposition theorem.

Helmholtz–Hodge Decomposition Theorem *A vector field* $\mathbf{w}$ *on* D *can be uniquely decomposed in the form*

$$\mathbf{w} = \mathbf{u} + \operatorname{grad} p, \tag{1.3.6}$$

where u *has zero divergence and is parallel to* ∂D*; that is,* $\mathbf{u} \cdot \mathbf{n} = 0$ *on* ∂D*.*

Proof First of all, we establish the orthogonality relation

$$\int_D \mathbf{u} \cdot \operatorname{grad} p \, dV = 0.$$

Indeed, by the identity

$$\operatorname{div}(p\mathbf{u}) = (\operatorname{div} \mathbf{u})p + \mathbf{u} \cdot \operatorname{grad} p,$$

the divergence theorem, and $\operatorname{div} \mathbf{u} = 0$, we get

$$\int_D \mathbf{u} \cdot \operatorname{grad} p \, dV = \int_D \operatorname{div}(p\mathbf{u}) \, dV = \int_{\partial D} p\mathbf{u} \cdot \mathbf{n} \, dA = 0,$$

because $\mathbf{u} \cdot \mathbf{n} = 0$ on ∂D. We use this orthogonality to prove uniqueness. Suppose the $\mathbf{w} = \mathbf{u}_1 + \operatorname{grad} p_1 = \mathbf{u}_2 + \operatorname{grad} p_2$. Then

$$0 = \mathbf{u}_1 - \mathbf{u}_2 + \operatorname{grad}(p_1 - p_2).$$

Taking the inner product with $\mathbf{u}_1 - \mathbf{u}_2$ and integrating, we get

$$0 = \int_D \left\{ \|\mathbf{u}_1 - \mathbf{u}_2\|^2 + (\mathbf{u}_1 - \mathbf{u}_2) \cdot \operatorname{grad}(p_1 - p_2) \right\} dV = \int_D \|\mathbf{u}_1 - \mathbf{u}_2\|^2 \, dV$$

by the orthogonality relation. It follows that $\mathbf{u}_1 = \mathbf{u}_2$, and so, $\operatorname{grad} p_1 = \operatorname{grad} p_2$ (which is the same thing as $p_1 = p_2 + \text{constant}$).

If $\mathbf{w} = \mathbf{u} + \operatorname{grad} p$, notice that $\operatorname{div} \mathbf{w} = \operatorname{div} \operatorname{grad} p = \Delta p$ and that $\mathbf{w} \cdot \mathbf{n} = \mathbf{n} \cdot \operatorname{grad} p$. We use this remark to prove existence. Indeed, given $\mathbf{w}$, let p be defined by the solution to the Neumann problem

$$\Delta p = \operatorname{div} \mathbf{w} \quad \text{in } D, \quad \text{with} \quad \frac{\partial p}{\partial n} = \mathbf{w} \cdot \mathbf{n} \quad \text{on } \partial D.$$

It is known[10] that the solution to this problem exists and is unique up to the addition of a constant to p. With this choice of p, define $\mathbf{u} = \mathbf{w} - \operatorname{grad} p$. Then, clearly $\mathbf{u}$ has the desired properties $\operatorname{div} \mathbf{u} = 0$, and also $\mathbf{u} \cdot \mathbf{n} = 0$ by construction of p. ■

The situation is shown schematically in Figure 1.3.2.

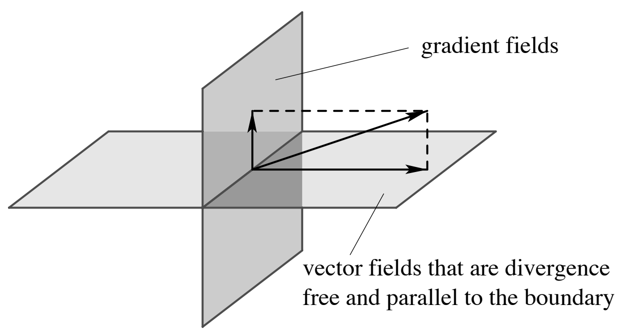

FIGURE 1.3.2. Decomposing a vector field into a divergence-free and gradient part.

It is natural to introduce the operator $\mathbb{P}$, an orthogonal projection operator, which maps $\mathbf{w}$ onto its divergence-free part $\mathbf{u}$. By the preceding theorem, $\mathbb{P}$ is well defined. Notice that by construction $\mathbb{P}$ is a linear operator and that

$$\mathbf{w} = \mathbb{P}\mathbf{w} + \operatorname{grad} p. \tag{1.3.7}$$

Also notice that

$$\mathbb{P}\mathbf{u} = \mathbf{u}$$

[10] See R. Courant and D. Hilbert [1953], *Methods of Mathematical Physics,* Wiley. The equation $\Delta p = f, \partial p/\partial n = g$ has a solution unique up to a constant if and only if $\int_D f dV = \int_{\partial D} g \, dA$. The divergence theorem ensures that this condition is satisfied in our case.

provided $\operatorname{div} \mathbf{u} = 0$ and $\mathbf{u} \cdot \mathbf{n} = 0$, and that

$$\mathbb{P}(\operatorname{grad} p) = 0.$$

Now we apply these ideas to the incompressible Navier–Stokes equations (1.3.5). If we apply the operator $\mathbb{P}$ to both sides, we obtain

$$\mathbb{P}(\partial_t \mathbf{u} + \operatorname{grad} p) = \mathbb{P}\left(-(\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{R}\Delta \mathbf{u}\right).$$

Because $\mathbf{u}$ is divergence-free and vanishes on the boundary, the same is true of $\partial_t \mathbf{u}$ (if $\mathbf{u}$ is smooth enough). Thus, by (1.3.7), $\mathbb{P}\partial_t \mathbf{u} = \partial_t \mathbf{u}$. Because $\mathbb{P}(\operatorname{grad} p) = 0$, we get

$$\partial_t \mathbf{u} = \mathbb{P}\left(-\mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{R}\Delta \mathbf{u}\right). \tag{1.3.8}$$

Although $\Delta \mathbf{u}$ is divergence free, it need not be parallel to the boundary and so we *cannot* simply write $\mathbb{P}\Delta \mathbf{u} = \mathbf{0}$. This form (1.3.8) of the Navier–Stokes equations eliminates the pressure and expresses $\partial_t \mathbf{u}$ in terms of $\mathbf{u}$ alone. The pressure can then be recovered as the gradient part of

$$-\mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{R}\Delta \mathbf{u}.$$

This form (1.3.8) of the equations is not only of theoretical interest, shedding light on the role of the pressure, but is of practical interest for numerical algorithms.[11]

The pressure in compressible flows is conceptually different than in incompressible flows just as it was in ideal flow. If we think of viscous flow as ideal flow with viscous effects added on, it is not unreasonable to assume that p is still a function of ρ.

A note of caution is appropriate here. The expressions for $p(\rho)$ used in practical situations are often borrowed from the science of equilibrium thermodynamics. It is not obvious that p as defined here (through equation (1.3.1)) is identical to p as defined in that other science. Not all quantities called p are equal. The use of expressions from equilibrium thermodynamics requires an additional physical justification, which is indeed often available, but which should not be forgotten.

According to the analysis given earlier, the pressure p in incompressible flow is determined by the equation of continuity $\operatorname{div} \mathbf{u} = 0$. To see why this

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[11] See, for instance, A. J. Chorin, *Math. Comp.* **23** [1969], 341-353 for algorithms, and D. Ebin and J. E. Marsden, *Ann. of Math.* **92** [1970], 102–163 for a theoretical investigation of the projection operator and the use of material coordinates.

is physically reasonable, consider a compressible flow with $p = p(\rho)$, where $p'(\rho) > 0$. If fluid flows into a given fixed volume V, the density in V will increase, and if $p'(\rho) > 0$, then p in V will also increase. If either the change in ρ is large enough or $p'(\rho)$ is large enough, $-\text{grad}\ p$ at the boundary of V will begin to point away from V, and through the term $-\text{grad}\ p$ in the equation for $\partial_t \mathbf{u}$, this will cause the fluid to flow away from V. Thus, the pressure controls and moderates the variations in density. If the density is to remain invariant, this must be accomplished by an appropriate p, that is, $\operatorname{div} \mathbf{u} = 0$ determines p.

In the Navier–Stokes equations for a viscous incompressible fluid, namely,

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \frac{1}{R}\Delta \mathbf{u},$$

we call

$$\frac{1}{R}\Delta \mathbf{u}, \quad \text{the } \textbf{\textit{diffusion}} \text{ or } \textbf{\textit{dissipation}} \text{ term,}$$

and

$$(\mathbf{u} \cdot \nabla)\mathbf{u}, \quad \text{the } \textbf{\textit{inertia}} \text{ or } \textbf{\textit{convective}} \text{ term.}$$

The equations say that $\mathbf{u}$ is convected subject to pressure forces and, at the same time, is dissipated. Suppose R is very small. If we write the equations in the form $\partial_t \mathbf{u} = \mathbb{P}(-\mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{R}\Delta \mathbf{u})$, we see that they are approximated by

$$\partial_t \mathbf{u} = \mathbb{P}\left(\frac{1}{R}\Delta \mathbf{u}\right),$$

that is,

$$\partial_t \mathbf{u} = -\operatorname{grad} p + \frac{1}{R}\Delta \mathbf{u} \qquad \text{and} \qquad \operatorname{div} \mathbf{u} = 0,$$

which are the ***Stokes' equations*** for incompressible flow. These are linear equations of "parabolic" type. For small R (i.e., slow velocity, large viscosity, or small bodies), the solution of the Stokes equation provides a good approximation to the solution of the Navier–Stokes equations. Later, we shall mostly be interested in flows with the large R; for these the inertial term is important and in some sense is dominant. We hesitate and say "in some sense" because no matter how small $(1/R)\Delta \mathbf{u}$ may be, it still produces a large effect, namely, the change in boundary conditions from $\mathbf{u} \cdot \mathbf{n} = 0$ when $(1/R)\Delta \mathbf{u}$ is absent to $\mathbf{u} = \mathbf{0}$ when it is present.

There is a major difference between the ideal and viscous flow with regard to the energy of the fluid. The viscous terms provide a mechanism by which macroscopic energy can be converted into internal energy. General principles of thermodynamics state that this energy transfer is one-way. In particular, for incompressible flow, we should have

$$\frac{d}{dt} E_{\text{kinetic}} \leq 0. \tag{1.3.9}$$

We calculate $(d/dt)E_{\text{kinetic}}$ for incompressible viscous flow using the transport theorem, as we did in §**1.1**. We get

$$\begin{aligned}\frac{d}{dt}E_{\text{kinetic}} &= \frac{d}{dt}\,\tfrac{1}{2}\int_D \rho\|\mathbf{u}\|^2 dV = \int_D \rho\mathbf{u}\cdot\frac{D\mathbf{u}}{Dt}\,dV\\ &= \int_D\left(-\mathbf{u}\cdot\nabla p + \frac{1}{R}\mathbf{u}\cdot\Delta\mathbf{u}\right)dV,\end{aligned}$$

by (1.3.3) and $\operatorname{div}\mathbf{u} = 0$. Because $\mathbf{u}$ is orthogonal to $\operatorname{grad} p$, we get

$$\frac{d}{dt}E_{\text{kinetic}} = \frac{1}{R}\int_D \mathbf{u}\cdot\Delta\mathbf{u}\,dV.$$

The vector identity $\operatorname{div}(f\mathbf{V}) = f\operatorname{div}\mathbf{V} + \mathbf{V}\cdot\nabla f$ gives

$$\begin{aligned}&\nabla\cdot(u\nabla u + v\nabla v + w\nabla w)\\ &\qquad = \nabla u\cdot\nabla u + \nabla v\cdot\nabla v + \nabla w\cdot\nabla w + u\Delta u + v\Delta v + w\Delta w.\end{aligned}$$

This equation, the divergence theorem, and the boundary condition $\mathbf{u} = \mathbf{0}$ on ∂D enable us to simplify the expression for $(d/dt)E_{\text{kinetic}}$ to

$$\frac{d}{dt}E_{\text{kinetic}} = -\mu\int_D \|\nabla\mathbf{u}\|^2\,dV, \tag{1.3.10}$$

where $\|\nabla\mathbf{u}\|^2 = \nabla\mathbf{u}\cdot\nabla\mathbf{u} = \|\nabla u\|^2 + \|\nabla v\|^2 + \|\nabla w\|^2$. Notice that (1.3.9) and (1.3.10) are compatible exactly when $\mu \geq 0$ (or, equivalently, $\nu \geq 0$ or $0 < R \leq \infty$). In other words, there is no such thing as "negative viscosity."

A similar analysis for compressible flow and making use of (1.3.2)' leads to the inequalities

$$\mu \geq 0 \quad \text{and} \quad \lambda + \tfrac{2}{3}\mu \geq 0$$

and with $\boldsymbol{\sigma}$ given by (1.3.2).[12]

At the end of §**1.1** we noted that ideal flow in a channel leads to unreasonable results. We now reconsider this example with viscous effects.

Example Consider stationary viscous incompressible flow between two stationary plates located at $y = 0$ and $y = 1$, as shown in Figure 1.3.3. We seek a solution for which $\mathbf{u}(x, y) = (u(x, y), 0)$ and p is only a function of x, with $p_1 = p(0), p_2 = p(L)$, and $p_1 > p_2$, so the fluid is "pushed" in the positive x direction. The incompressible Navier–Stokes equations are

$$\partial_x u = 0 \qquad \text{(incompressibility)}$$

and

$$0 = -u\,\partial_x u - \partial_x p + \frac{1}{R}\left[\partial_x^2 u + \partial_y^2 u\right]$$

[12] See, for example, S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-uniform Gases*, Cambridge University Press, 1958.

with boundary conditions $u(x,0) = u(x,1) = 0$. Because $\partial_x u = 0, u$ is only a function of y and thus, writing $u(x,y) = u(y)$, we obtain

$$p' = \frac{1}{R}u''.$$

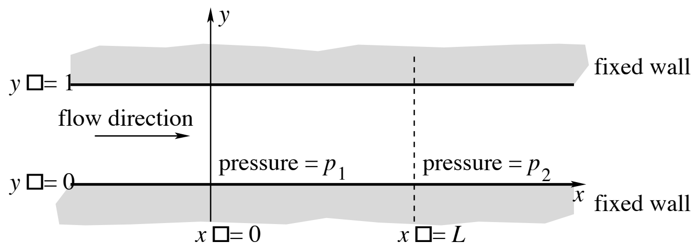

FIGURE 1.3.3. Flow between two parallel plates; the fluid is pushed from left to right and correspondingly, $p_1 > p_2$.

Because each side depends on different variables,

$$p' = \text{constant}, \quad \frac{1}{R}u'' = \text{constant}.$$

Integration gives

$$p(x) = p_1 - \frac{\Delta p}{L}x, \qquad \Delta p = p_1 - p_2,$$

and

$$u(y) = y(1-y)R\frac{\Delta p}{2L}.$$

Notice that the velocity profile is a parabola (Figure 1.3.4).

The presence of viscosity allows the pressure forces to be balanced by the term $\frac{1}{R}u''(y)$ and allows the fluid to achieve a stationary state. We saw that this was not possible for ideal flow. ♦

Next we consider the vorticity equation for (homogeneous) viscous incompressible flow. In the two-dimensional case we proved in §**1.2** (see equation (1.2.12)) that for isentropic ideal plane flow, $D\xi/Dt = 0$. The derivation is readily modified to cover viscous incompressible flow; the result is

$$\frac{D\xi}{Dt} = \frac{1}{R}\Delta\xi. \tag{1.3.11}$$

This shows that the vorticity is diffused by viscosity as well as being tranported by the flow. Introduce the stream function $\psi(x,y,t)$ by means of

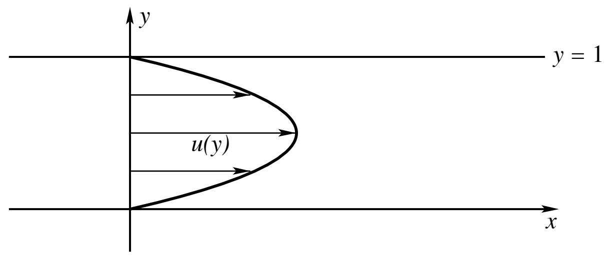

FIGURE 1.3.4. Viscous flow between two plates.

$(1.2.15)_2$ and $(1.2.15)_3$ as before. We saw that we could impose the boundary condition $\psi = 0$ on ∂D. Now, however, the no-slip condition $\mathbf{u} = \mathbf{0}$ on ∂D implies that

$$\partial_x \psi = 0 = \partial_y \psi \quad \text{on } \partial D$$

by $(1.2.15)_3$. Because $\psi = 0$ on ∂D implies that the tangential derivative of ψ on ∂D vanishes, we get the extra boundary condition

$$\frac{\partial \psi}{\partial \mathbf{n}} = 0 \quad \text{on } \partial D$$

This extra condition should be somewhat mystifying; certainly we cannot impose it when we solve $\Delta\psi = -\xi, \psi = 0$ on ∂D, because this problem already has a solution. Thus, it is not clear how to get the system

$$\left.\begin{array}{c} \dfrac{D\xi}{Dt} = \dfrac{1}{R}\Delta\xi, \\ \Delta\psi = -\xi, \qquad \psi = 0 \quad \text{on } \partial D, \\ u = \partial_y\psi, \qquad v = -\partial_x\psi \end{array}\right\} \tag{1.3.12}$$

to work. We shall study this problem in §**2.2**.

For three-dimensional viscous incompressible flow, the vorticity equation is

$$\frac{D\boldsymbol{\xi}}{Dt} - (\boldsymbol{\xi}\cdot\nabla)\mathbf{u} = \frac{1}{R}\Delta\boldsymbol{\xi}. \tag{1.3.13}$$

Thus, vorticity is convected, stretched, and diffused. (The left-hand side of (1.3.13) is called the ***Lie derivative***. It is this *combination*, rather than each term *separately*, that makes coordinate independent sense.) Here the problems with getting a system like (1.3.12) are even worse; even in the isentropic case we had trouble with (1.2.16) because of boundary conditions.

For viscous flow, circulation is no longer a constant of the motion. One might suspect from (1.3.13) that if $\boldsymbol{\xi} = \mathbf{0}$ at $t = 0$, then $\boldsymbol{\xi} = \mathbf{0}$ for all time. However, this is not true: viscous flow allows for the generation of vorticity.

This is possible because of the difference in boundary conditions between ideal and viscous flows. The mechanism of vorticity generation is related to the difficulties with the boundary conditions in equations (1.3.12) and will be discussed in §**2.2**.

For many of our discussions we have made the assumption of incompressibility. We now give a heuristic analysis of when such an assumption will be reasonable and when, instead, the compressible equations should be used. We shall do this in the context of isentropic stationary flows for simplicity. Assume that we have an ***equation of state***

$$p = p(\rho), \qquad p'(\rho) > 0.$$

Define

$$c = \sqrt{p'(\rho)}.$$

For reasons that will become clear later, c is called the ***sound speed*** of the fluid. Thus, we have

$$c^2 d\rho = dp. \tag{1.3.14}$$

Let $u = \|\mathbf{u}\|$ be the flow speed. One calls $M = u/c$ the (local) ***Mach number*** of the flow; it is a function of position in the flow. From Bernoulli's theorem proved in §**1.1**,

$$\frac{u^2}{2} + \int \frac{dp}{\rho(p)} = \text{constant on streamlines.} \tag{1.3.15}$$

Also, differentiating the continuity equation in the form (1.2.10) along streamlines gives

$$0 = J d\rho + \rho\, dJ, \tag{1.3.16}$$

where J is the Jacobian of the flow map. Combining (1.3.14), (1.3.15), and (1.3.16) we get

$$\frac{dJ}{J} = -M\frac{du}{c}.$$

The flow will be approximately incompressible if J changes only by a small amount along streamlines. Thus, a steady flow can be viewed as incompressible if the flow speed is much less than the sound speed,

$$u \ll c, \quad \text{i.e.,} \quad M \ll 1,$$

or if changes in the speed along streamlines are very small compared to the sound speed.

For example, for equations of state of the kind associated with ideal gases,

$$p = A\rho^\gamma, \qquad \gamma > 1,$$

we have

$$c = \sqrt{\frac{\gamma p}{\rho}}$$

so the flow will be approximately incompressible if γ is very large.

For nonsteady flow one also needs to know that

$$\frac{l}{\tau} \ll c,$$

where l is a characteristic length and τ is a characteristic time over which the flow picture changes appreciably.[13] The presence of viscosity does not alter these conclusions significantly.

Exercises

◇ **Exercise 1.3-1** Find a stationary viscous incompressible flow in a circular pipe with radius $a > 0$ and with pressure gradient ∇p.

◇ **Exercise 1.3-2** Show that the incompressible Navier–Stokes equations in cylindrical coordinates are

(i) $$\rho\left(\frac{Dv_r}{Dt} - \frac{v_\theta^2}{r}\right) = \rho f_r - \frac{\partial p}{\partial r} + \mu\left(\Delta v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right).$$

(ii) $$\rho\left(\frac{Dv_\theta}{Dt} + \frac{v_r v_\theta}{r}\right) = \rho f_\theta - \frac{1\partial p}{r\partial\theta} + \mu\left(\Delta v_\theta + \frac{2\partial v_2}{r^2\partial\theta} - \frac{v_\theta}{r^2}\right).$$

(iii) $$\rho\frac{Dv_z}{Dt} = \rho f_z - \frac{\partial p}{\partial z} + \mu\Delta v_z,$$

where
$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}$$

and
$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_r\frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial\theta} + v_z\frac{\partial}{\partial z}.$$

◇ **Exercise 1.3-3 Flow in an infinite pipe**.

(i) ***Poiseuille flow***. Work in cylindrical coordinates with a pipe of radius a aligned along the z-axis. The no-slip boundary condition is

[13] Theoretical work on the limit $c \to \infty$ is given by D. Ebin, *Ann. Math.* **141** [1977], 105, and S. Klainerman and A. Majda, *Comm. Pure Appl. Math.*, **35** [1982], 629. Algorithms for solving the equations for incompressible flow by exploiting the regularity of the limit $c \to \infty$ can be found in A. J. Chorin, *J. Comp. Phys.* **12** [1967], 1.

$\mathbf{v} = \mathbf{0}$ when $r = a$. Assume the solution takes the form $p = Cz$, C constant, $v_z = v_z(r)$, and $v_r = v_\theta = 0$. Using Exercise 1.3-2, obtain

$$C = \mu \Delta v_z = \mu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) \right).$$

Integration yields

$$v_z = -\frac{C}{4\mu} r^2 + A \log r + B,$$

where A, B are constants. Because we require that the solution be bounded, A must be 0, because $\log r \to -\infty$ as $r \to 0$. Use the no-slip condition to determine B and obtain

$$v_z = \frac{C}{4\mu}(a^2 - r^2).$$

(ii) Show that the ***mass flow rate*** $Q = \int_s \rho v_z \, dA$ through the pipe is $Q = \rho\pi C a^4/8\mu$. This is the so-called ***fourth-power law***.

(iii) Determine the pressure on the walls.

◇ **Exercise 1.3-4** Compute the solution to the problem of stationary viscous flow between two concentric cylinders and determine the pressure on the walls. (Hint: Proceed as above, but retain the log term.)

2
Potential Flow and Slightly Viscous Flow

The goal of this chapter is to present a deeper study of the relationship between viscous and nonviscous flows. We begin with a more detailed study of inviscid irrotational flows, that is, potential flows. Then we go on to study boundary layers, where the main difference between slightly viscous and inviscid flows originates.

This is further developed in the third section using probabilistic methods. For most of this chapter we will study incompressible flows. A detailed study of some special compressible flows is the subject of Chapter **3**.

2.1 Potential Flow

Throughout this section, all flows are understood to be ideal (*i.e.,* inviscid); in other words, either incompressible and nonviscous or isentropic and nonviscous. Although we allow both, our main emphasis will be on the incompressible case.

A flow with zero vorticity is called ***irrotational***. For ideal flow, this holds for all time if it holds at one time by the results of §**1.2**. An inviscid, irrotational flow is called a ***potential flow***. A domain D is called ***simply connected*** if any continuous closed curve in D can be continuously shrunk to a point without leaving D. For example, in space, the exterior of a solid sphere is simply connected, whereas in the plane the exterior of a solid disc is not simply connected.

For irrotational flow in a simply connected region D, there is a scalar function $\varphi(x,t)$ on D called the ***velocity potential*** such that for each t, $\mathbf{u} = \operatorname{grad}\varphi$. In particular, it follows that the circulation around any closed curve C in D is zero. Using the identity

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \tfrac{1}{2}\nabla\left(\|\mathbf{u}\|^2\right) - \mathbf{u}\times(\nabla\times\mathbf{u}), \tag{2.1.1}$$

we can write the equations for isentropic potential flow in the form

$$\partial_t\mathbf{u} + \tfrac{1}{2}\nabla(\|\mathbf{u}\|^2) = -\operatorname{grad} w,$$

where w is the enthalpy, as in §**1.1**. Substituting $\mathbf{u} = \operatorname{grad}\varphi$, we obtain

$$\operatorname{grad}\left(\partial_t\varphi + \tfrac{1}{2}\|\mathbf{u}\|^2 + w\right) = 0,$$

and thus

$$\partial_t\varphi + \tfrac{1}{2}\|\mathbf{u}\|^2 + w = \text{ constant in space.} \tag{2.1.2}$$

In particular, if the flow is stationary,

$$\tfrac{1}{2}\|\mathbf{u}\|^2 + w = \text{ constant in space.}$$

For the last equation to hold, simple connectivity of D is unnecessary. The version of Bernoulli's theorem given in §**1.1** concluded that $\frac{1}{2}\|\mathbf{u}\|^2 + w$ was constant on streamlines. The stronger conclusion here results from the additional irrotational hypothesis, $\boldsymbol{\xi} = \mathbf{0}$. For homogeneous incompressible ideal flow, note that $w = p/\rho_0$ from the definition of w.

For potential flow in nonsimply connected domains, it can occur that the circulation Γ_C around a closed curve C is nonzero. For instance, consider

$$\mathbf{u} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

outside the origin. If the contour C can be deformed within D to another contour C', then $\Gamma_C = \Gamma_{C'}$; see Figure 2.1.1.

The reason is that basically $C \cup C'$ forms the boundary of a surface Σ in D. Stokes' theorem then gives

$$\int_\Sigma \boldsymbol{\xi}\cdot d\mathbf{A} = \int_C \mathbf{u}\cdot d\mathbf{s} - \int_{C'} \mathbf{u}\cdot d\mathbf{s} = \Gamma_C - \Gamma_{C'}$$

and because $\boldsymbol{\xi} = \mathbf{0}$ in D, we get $\Gamma_C = \Gamma_{C'}$. (A more careful argument proving the invariance of Γ_C under deformation is given in books on complex variables.) Notice that from §**1.2**, the circulation around a curve is constant in time. Thus, the circulation around an obstacle in the plane is well-defined and is constant in time.

Next, consider incompressible potential flow in a simply connected domain D. From $\mathbf{u} = \operatorname{grad}\varphi$ and $\operatorname{div}\mathbf{u} = 0$, we have

$$\Delta\varphi = 0.$$

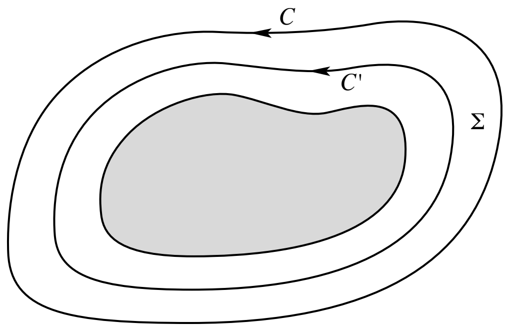

FIGURE 2.1.1. The circulations about C and C' are equal if the flow is potential in Σ.

Let the velocity of ∂D be specified as $\mathbf{V}$, so

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{V} \cdot \mathbf{n}.$$

Thus, φ solves the Neumann problem:

$$\Delta\varphi = 0, \qquad \frac{\partial\varphi}{\partial n} = \mathbf{V} \cdot \mathbf{n}. \tag{2.1.3}$$

If φ is a solution, then $\mathbf{u} = \operatorname{grad}\varphi$ is a solution of the stationary homogeneous Euler equations, *i.e.,*

$$\begin{aligned} \rho(\mathbf{u} \cdot \nabla)\mathbf{u} &= -\operatorname{grad} p, \\ \operatorname{div}\mathbf{u} &= 0, \\ \mathbf{u} \cdot \mathbf{n} &= \mathbf{V} \cdot \mathbf{n} \quad \text{on } \partial D, \end{aligned} \tag{2.1.4}$$

where $p = -\rho\|\mathbf{u}\|^2/2$. This follows from the identity (2.1.1). Therefore, solutions of (2.1.3) are in one-to-one correspondence with irrotational solutions of (2.1.4) (with φ determined only up to an additive constant) on simply connected regions. This observation leads to the following.

Theorem *Let D be a simply connected, bounded region with prescribed velocity $\mathbf{V}$ on ∂D. Then*

i *there is exactly one potential homogeneous incompressible flow* (*satisfying* (2.1.4)) *in D if and only if $\int_{\partial D} \mathbf{V} \cdot \mathbf{n}\, dA = 0$;*

ii *this flow is the minimizer of the kinetic energy function*

$$E_{\text{kinetic}} = \frac{1}{2}\int_D \rho\,\|\mathbf{u}\|^2\, dV,$$

among all divergence-free vector fields $\mathbf{u}'$ on D satisfying $\mathbf{u}' \cdot \mathbf{n} = \mathbf{V} \cdot \mathbf{n}$.

Proof

i The Neumann problem (2.1.3) has a solution if and only if the obvious necessary condition $\int_{\partial D} \mathbf{V} \cdot \mathbf{n}\, dA = 0$ holds, as was mentioned earlier. We can demonstrate the uniqueness of $\mathbf{u}$ directly as follows: Let $\mathbf{u}$ and $\mathbf{u}'$ be two solutions, and let $\mathbf{v} = \mathbf{u} - \mathbf{u}', \psi = \varphi - \varphi'$. Then $\Delta\psi = 0, \partial\psi/\partial n = 0$, and $\mathbf{v} = \operatorname{grad}\psi$. Hence,

$$\int_D \operatorname{div}(\psi\mathbf{v})\, dV = \int_D \mathbf{v} \cdot \operatorname{grad}\psi\, dV + \int_D \psi \operatorname{div}\mathbf{v}\, dV = \int_D \mathbf{v} \cdot \mathbf{v}\, dV.$$

On the other hand,

$$\int_D \operatorname{div}(\psi\mathbf{v})\, dV = \int_{\partial D} \psi\mathbf{v} \cdot \mathbf{n}\, dA = 0.$$

Thus, $\int_D \|\mathbf{v}\|^2 dV = 0$ and $\mathbf{v} = 0$, that is, $\mathbf{u} = \mathbf{u}'$.

ii Let $\mathbf{u}$ solve (2.1.4) and let $\mathbf{u}'$ be divergence free and $\mathbf{u}' = \mathbf{n} = \mathbf{V} \cdot \mathbf{n}$. Let $\mathbf{v} = \mathbf{u} - \mathbf{u}'$; then $\operatorname{div}\mathbf{v} = 0$ and $\mathbf{v} \cdot \mathbf{n} = 0$ on ∂D. Therefore,

$$\begin{aligned} E_{\text{kinetic}} - E'_{\text{kinetic}} &= \tfrac{1}{2} \int_D \rho(\|\mathbf{u}\|^2 - \|\mathbf{u}'\|^2)\, dV \\ &= -\tfrac{1}{2} \int_D \rho\|\mathbf{u} - u'\|^2 dV + \int_D \rho(\mathbf{u} - u') \cdot \mathbf{u}\, dV \\ &\le \int_D \rho\mathbf{v} \cdot \operatorname{grad}\varphi\, dV = 0. \end{aligned}$$

The last equality follows by the orthogonality relation proved in §**1.3**. Thus,

$$E_{\text{kinetic}} \le E'_{\text{kinetic}}$$

as claimed. ∎

Notice, in particular, that the only incompressible potential flow in a bounded region with fixed boundary is the trivial flow $\mathbf{u} = \mathbf{0}$. For unbounded regions this is not true without a careful specification of what can happen at infinity; the above uniqueness proof is valid only if the integration by parts (i.e., use of the divergence theorem) can be justified. For example, in polar coordinates in the plane,

$$\varphi(r, \theta) = \left(r + \frac{1}{r}\right) \cos\theta$$

solves (2.1.3) with $\partial\varphi/\partial n = 0$ on the unit circle and on the x-axis. It represents a nontrivial irrotational potential flow on the simply connected

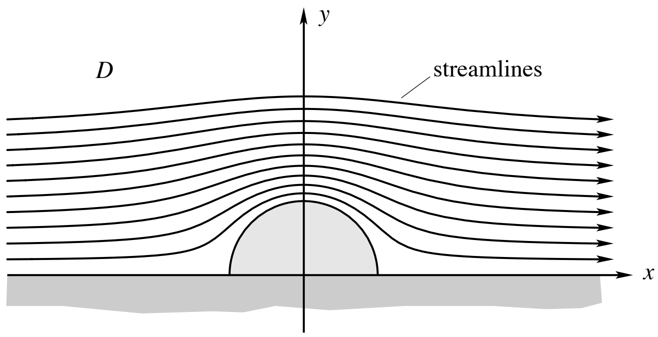

FIGURE 2.1.2. Potential flow in the upper half-plane outside the unit circle.

region D shown in Figure 2.1.2. This flow may be arrived at by the methods of complex variables to which we will now turn.

Incompressible potential flow is very special, but is a key building block for understanding complicated flows. For plane flows the methods of complex variables are useful tools.

Let D be a region in the plane and suppose $\mathbf{u} = (u, v)$ is incompressible, that is,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.1.5}$$

and is irrotational, that is,

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0. \tag{2.1.6}$$

Let

$$F = u - iv, \tag{2.1.7}$$

which is called the ***complex velocity***. Equations (2.1.5) and (2.1.6) are exactly the Cauchy-Riemann equations for F, and so F is an analytic function on D. Conversely, given any analytic function F, $u = \operatorname{Re} F$ and $v = -\operatorname{Im} F$ define an incompressible (stationary) potential flow.

If F has a primitive, $F = dW/dz$, then we call W the ***complex potential***. (If one allows multivalued functions, W will always exist, but such a convention could cause confusion.) Write $W = \varphi + i\psi$. Then (2.1.7) is equivalent to

$$u = \partial_x \varphi = \partial_y \psi \quad \text{and} \quad v = \partial_y \varphi = -\partial_x \psi,$$

that is, $\mathbf{u} = \operatorname{grad} \varphi$ and ψ is the stream function. In what follows, however, we do not and must not assume a (single-valued) W exists.

Consider a flow in the exterior of an obstacle $\mathcal{B}$ (Figure 2.1.3).

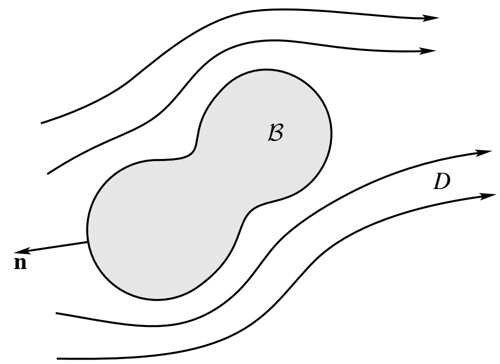

FIGURE 2.1.3. Flow around an obstacle.

The force on the body equals the force exerted on $\partial\mathcal{B}$ by the pressure, that is,

$$\mathcal{F} = -\int_{\partial\mathcal{B}} p\mathbf{n}\, ds, \tag{2.1.8}$$

which means that for any fixed vector $\mathbf{a}$,

$$\mathcal{F} \cdot \mathbf{a} = -\int_{\partial\mathcal{B}} p\mathbf{n} \cdot \mathbf{a}\, ds.$$

Formula (2.1.8) was already discussed at length in §**1.1**. We next prove a theorem that gives a convenient expression for $\mathcal{F}$.

Blasius' Theorem *For incompressible potential flow exterior to a body $\mathcal{B}$ (with rigid boundary) and complex velocity F, the force $\mathcal{F}$ on the body is given by*

$$\mathcal{F} = \frac{-i\rho}{2} \overline{\left[\int_{\partial\mathcal{B}} F^2\, dz\right]} \tag{2.1.9}$$

where the overbar denotes complex conjugation and where the vector $\mathcal{F}$ is identified with a complex number in the standard way; i.e., (x, y) is identified with $z = x + iy$.

Proof If $dz = dx + i\, dy$ represents an infinitesimal displacement along the boundary curve $C = \partial\mathcal{B}$, then $(1/i)dz = dy - i\, dx$ represents a normal displacement. Thus, by (2.1.8)

$$\mathcal{F} = -\int_C p\, dy + i\int_C p\, dx = i\int_C p(dx + i\, dy).$$

As we observed in (2.1.4),

$$p = \frac{-\rho(u^2 + v^2)}{2}, \quad \text{and therefore} \quad \mathcal{F} = \frac{-i\rho}{2}\int_C (u^2 + v^2)\, dz.$$

On the other hand, $F^2 = (u - iv)^2 = u^2 - v^2 - 2iuv$, and because $\mathbf{u}$ is parallel to the boundary, we get $u\,dy = v\,dx$. Thus,

$$F^2 dz = (u^2 - v^2 - 2iuv)(dx + i\,dy) = (u^2 + v^2)(dx - i\,dy),$$

and because $u^2 + v^2$ is real, $\overline{F^2\,dz} = (u^2 + v^2)\,dz$. ■

This formula will be used to prove the following (Figure 2.1.4):

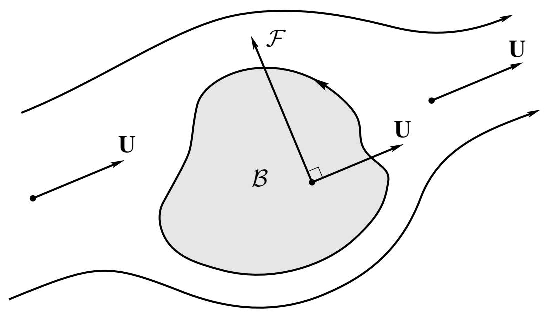

FIGURE 2.1.4. The Kutta–Joukowski theorem gives the force exerted on $\mathcal{B}$.

Kutta–Joukowski Theorem *Consider incompressible potential flow exterior to a region $\mathcal{B}$. Let the velocity field approach the constant value $(U, V) = \mathbf{U}$ at infinity. Then the force exerted on $\mathcal{B}$ is given by*

$$\mathcal{F} = -\rho\Gamma_C\|\mathbf{U}\|\mathbf{n}, \tag{2.1.10}$$

where Γ_C is the circulation around $\mathcal{B}$ and $\mathbf{n}$ is a unit vector orthogonal to $\mathbf{U}$.

Proof By assumption, the complex velocity F is an analytic function outside the body $\mathcal{B}$. It may therefore, be expanded in a Laurent series. Because F tends to a constant $\mathbf{U}$ at infinity, no positive powers of z can occur in the expansion. In other words, F has the form

$$F = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3} + \cdots$$

valid outside any disc centered at the origin and containing $\mathcal{B}$. Because $\mathbf{U}$ is the velocity at infinity, $a_0 = U - iV$. By Cauchy's theorem,

$$\int_C F\,dz = 2\pi a_1 i,$$

where $C = \partial\mathcal{B}$. (The integral is unchanged if we change C to a circle of large radius.) However,

$$\int_C F\,dz = \int_C (u - iv)(dx + i\,dy) = \int_C u\,dx + v\,dy = \int_C \mathbf{u}\cdot d\mathbf{s} = \Gamma_C$$

because $u\,dy = v\,dx$, *i.e.,* $\mathbf{u}$ is parallel to $\partial\mathcal{B}$. Therefore,

$$a_1 = \frac{\Gamma_C}{2\pi i}.$$

Squaring F gives the Laurent expansion

$$F^2 = a_0^2 + \frac{2a_0a_1}{z} + \frac{2a_0a_2 + a_1^2}{z^2} + \cdots .$$

By Blasius' theorem and Cauchy's theorem,

$$\mathcal{F} = -\frac{i\rho}{2}\overline{\int_C F^2\,dz} = -\frac{i\rho}{2}\cdot\overline{(2\pi i\cdot 2a_0a_1)} = \rho\Gamma_C(V - iU)$$

which proves the theorem. ■

Notice that the force exerted on the body $\mathcal{B}$ by the flow is normal to the direction of flow and is proportional to the circulation around the body. In any case, the body experiences no drag (i.e., no force opposing the flow) in contradiction with our intuition and with experiment. The difficulty, of course, stems from the fact that we have neglected viscosity. (We shall remedy this in the succeeding two sections.) Even worse, if $\Gamma_C = 0$, there is no net force on the body at all, a fact hard to reconcile with our intuition even for ideal flow. This result is called ***d'Alembert's paradox***.

Example 1 For a complex number $\alpha = U - iV$, let $W(z) = \alpha z$. Thus, $F(x) = \alpha$, so the velocity field is $\mathbf{u} = (U, V)$. This is two-dimensional flow moving with constant velocity in the direction (U, V). ♦

Example 2 Let $\mathcal{B}$ be the disc of radius $a > 0$ centered at the origin in the complex plane, and let

$$W(z) = U\left(z + \frac{a^2}{z}\right) \tag{2.1.11}$$

for a positive constant U. The complex velocity is

$$F(z) = W'(z) = U\left(1 - \frac{a^2}{z^2}\right), \tag{2.1.12}$$

which approaches U at ∞. The velocity potential φ and the stream function ψ are determined by $W = \varphi + i\psi$. To verify that the flow is tangent to the circle $|z| = a$, we need only to show that $\psi = $ constant when $|z| = a$. In fact, for $|z|^2 = z\bar{z} = a^2$, we have from (2.1.1),

$$W(z) = U(z + \bar{z}),$$

so W is real on $|z| = a$, that is, $\psi = 0$ on $|z| = a$. The flow is shown in Figure 2.1.5.

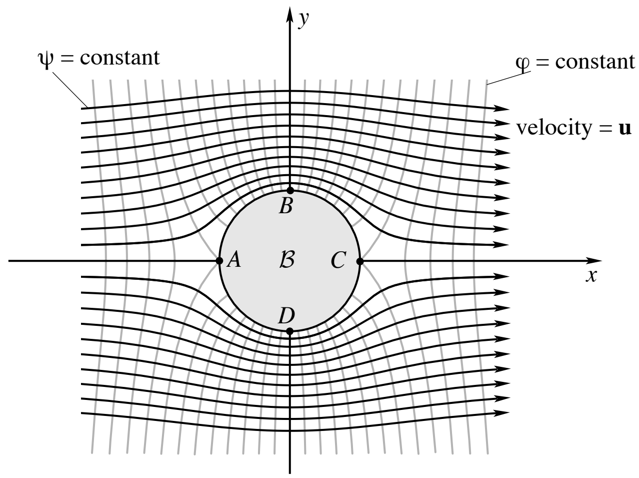

FIGURE 2.1.5. Potential flow around a disc.

From (2.1.12) with $z = ae^{i\theta}$, that is, z on $\partial\mathcal{B}$, we find

$$F(z) = U\left(1 - \frac{a^2}{a^2 e^{2i\theta}}\right) = U(1 - \cos 2\theta + i \sin 2\theta).$$

Thus, the velocity is zero at A and C; that is, A and C are stagnation points. The velocity reaches a maximum at B and D. By Bernoulli's theorem, we can write

$$p = -\frac{\rho}{2}\|\mathbf{u}\|^2 + \text{constant};$$

thus, the pressure at A and C is maximum and is a minimum at B and D. The disc has zero circulation because $F = W'$ and W is single-valued.

If W is any analytic function defined in the whole plane, then

$$\tilde{W}(z) = W(z) + \overline{W\left(\frac{a^2}{z}\right)}, \qquad |z| \geq a$$

is a potential describing a flow exterior to the disc of radius $a > 0$, but possibly with a more complicated behavior at infinity. This is proved along the same lines as in the argument just presented. ♦

Example 3 In §**1.2** we proved that choosing ψ to be an arbitrary increasing function of r alone yields a flow that is incompressible and has vorticity $\xi = -\Delta\psi$. If we can arrange for ψ to be the imaginary part of an analytic function, then the flow will be irrotational as well. The function

$$W(z) = \frac{\Gamma}{2\pi i} \log z \tag{2.1.13}$$

has this property, because $\log z = \log|z| + i \arg z$. Of course, $W(z)$ is not single-valued, but the complex velocity

$$F(z) = \frac{\Gamma}{2\pi i z} \tag{2.1.14}$$

is analytic and single-valued outside $z = 0$. The circulation is indeed Γ. Note that the velocity field is zero at infinity. For incompressible potential flow about a disc of radius a centered at z_0, we need only choose

$$W(z) = \frac{\Gamma}{2\pi i} \log(z - z_0).$$

The boundary conditions are satisfied because ψ is constant on any circle centered at z_0 (see Figure 2.1.6). The incompressible potential flow with

$$W(z) = \frac{\Gamma}{2\pi i} \log(z - z_0)$$

will be called a ***potential vortex*** at z_0. ♦

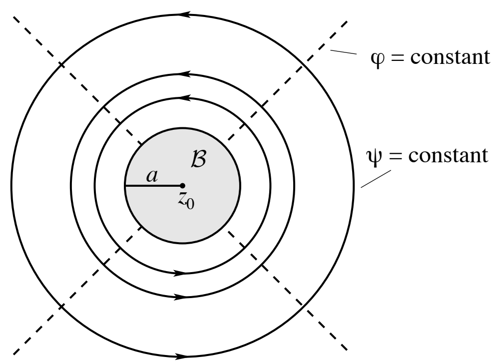

FIGURE 2.1.6. Potential vortex flow centered at z_0.

Example 4 We combine Examples 2 and 3 by forming

$$W(z) = U\left(z + \frac{a^2}{z}\right) + \frac{\Gamma}{2\pi i} \log z, \tag{2.1.15}$$

where $|z| \geq a$. Because ψ is constant on the boundary for each flow separately, it is also true for W given by (2.1.15). Thus, we get an incompressible potential flow on the exterior of the disc $|z| \leq a$ with circulation Γ around the disc. The velocity field is $(U, 0)$ at infinity (therefore, the

Kutta–Joukowski theorem applies). On the surface of the disc the velocity $\mathbf{u} = \operatorname{grad}\varphi$ is tangent to the disc and is given in magnitude by

$$\text{velocity} = \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\bigg|_{r=a}.$$

Here $\varphi = \operatorname{Re} W$, so that

$$\varphi(r,\theta) = U\cos\theta\left(r + \frac{a^2}{r}\right) + \frac{\Gamma\theta}{2\pi},$$

and thus

$$\text{velocity} = \frac{1}{a}\,\frac{\partial\varphi}{\partial\theta}\bigg|_{r=a} = -2U\sin\theta + \frac{\Gamma}{2\pi a}.$$

If $|\Gamma| < 4\pi a\,U$, there are two stagnation points A and C defined by

$$\sin\theta = \frac{\Gamma}{4\pi a U}$$

on the boundary, where the pressure is highest. See Figure 2.1.7.

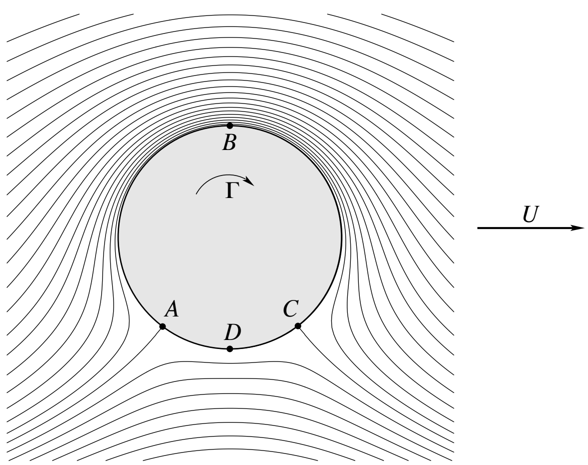

FIGURE 2.1.7. Flow around a disc with circulation.

This example helps to explain the Kutta–Joukowski theorem; note that the vertical lift may be attributed to the higher pressure at A and C. The symmetry in the y-axis means that there is no drag. ♦

D'Alembert's Paradox in Three Dimensions *In the case of steady incompressible potential flow around an obstacle in three dimensions with constant velocity U at infinity, not only can there be no drag, there can be no lift either.*

The difference with the two-dimensional case is explained by the fact that the exterior of a body in three-space is simply connected, whereas this is not true in two dimensions. We will not present the detailed proof of d'Alembert's paradox here, but we can give the idea.

Recall that the solution of $\Delta\varphi = -\rho$ in space is

$$\varphi(\mathbf{x}) = \frac{1}{4\pi}\int \frac{\rho(\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|}\,dV(\mathbf{y}),$$

that is, φ is the potential due to a charge distribution ρ. Notice that if ρ is concentrated in a finite region, then

$$\varphi(\mathbf{x}) = O\left(\frac{1}{r}\right),$$

where $r = \|\mathbf{x}\|$, that is,

$$|\varphi(\mathbf{x})| \le \frac{\text{constant}}{r}$$

for large r. In fact, as we know physically, $\varphi(\mathbf{x}) \approx Q/4\pi r$ for r large, where $Q = \int \rho(\mathbf{y})\,dV(\mathbf{y})$ is the total charge. If $Q = 0$, then $\varphi(\mathbf{x}) = O(1/r^2)$ because the first term in the expansion in powers of $1/r$ is now missing.

For an incompressible potential flow there will be a potential φ, that is, $\mathbf{u} = \operatorname{grad}\varphi$ (because the exterior of the body is simply connected). The potential satisfies

$$\Delta\varphi = 0, \quad \nabla\varphi = \mathbf{U} \text{ at } \infty,$$

and

$$\frac{\partial\varphi}{\partial n} = 0 \text{ on the boundary of the obstacle.}$$

The solution here can then be shown to satisfy

$$\varphi(\mathbf{x}) = \mathbf{U}\cdot\mathbf{x} + O\left(\frac{1}{r}\right)$$

as in the potential case above. However, there is an integral condition analogous to $Q = 0$, namely, the net outflow at ∞ should be zero. This means

$$\varphi(\mathbf{x}) = \mathbf{U}\cdot\mathbf{x} + O\left(\frac{1}{r^2}\right).$$

Hence,

$$\mathbf{u} = \mathbf{U} + O(r^{-3}). \tag{2.1.16}$$

Because $p = -\rho v^2/2$, we also have $p = p_0 + O(r^{-3})$. (To see that this is true, write $\|\mathbf{u}\|^2 = U^2 + (\mathbf{u}-\mathbf{U})\cdot(\mathbf{u}+\mathbf{U})$). The force on the body $\mathcal{B}$ is

$$\mathcal{F} = -\int_{\partial\mathcal{B}} p\mathbf{n}\,dA.$$

Let Σ be a surface containing $\mathcal{B}$. Because $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\mathcal{B}$ and the flow is steady, equation (BM3) from §**1.1** applied to the region between $\mathcal{B}$ and Σ shows that

$$\mathcal{F} = -\int_\Sigma (\rho(\mathbf{u} \cdot \mathbf{n})\mathbf{u} + p\mathbf{n})\, dA.$$

We are free to choose Σ to be a sphere of large radius R enclosing the obstacle. Then

$$\begin{aligned}\mathcal{F} &= -\int_\Sigma (p_0\mathbf{n} + \rho(\mathbf{U} \cdot \mathbf{n})\mathbf{U})\, dA + (\operatorname{area}\Sigma) \cdot O(R^{-3}) \\ &= \mathbf{0} + O(R^{-1}) \to 0 \text{ as } R \to \infty.\end{aligned}$$

Hence, $\mathcal{F} = \mathbf{0}$.

One may verify d'Alembert's paradox directly for flow past a sphere of radius $a > 0$. In this case

$$\varphi = -\frac{a^3}{2r^2}\mathbf{U} \cdot \mathbf{n} + \mathbf{x} \cdot \mathbf{U},$$

where $\mathbf{n} = \frac{\mathbf{x}}{\|\mathbf{x}\|}$, and

$$u = -\frac{a^3}{2r^2}\left[3\mathbf{n}(\mathbf{U} \cdot \mathbf{n}) - \mathbf{U}\right] + \mathbf{U},$$

where $\mathbf{U}$ is the velocity at infinity. We leave the detailed verification to the reader.[1] ♦

Next we will discuss a possible mechanism, ultimately to be justified by the presence of viscosity, by which one can avoid d'Alembert's paradox. An effort to resolve the paradox is of course prompted by the fact that real bodies in fluids do experience drag.

By an ***almost potential flow***, we mean a flow in which vorticity is concentrated in some thin layers of fluid; the flow is potential outside these thin layers, but there is a mechanism for producing vorticity near boundaries. For example, one can postulate that the flow past the obstacle shown in Figure 2.1.8 produces an almost potential flow with vorticity produced at the boundary and concentrated on two streamlines emanating from the body.

We image different potential flows in the two regions separated by these streamlines with the velocity field discontinuous across them. For such a model, the Kutta–Joukowski theorem does not apply and the drag may be different from zero. There are a number of situations in engineering where real flows can be usefully idealized as "nearly potential." These situations arise in particular when one considers "streamlined" bodies, that is, bodies

[1] See L. Landau and E. Lifschitz [1959] *Fluid Mechanics,* Pergamon, p. 34 for more information.

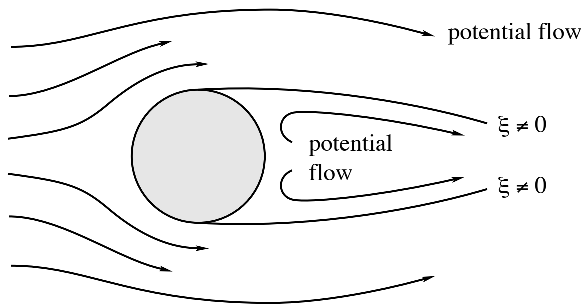

FIGURE 2.1.8. Almost potential flow has vorticity concentrated on two curves.

so shaped as to reduce their drag. The discussion of such bodies, their design, and the validity of potential approximation to the flow around them are outside the scope of this book.

Next we shall examine a model of incompressible inviscid flow inspired by the idea of an almost potential flow and Example 3.

We imagine the vorticity in a fluid is concentrated in N ***vortices*** (i.e., points at which the vorticity field is singular), located at $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$ in the plane (Figure 2.1.9). The stream function of the jth vortex, ignoring the other vortices for a moment, is by Example 3,

$$\psi_j(\mathbf{x}) = -\frac{\Gamma_j}{2\pi} \log \|\mathbf{x} - \mathbf{x}_j\|. \tag{2.1.17}$$

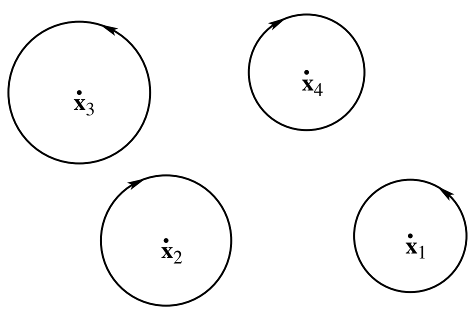

FIGURE 2.1.9. The flow generated by point vortices in the plane.

As the fluid moves according to Euler's equations, the circulations Γ_j associated with each vortex will remain constant. The vorticity field produced by the jth vortex can be written as

$$\xi_j = -\Delta\psi_j = \Gamma_j \delta(\mathbf{x} - \mathbf{x}_j),$$

where δ is the Dirac δ function. This equation arises from the fact, which we just accept, that the ***Green's function*** for the Laplacian in the plane is

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{2\pi} \log \|\mathbf{x} - \mathbf{x}'\|,$$

that is, G satisfies

$$\Delta_{\mathbf{x}} G(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}').$$

The solution of $\Delta\psi = -\xi$ is then given by the superposition

$$\psi(\mathbf{x}) = -\int \xi(\mathbf{x}') G(\mathbf{x} - \mathbf{x}')\, d\mathbf{x}'$$

which in our case reduces to $\psi(\mathbf{x}) = \sum_{j=1}^{N} \psi_j(\mathbf{x})$, where

$$\psi_j(\mathbf{x}) = -\frac{1}{2\pi} \Gamma_j \log \|\mathbf{x} - \mathbf{x}_j\|.$$

The velocity field induced by the jth vortex (again ignoring the other vortices) is given by

$$\mathbf{u}_j = (\partial_y \psi_j, -\partial_x \psi_j) = \left(-\frac{\Gamma_j}{2\pi} \left(\frac{y - y_j}{r^2} \right), \frac{\Gamma_j}{2\pi} \left(\frac{x - x_j}{r^2} \right) \right), \qquad (2.1.18)$$

where $r = \|\mathbf{x} - \mathbf{x}_j\|$. Let the vortices all move according to the velocity field

$$\mathbf{u}(\mathbf{x}, t) = \sum_{j=1}^{N} \mathbf{u}_j(\mathbf{x}, t),$$

where $\mathbf{u}_j$ is given by (2.1.18) except we now allow, as we must, the centers of the vortices $\mathbf{x}_j, j = 1, \ldots, N$ to move. Each one ought to move as if convected by the net velocity field of the other vortices. Therefore, by (2.1.18), $\mathbf{x}_j$ moves according to the equations

$$\frac{dx_j}{dt} = -\frac{1}{2\pi} \sum_{i \neq j} \frac{\Gamma_i (y_j - y_i)}{r_{ij}^2} \quad \text{and} \quad \frac{dy_j}{dt} = \frac{1}{2\pi} \sum_{i \neq j} \frac{\Gamma_i (x_j - x_i)}{r_{ij}^2}, \qquad (2.1.19)$$

where $r_{ij} = \|\mathbf{x}_i - \mathbf{x}_j\|$.

Let us summarize the construction of the flows we are considering: choose constants $\Gamma_1, \ldots, \Gamma_N$ and initial points $\mathbf{x}_1 = (x_1, y_1), \ldots, \mathbf{x}_N = (x_N, y_N)$ in the plane. Let these points move according to the equations (2.1.19). Define $\mathbf{u}_j$ by (2.1.18) and let

$$\mathbf{u}(\mathbf{x}, t) = -\sum_{j=1}^{N} \mathbf{u}_j(\mathbf{x}, t).$$

This construction produces formal solutions of Euler's equation ("formal" because the meaning of δ-function solutions of nonlinear equations is not obvious). These solutions have the property that the circulation theorem is satisfied. If C is a contour containing l vortices at $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_l$, then $\Gamma_C = -\sum_{i=1}^{l} \Gamma_i$ and Γ_C is invariant under the flow. However, the relationship between these solutions and bona fide solutions of Euler's equations is not readily apparent. Such a relationship can, however, be established rigorously and such vortex systems do contain significant information about the solutions of Euler's equations under a wide variety of conditions.[2]

An important property of the equations is that they form a ***Hamiltonian system***. Define

$$H = -\frac{1}{4\pi} \sum_{i \neq j} \Gamma_i \Gamma_j \log \|\mathbf{x}_i - \mathbf{x}_j\|. \tag{2.1.20}$$

First of all, it is easy to check that (2.1.19) is equivalent to

$$\Gamma_j \frac{dx_j}{dt} = \frac{\partial H}{\partial y_j}, \quad \Gamma_j \frac{dy_j}{dt} = -\frac{\partial H}{\partial x_j}, \tag{2.1.21}$$

where $j = 1, \ldots, N$ (there is no sum on j). Introduce the new variables

$$x_i' = \sqrt{|\Gamma_i|}\, x_i, \qquad y_i' = \sqrt{|\Gamma_i|}\, \operatorname{sgn}(\Gamma_i) y_i, \qquad i = 1, \ldots, N,$$

where $\operatorname{sgn}(\Gamma_i)$ is 1 if $\Gamma_i > 0$, and is -1 otherwise. Then (2.1.19) is equivalent to the following system of Hamiltonian equations

$$\frac{dx_i'}{dt} = \frac{\partial H}{\partial y_i'}, \qquad \frac{dy_i'}{dt} = -\frac{\partial H}{\partial x_i'}, \qquad i = 1, \ldots, N, \tag{2.1.22}$$

with Hamiltonian H and generalized coordinates (x_i', y_i'). As in elementary mechanics,

$$\frac{dH}{dt} = \sum_{i=1}^{N} \frac{\partial H}{\partial x_i'} \frac{dx_i'}{dt} + \sum_{i=1}^{N} \frac{\partial H}{\partial y_i'} \frac{dy_i'}{dt} = 0,$$

that is, H is a constant of the motion. A consequence of this fact is that if the vortices all have the same sign they cannot collide. If $\|\mathbf{x}_i - \mathbf{x}_j\| \neq 0, i \neq j$ at $t = 0$, then this remains so for all time because if $\|\mathbf{x}_i - \mathbf{x}_j\| \to 0, H$ becomes infinite.

This Hamiltonian system is of importance in understanding how vorticity evolves and organizes itself.[3]

[2] See C. Anderson and C. Greengard, *On Vortex Methods, SIAM J. Sci. Statist. Comput.* **22** [1985], 413.

[3] The Euler equations themselves form a Hamiltonian system (this is explained, along with references, in R. Abraham and J. E. Marsden, *Foundations of Mechanics,* 2nd Edition [1978]), and the Hamiltonian nature of the vortex approximation is consistent with this. See also J. E. Marsden and A. Weinstein, Coadjoint orbits, vortices and Clebsch variables for incompressible fluids, *Physica* **7D** [1983], 305–323.

Let us now generalize the situation and imagine our N vortices moving in a domain D with boundary ∂D. We can go through the same construction as before, but we have to modify the flow $\mathbf{u}_j$ of the jth vortex in such a way that $\mathbf{u} \cdot \mathbf{n} = 0$, that is, the boundary conditions appropriate to the Euler equations hold. We can arrange this by adding a potential flow $\mathbf{v}_j$ to $\mathbf{u}_j$ such that $\mathbf{u}_j \cdot \mathbf{n} = -\mathbf{v}_j \cdot \mathbf{n}$ on ∂D. In other words, we choose the stream function ψ_j for the jth vortex to solve

$$\Delta \psi_j = -\xi_j = -\Gamma_j \delta(\mathbf{x} - \mathbf{x}_j) \quad \text{with } \frac{\partial \psi_j}{\partial n} = 0 \text{ on } \partial D.$$

This is equivalent to requiring $\psi_j(\mathbf{x}) = -\Gamma_j G(\mathbf{x}, \mathbf{x}_j)$ where G is the Green's function for the Neumann problem for the region D. This procedure will appropriately modify the function $(1/2\pi) \log \|\mathbf{x} - \mathbf{x}_j\|$ and allow the analysis to go through as before.

For example, suppose D is the upper half-plane $y \geq 0$. Then we get G by the reflection principle:

$$G(\mathbf{x}, \mathbf{x}_j) = \frac{1}{2\pi} \left(\log \|\mathbf{x} - \mathbf{x}_j\| + \log \|\mathbf{x} - \hat{\mathbf{x}}_j\| \right),$$

where $\hat{\mathbf{x}}_j = (x_j, -y_j)$ is the reflection of $\mathbf{x}_j$ across the x-axis (see Figure 2.1.10). For the Neumann-Green's functions for other regions the reader may consult textbooks on partial differential equations.

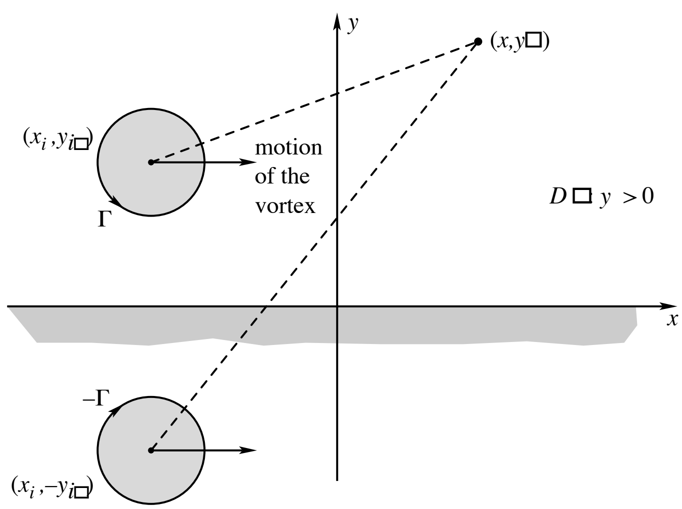

FIGURE 2.1.10. The stream function at (x, y) is the superposition of those due to vortices with opposite circulations located at (x_i, y_i) and $(x_i, -y_i)$.

Consider again Euler's equations in the form

$$\Delta\psi = -\xi, \quad u = \partial_y \psi, \quad v = -\partial_x \psi, \quad \frac{D\xi}{Dt} = 0.$$

One can write

$$\psi = -\int \xi(\mathbf{x}')G(\mathbf{x}, \mathbf{x}')\, d\mathbf{x}',$$

where $G(\mathbf{x}, \mathbf{x}') = \frac{1}{2}\pi \log \|\mathbf{x} - \mathbf{x}'\|$, and set $u = \partial_y \psi$, $v = -\partial_x \psi$. The resulting equations resemble the equations just derived for a system of point vortices. The integral for ψ here resembles the formula for ψ in the point vortex system somewhat as an integral resembles one of its Riemann sum approximations. This suggests that an ideal incompressible flow can be approximated by the motion of a system of point vortices. There are in fact theorems along these lines.[4] Vortex systems provide both a useful heuristic tool in the analysis of the general properties of the solutions of Euler's equations, and a useful starting point for the construction of practical algorithms for solving these equations in specific situations.

One can ask if there is a similar construction in three dimensions. First of all, one can seek an analogue of the superposition of stream functions from point potential vortices. Given $\mathbf{u}$ satisfying $\operatorname{div} \mathbf{u} = 0$, there is a vector field $\mathbf{A}$ such that $\operatorname{div} \mathbf{A} = 0$ and such that $\mathbf{u} = \operatorname{curl} \mathbf{A}$, and therefore $\Delta \mathbf{A} = -\boldsymbol{\xi}$. In three dimensions, Green's function for the Laplacian is given by

$$G(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\pi} \frac{1}{\|\mathbf{x} - \mathbf{x}'\|}, \qquad \mathbf{x} \neq \mathbf{x}'.$$

Then we can represent $\mathbf{A}$ in terms of $\boldsymbol{\xi}$ by

$$\mathbf{A} = -\frac{1}{4\pi} \int \frac{\boldsymbol{\xi}(\mathbf{x}')}{s}\, dV(\mathbf{x}'),$$

where $s = \|\mathbf{x} - \mathbf{x}'\|$, and where $dV(\mathbf{x}')$ is the usual volume element in space. It is easy to check that $\mathbf{A}$ defined by the above integral satisfies the normalization condition $\operatorname{div} \mathbf{A} = 0$. Thus, because $\mathbf{u} = \operatorname{curl} \mathbf{A}$, we obtain

$$\mathbf{u}(\mathbf{x}) = \frac{1}{4\pi} \int \frac{\mathbf{s} \times \boldsymbol{\xi}'}{s^3}\, dV(\mathbf{x}'),$$

where $\mathbf{s} = \mathbf{x} - \mathbf{x}$ and $\boldsymbol{\xi}' = \boldsymbol{\xi}(\mathbf{x}')$. Suppose that we have a vortex line C in space with circulation Γ (see Figure 2.1.11) and we assume that the

[4] The discrete vortex method is discussed in L. Onsager, *Nuovo Cimento* **6** (Suppl.) [1949], 229; A. J. Chorin, *J. Fluid Mech.* **57** [1973], 781; and A. J. Chorin, *SIAM J. Sci. Statist. Comput.* **1** [1980], 1. Convergence of solutions of the discrete vortex equations to solutions of Euler's equations as $N \to \infty$ is discussed in O. H. Hald, *SIAM J. Numer. Anal.* **16** [1979], 726; T. Beale and A. Majda, *Math. Comp.* **39** [1982], 1–28, 29–52; and K. Gustafson and J. Sethian, *Vortex Flows,* SIAM Publications, 1991.

vorticity field $\boldsymbol{\xi}$ is concentrated on C only, that is, the flow is potential outside the filament C. Then $\mathbf{u}(\mathbf{x})$ can be written as

$$\mathbf{u}(\mathbf{x}) = \frac{1}{4\pi} \int_C \frac{\mathbf{s} \times \Gamma\, d\mathbf{s}}{s^3}$$

where ds is the line element on C.

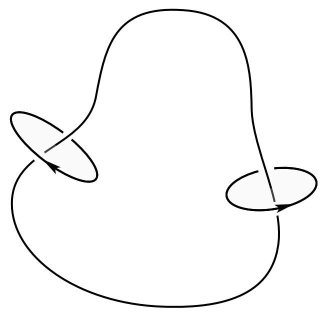

FIGURE 2.1.11. The flow induced by a vortex filament.

Exercises

◇ **Exercise 2.1-1** If $f : D \to D'$ is a conformal transformation (an analytic function that is one to one and onto), it can be used to transform one complex potential to another. Use $f(z) = z + a^2/z$ (which takes the exterior of the disc of radius a in the upper half-plane to the upper half-plane) and the complex potential in the upper half-plane to generate formula (2.1.11).

◇ **Exercise 2.1-2** Let $F(z) = z^2$ be a complex potential in the first quadrant. Sketch some streamlines and the curves $\phi =$ constant, $\psi =$ constant, where $F = \phi + i\psi$. What is the force exerted on the walls?

◇ **Exercise 2.1-3** Use conformal maps to find a formula for potential flow over the plate in Figure 2.1.12. What is the force exerted on this plate?

◇ **Exercise 2.1-4** Let a spherical object move through a fluid in $\mathbb{R}^3$. For slow velocities, assume Stokes' equations apply. Take the point of view that the object is stationary and the fluid streams by. The setup for the boundary value problem is as follows: given $\mathbf{U} = (U, 0, 0), U$ constant, find $\mathbf{u}$ and p such that Stokes' equation holds in the region exterior to a sphere of radius R, $\mathbf{u} = \mathbf{0}$ on the boundary of the sphere and $\mathbf{u} = \mathbf{U}$ at infinity. The solution to this problem (in spherical coordinates centered in the object)

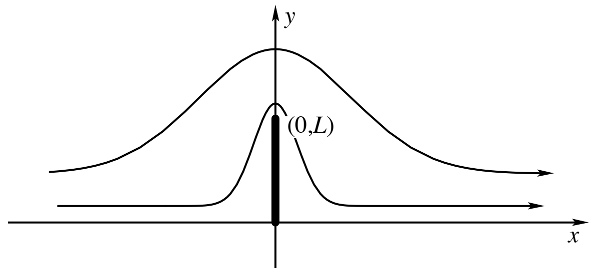

FIGURE 2.1.12. Flow over a vertical plate.

is called ***Stokes' Flow***:

$$\begin{aligned}\mathbf{u} &= -\frac{3}{4}R\,\frac{\mathbf{U}+\mathbf{n}(\mathbf{U}\cdot\mathbf{n})}{r} - \frac{1}{4}R^3\,\frac{\mathbf{U}-3\mathbf{n}(\mathbf{U}\cdot\mathbf{n})}{r^3} + \mathbf{U},\\ p &= p_0 - \frac{3}{2}\nu\,\frac{\mathbf{U}\cdot\mathbf{n}}{r^2}\,R,\end{aligned} \tag{2.1.23}$$

where p_0 is constant and $\mathbf{n} = \mathbf{r}/r$.

(a) Verify this solution.

(b) Show that the drag is $6\pi R\nu U$ and there is no lift.

(c) Show there is net outflow at infinity (an infinite wake).

◇ **Exercise 2.1-5** Because of the difficulties in Exercise 2.1-4, Oseen in 1910 suggested that Stokes' equations be replaced by

$$-\nu\Delta\mathbf{u} + (\mathbf{U}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\,\operatorname{grad} p,$$

with $\operatorname{div}\mathbf{u} = 0$, where $\mathbf{u}$ represents the true velocity minus $\mathbf{U}$. This amounts to linearizing the Navier–Stokes equations about $\mathbf{U}$, whereas Stokes' equations may be viewed as a linearization about $\mathbf{0}$. One would thus conjecture that Oseen's equations are good where the flow is close to the free stream velocity $\mathbf{U}$ (away from the sphere) and that Stokes' equations are good where the velocity is $\mathbf{0}$ (near the sphere). The solution of Oseen's equations in the region exterior to a sphere in $\mathbb{R}^3$ can be found in Lamb's book. Show that drag on the sphere for the Oseen solution is $F = 6\pi RU\nu(1+\frac{3}{8}R)$, where $R = UR/\nu$ is the Reynolds number. Thus, there is a difference of the order R in the Stokes and Oseen drag forces.

Notes on Exercise 2.1-4 and Exercise 2.1-5: If D is bounded with smooth boundary, then there exists at most one solution to Stokes' equations. See Ladyzhenskaya's book listed in the Preface. In the exterior of a bounded region in $\mathbb{R}^3$ there exists a unique solution to Stokes' equations. The situation in $\mathbb{R}^2$ is different; in fact, we have the following strange situation:

Stokes' Paradox *There is no solution to Stokes' equations in* $\mathbb{R}^2$ *in the region exterior to a disc (with reasonable boundary conditions).*[5]

Stokes' paradox does not apply to the Oseen or Navier–Stokes equations in $\mathbb{R}^2$ or $\mathbb{R}^3$. However, Filon in 1927 pointed out that for other reasons Oseen's equations also lead to unacceptable results. The example he gives is a skewed ellipse in a free stream. Computation of the moment exerted on the ellipse reveals that it is infinite! This is not so surprising in view of the fact that Oseen's equations represent linearization about the free stream. One cannot expect them to give good results around the obstacle because the equations contain errors there of order U^2.

2.2 Boundary Layers

Consider the Navier–Stokes equations

$$\left.\begin{aligned} \partial_t \mathbf{u} + (\mathbf{u}\cdot\bigtriangledown)\mathbf{u} &= -\operatorname{grad} p + \frac{1}{R}\bigtriangleup \mathbf{u}, \\ \operatorname{div}\mathbf{u} &= 0, \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \partial D, \end{aligned}\right\} \tag{2.2.1}$$

and assume the Reynolds number R is large. We ask how different a flow governed by (2.2.1) is from one governed by the Euler equations for incompressible ideal flow:

$$\left.\begin{aligned} \partial_t \mathbf{u} + (\mathbf{u}\cdot\bigtriangledown)\mathbf{u} &= -\operatorname{grad} p, \\ \operatorname{div}\mathbf{u} &= 0, \\ \mathbf{u}\cdot\mathbf{n} &= 0 \quad \text{on } \partial D. \end{aligned}\right\} \tag{2.2.2}$$

Imagine that both flows coincide at $t = 0$ and, say, are irrotational, that is, $\boldsymbol{\xi} = \mathbf{0}$. Thus, under (2.2.2) the flow stays irrotational, and thus is a potential flow. However, we claim that the presence of the (small) viscosity term $(1/R)\bigtriangleup \mathbf{u}$ and the difference in the boundary conditions have the following effects:

1. The flow governed by (2.2.2) is drastically modified near the wall in a region with thickness proportional to $1/\sqrt{R}$.

2. The region in which the flow is modified may separate from the boundary.

3. This separation will act as a source of vorticity.

[5] See Birkhoff's book, and J. Heywood, *Arch. Rational Mech. Anal.* **37** [1970], 48–60, and *Acta Math.* **129** [1972], 11–34.

1. $\nabla(f+g) = \nabla f + \nabla g$
2. $\nabla(cf) = c\nabla f$, for a constant c
3. $\nabla(fg) = f\nabla g + g\nabla f$
4. $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\nabla f - f\nabla}{g^2}$
5. $\operatorname{div}(\mathbf{F}+\mathbf{G}) = \operatorname{div}\mathbf{F} + \operatorname{div}\mathbf{G}$
6. $\operatorname{curl}(\mathbf{F}+\mathbf{G}) = \operatorname{curl}\mathbf{F} + \operatorname{curl}\mathbf{G}$
7. $\nabla(\mathbf{F}\cdot\mathbf{G}) = (\mathbf{F}\cdot\nabla)\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F} + \mathbf{F}\times\operatorname{curl}\mathbf{G} + \mathbf{G}\times\operatorname{curl}\mathbf{F}$
8. $\operatorname{div}(f\mathbf{F}) = f\operatorname{div}\mathbf{F} + \mathbf{F}\cdot\nabla f$
9. $\operatorname{div}(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot\operatorname{curl}\mathbf{F} - \mathbf{F}\cdot\operatorname{curl}\mathbf{G}$
10. $\operatorname{div}\operatorname{curl}\mathbf{F} = 0$
11. $\operatorname{curl}(f\mathbf{F}) = f\operatorname{curl}\mathbf{F} + \nabla f\times\mathbf{F}$
12. $\operatorname{curl}(\mathbf{F}\times\mathbf{G}) = \mathbf{F}\ \operatorname{div}\ \mathbf{G} - \mathbf{G}\ \operatorname{div}\ \mathbf{F} + (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G}$
13. $\operatorname{curl}\ \operatorname{curl}\mathbf{F} = \operatorname{grad}\operatorname{div}\ \mathbf{F} - \nabla^2\mathbf{F}$
14. $\operatorname{curl}\nabla f = 0$
15. $\nabla(\mathbf{F}\cdot\mathbf{F}) = 2(\mathbf{F}\cdot\nabla)\mathbf{F} + 2\mathbf{F}\times(\operatorname{curl}\mathbf{F})$
16. $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2(\nabla f\cdot\nabla g)$
17. $\operatorname{div}(\nabla f\times\nabla g) = 0$
18. $\nabla\cdot(f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$
19. $\mathbf{H}\cdot(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot(\mathbf{H}\times\mathbf{F}) = \mathbf{F}\cdot(\mathbf{G}\times\mathbf{H})$
20. $\mathbf{H}\cdot((\mathbf{F}\times\nabla)\times\mathbf{G}) = ((\mathbf{H}\cdot\nabla)\mathbf{G})\cdot\mathbf{F} - (\mathbf{H}\cdot\mathbf{F})(\nabla\cdot\mathbf{G})$
21. $\mathbf{F}\times(\mathbf{G}\times\mathbf{H}) = (\mathbf{F}\cdot\mathbf{H})\mathbf{G} - \mathbf{H}(\mathbf{F}\cdot\mathbf{G})$

Notes.

In identity 7, $\mathbf{V} = (\mathbf{F}\cdot\nabla)\mathbf{G}$ has components $\mathbf{V}_i = \mathbf{F}\cdot(\nabla G_i)$, for $i = 1, 2, 3$, where $\mathbf{G} = (G_1, G_2, G_3)$.

In identity 13, the vector field $\nabla^2\mathbf{F}$ has components $\nabla^2 F_i$, where $\mathbf{F} = (F_1, F_2, F_3)$.

In identity 20, $(\mathbf{F}\times\nabla)\times\mathbf{G}$ means ∇ is to operate only on $\mathbf{G}$ in the following way: To calculate $(\mathbf{F}\times\nabla)\times\mathbf{G}$, we define $(\mathbf{F}\times\nabla)\times\mathbf{G} = \mathbf{U}\times\mathbf{G}$

where we define $\mathbf{U} = \mathbf{F} \times \nabla$ by:

$$\mathbf{U} = \mathbf{F} \times \nabla = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ F_1 & F_2 & F_3 \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \end{vmatrix}.$$

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