Introduzione alle equazioni alle derivate parziali,
Laurea Magistrale in Matematica

The Dirichlet problem in unbounded domains.

In this section we consider the Dirichlet problem in the complementary of a set $\Omega \subset \mathbb{R}^n$, which is assumed to be a bounded open set with Lipschitz boundary.

**Definition.** A open set $\Omega \subseteq \mathbb{R}^n$ has Lipschitz boundary if at every $x \in \partial \Omega$ there exists a neighbourhood $U$ of $x$, a open bounded set $D \subseteq \mathbb{R}^{n-1}$ and a Lipschitz function $\phi : D \to \mathbb{R}$ such that (up to a suitable orthogonal transformation of coordinates)

$$\partial \Omega \cap U = \{(x, \phi(x)) \mid x \in D\}.$$ 

**Remark.** If $\Omega$ is a open set with Lipschitz boundary, then at every point of the boundary of $\Omega$ the exterior and interior cone condition is satisfied. So, every point in $\partial \Omega$ is regular for the Laplacian (both for the Dirichlet problem in the set $\Omega$ and for the Dirichlet problem in the set $\mathbb{R}^n \setminus \Omega$).

**Theorem 1.** Assume $n \geq 3$. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and $g \in C(\partial \Omega)$. Then there exists a unique solution $u \in C^\infty(\mathbb{R}^n \setminus \overline{\Omega}) \cap C(\mathbb{R}^n \setminus \Omega)$ of the exterior Dirichlet problem

$$\begin{cases}
-\Delta u = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega} \\
u(x) = g(x) & \text{in } \partial \Omega \\
\lim_{|x| \to +\infty} u(x) = 0.
\end{cases}$$

Moreover there exists $c > 0$ such that,

$$|u(x)| \leq \frac{c}{|x|^\frac{n}{2}}.$$ 

**Proof.** We divide the proof in various steps.

For simplicity we assume that $\mathbb{R}^n \setminus \Omega$ is connected. If it is not true, all the arguments can be carried out on every connected component of $\mathbb{R}^n \setminus \Omega$.

**Uniqueness.** Assume $u, v$ are two solutions to (1), then $w = u - v$ is a continuous function in $\mathbb{R}^n \setminus \Omega$ such that $w = 0$ on $\partial \Omega$ and $\lim_{|x| \to +\infty} w(x) = 0$. Then by Weierstrass theorem, either $w \equiv 0$ or $w$ admits a positive maximum and/or a negative minimum in $\mathbb{R}^n \setminus \Omega$. But $w$ is harmonic in $\mathbb{R}^n \setminus \overline{\Omega}$, and then by strong maximum principle, the only possibility is that $w \equiv 0$.

**Existence of a solution.**

Let $r > 0$ such that $B(0, r) \supset \Omega$. For every $R > r$, let $u_R \in C^\infty(B(0, R) \setminus \overline{\Omega}) \cap C(B(0, R) \setminus \Omega)$ be the solution of the approximating problem

$$\begin{cases}
-\Delta u_R = 0 & \text{in } B(0, R) \setminus \overline{\Omega} \\
u_R(x) = g(x) & \text{in } \partial \Omega \\
u_R(x) = 0 & \text{in } \partial B(0, R).
\end{cases}$$

Note that this solution exists by Perron-Wiener theorems (all the boundary points of $\mathbb{R}^n \setminus \Omega$ are regular for the Laplacian) and it is unique by Maximum principle. Extend the function $u_R$ to a function $u_R : \mathbb{R}^n \setminus \Omega \to \mathbb{R}$ by putting $u_R(y) = 0$ if $|y| > R$. So $u_R \in C^\infty(B(0, R) \setminus \overline{\Omega}) \cap C(\mathbb{R}^n \setminus \Omega)$.

Observe that by weak maximum principle we have that $\|u_R\|_{\infty} = \|g\|_{\infty}$ so $(u_R)_{R > r}$ is an equibounded family of continuous functions, harmonic in sets $B(0, R) \setminus \overline{\Omega}$.

Moreover, by Cauchy estimates on harmonic functions, for every compact $K \subseteq \mathbb{R}^n \setminus \overline{\Omega}$, there exists $C = C_K$ such that $\sup_K |D u_R| \leq C \|g\|_{\infty}$, for any $R$ such that $K \subset B(0, R)$.

Then, by Ascoli Arzela theorem and a standard diagonalization procedure, we can extract from $u_R$ a subsequence (which we will continue to denote as $u_R$) converging locally uniformly in $\mathbb{R}^n \setminus \overline{\Omega}$ to a function $u \in C(\mathbb{R}^n \setminus \overline{\Omega})$. We can extend continuously $u$ up to $\partial \Omega$ putting $u = g$. 


Finally, \( u \) is harmonic in \( \mathbb{R}^n \setminus \overline{\Omega} \). In fact, fix \( x \in \mathbb{R}^n \setminus \overline{\Omega} \) and a open neighbourhood \( U \) of \( x \) in \( \mathbb{R}^n \setminus \overline{\Omega} \), then there exists \( R_0 \) such that for all \( R > R_0 \), \( U \subset B(0, R) \). In \( U \), every function \( u_R \) with \( R > R_0 \) is harmonic, and then also \( u \) is harmonic in \( U \) since it is the uniform limit of harmonic functions.

In conclusion, \( u \in C^1(\mathbb{R}^n \setminus \overline{\Omega}) \cap C(\mathbb{R}^n \setminus \Omega) \) solves
\[
\begin{cases}
-\Delta u = 0 & x \in \mathbb{R}^n \setminus \overline{\Omega} \\
u(x) = g(x) & x \in \partial\Omega.
\end{cases}
\]

**Behaviour at \( \infty \).** It remains to check that the function \( u \) we have constructed vanishes at infinity. Consider \( r_0 \) as before and \( R > 0 \). Observe that
\[
w_{r,R} = \frac{r^{n-2}}{|x|^{n-2}} \frac{R^{n-2} - |x|^{n-2}}{R^{n-2} - r^{n-2}}
\]
is the solution of the Dirichlet problem
\[
\begin{cases}
-\Delta w_{r,R} &= 0 & r < |x| < R \\
w_{r,R}(x) &= 1 & |x| = r \\
w_{r,R}(x) &= 0 & |x| = R.
\end{cases}
\]
(Exercise: check it!).

By weak Maximum principle (applied in the set \( r \leq |x| \leq R \)), we get that
\[
|u_R(x)| \leq \|g\|_{\infty} w_{r,R}(x) = \|g\|_{\infty} \frac{r^{n-2}}{|x|^{n-2}} \frac{R^{n-2} - |x|^{n-2}}{R^{n-2} - r^{n-2}} \quad r \leq |x| \leq R.
\]

Passing to the limit \( R \to +\infty \) in this inequality (along the converging subsequence \( u_R \) obtained in the previous step), we get
\[
|u(x)| \leq \|g\|_{\infty} \frac{r^{n-2}}{|x|^{n-2}} \quad \forall |x| \geq r
\]
which concludes the proof. \( \square \)

**Remark.** The solution \( u \) of (1) with boundary data \( g \equiv 1 \) is called (conductor) potential of \( \Omega \) and the capacity of \( \Omega \) is defined as
\[
cap(\Omega) = \lim_{|x| \to +\infty} \frac{u(x)|x|^{n-2}}{n(n-2)\omega_n}.
\]
Capacity can also be characterized as follows
\[
cap(\Omega) = \inf_{u \in \mathcal{A}} \int_{\mathbb{R}^n \setminus \Omega} |Du|^2 \, dx
\]
where \( \mathcal{A} = \{ u \in C^1(\mathbb{R}^n \setminus \Omega) \mid u(x) = 1 \text{ if } x \in \partial \Omega \text{ and } \lim_{|x| \to +\infty} u(x) = 0 \} \).

Note that if \( u \in \mathcal{A} \) is a minimum and \( \Omega \) has \( C^1 \) boundary, then
\[
cap(\Omega) = \int_{\mathbb{R}^n \setminus \Omega} |Du|^2 \, dx = - \int_{\partial \Omega} \frac{\partial u}{\partial n} \, dS
\]
where \( n \) is the exterior normal to \( \Omega \) (check it!)

The definition of capacity can be extended to domains with non smooth boundaries (by approximation arguments) and is useful to characterize the \( \Delta \)-regular points of the boundary (see the Wiener criterion, e.g. [Gilbarg, Trudinger]).
We consider now the case of \( \mathbb{R}^2 \).

**Theorem 2.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded open set with Lipschitz boundary and \( g \in C(\partial \Omega) \). Then there exists a unique solution \( u \in C^\infty(\mathbb{R}^n \setminus \overline{\Omega}) \cap C(\mathbb{R}^n \setminus \Omega) \) of the exterior Dirichlet problem

\[
\begin{cases}
-\Delta u = 0 & x \in \mathbb{R}^n \setminus \overline{\Omega} \\
u(x) = g(x) & x \in \partial \Omega \\
\lim_{|x| \to +\infty} \frac{u(x)}{\log |x|} = 0.
\end{cases}
\] (5)

Moreover \( u \) is bounded.

**Remark.** For \( n = 2 \) we weaken the condition to vanish at infinity, by requiring just the condition to grow less than logarithmically. In particular this implies that there exists a unique bounded solution to (5). In \( \mathbb{R}^n \) with \( n \geq 3 \), we cannot expect a similar result to hold. Observe e.g. that both \( u(x) = r^{n-2}|x|^{2-n} \) and \( v(x) \equiv 1 \) are bounded solutions of the Dirichlet problem in the exterior of the ball of radius \( r \)

\[
\begin{cases}
-\Delta u = 0 & |x| > r \\
u(x) = 1 & |x| = r.
\end{cases}
\]

**Proof.** As for the existence of a solution to

\[
\begin{cases}
-\Delta u = 0 & x \in \mathbb{R}^n \setminus \overline{\Omega} \\
u(x) = g(x) & x \in \partial \Omega.
\end{cases}
\]

the proof is exactly the same as in the proof of Theorem 1. Moreover \( |u(x)| \leq ||g||_\infty \) for every \( x \in \mathbb{R}^n \setminus \Omega \). So \( u \) is bounded.  

So, it remains to prove the uniqueness. Assume \( u, v \) are two solutions to (5). Then \( w = u - v \) is harmonic in \( \mathbb{R}^n \setminus \overline{\Omega} \), \( w = 0 \) on \( \partial \Omega \) and \( \lim_{|x| \to +\infty} \frac{w(x)}{\log |x|} = 0 \).

Take \( x_0 \in \Omega \) and \( r > 0 \) such that \( B(x_0, r) \subset \subset \Omega \). Observe that also

\[
\lim_{|x| \to +\infty} \frac{w(x)}{|x| - |x_0| - \log r} = 0.
\]

Exploiting this last property we get that for every \( \varepsilon > 0 \) there exists \( M > 0 \) such that \( \Omega \subset B(0, M) \) and

\[
|w(x)| \leq \varepsilon \log \left(\frac{|x - x_0|}{r}\right) \quad \forall |x| \geq M.
\]

Note that the function \( \varepsilon \log \left(\frac{|x - x_0|}{r}\right) \) is harmonic in \( \mathbb{R}^2 \setminus \Omega \) and moreover on \( \partial \Omega, \varepsilon \log \left(\frac{|x - x_0|}{r}\right) > 0 \) (by our choice of \( r \)). Then, by weak Maximum principle (applied to the set \( B(0, M) \setminus \Omega \)) we get that

\[
|w(x)| \leq \varepsilon \log \left(\frac{|x - x_0|}{r}\right) \quad \forall x \in \mathbb{R}^2 \setminus \Omega.
\]

We conclude by the arbitrariness of \( \varepsilon \) that \( w = 0 \). 

\[ \square \]