Introduzione alle equazioni alle derivate parziali,
Laurea Magistrale in Matematica

The Dirichlet problem in unbounded domains.

In this section we consider the Dirichlet problem in the complementary of a set $\Omega \subset \mathbb{R}^n$, which is assumed to be a bounded open set with Lipschitz boundary.

Definition. *A open set $\Omega \subseteq \mathbb{R}^n$ has Lipschitz boundary if at every $x \in \partial\Omega$ there exists a neighbourhood U of x, a open bounded set $D \subseteq \mathbb{R}^{n-1}$ and a Lipschitz function $\phi : D \to \mathbb{R}$ such that (up to a suitable orthogonal transformation of coordinates)*

$$\partial\Omega \cap U = \{(x, \phi(x)) \; |x \in D\}.$$

Remark. If Ω is a open set with Lipschitz boundary, then at every point of the boundary of Ω the exterior and interior cone condition is satisfied. So, every point in $\partial\Omega$ is regular for the Laplacian (both for the Dirichlet problem in the set Ω and for the Dirichlet problem in the set $\mathbb{R}^n \setminus \Omega$.

Theorem 1. *Assume $n \geq 3$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and $g \in \mathcal{C}(\partial\Omega)$. Then there exists a unique solution $u \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \overline{\Omega}) \cap \mathcal{C}(\mathbb{R}^n \setminus \Omega)$ of the exterion Dirichlet problem*

$$\begin{cases} -\Delta u = 0 & x \in \mathbb{R}^n \setminus \overline{\Omega} \\ u(x) = g(x) & x \in \partial\Omega \\ \lim_{|x| \to +\infty} u(x) = 0. \end{cases} \tag{1}$$

Moreover there exists $c > 0$ such that,

$$|u(x)| \leq \frac{c}{|x|^{n-2}}.$$

Proof. We divide the proof in various steps.

For simplicity we assume that $\mathbb{R}^n \setminus \Omega$ is connected. If it is not true, all the arguments can be carried out on every connected component of $\mathbb{R}^n \setminus \Omega$.

Uniqueness. Assume u, v are two solutions to (1), then $w = u - v$ is a continuous function in $\mathbb{R}^n \setminus \Omega$ such that $w = 0$ on $\partial\Omega$ and $\lim_{|x| \to +\infty} w(x) = 0$. Then by Weierstrass theorem, either $w \equiv 0$ or w admits a positive maximum and/or a negative minimum in $\mathbb{R}^n \setminus \Omega$. But w is harmonic in $\mathbb{R}^n \setminus \overline{\Omega}$, and then by strong maximum principle, the only possibility is that $w \equiv 0$.

Esistence of a solution.

Let $r > 0$ such that $B(0, r) \supset \Omega$. For every $R > r$, let $u_R \in \mathcal{C}^\infty(B(0, R) \setminus \overline{\Omega}) \cap \mathcal{C}(\overline{B(0, R)} \setminus \Omega)$ be the solution of the approximating problem

$$\begin{cases} -\Delta u_R = 0 & x \in B(0, R) \setminus \overline{\Omega} \\ u_R(x) = g(x) & x \in \partial\Omega \\ u_R(x) = 0 & x \in \partial B(0, R). \end{cases} \tag{2}$$

Note that this solution exists by Perron-Wiener theorems (all the boundary points of $\mathbb{R}^n \setminus \Omega$ are regular for the Laplacian) and it is unique by Maximum principle. Extend the function u_R to a function $u_R : \mathbb{R}^n \setminus \Omega \to \mathbb{R}$ by putting $u_R(y) = 0$ if $|y| > R$. So $u_R \in \mathcal{C}^\infty(B(0, R) \setminus \overline{\Omega}) \cap \mathcal{C}(\mathbb{R}^n \setminus \Omega)$.

Observe that by weak maximum principle we have that $\|u_R\|_\infty = \|g\|_\infty$ so $(u_R)_{R>r}$ is a equibounded family of continuous functions, harmonic in sets $B(0, R) \setminus \overline{\Omega}$.

Moreover, by Cauchy estimates on harmonic functions, for every compact $K \subseteq \mathbb{R}^n \setminus \overline{\Omega}$, there exists $C = C_K$ such that $\sup_K |Du_R| \leq C\|g\|_\infty$, for any R such that $K \subset B(0, R)$.

Then, by Ascoli Arzelà theorem and a standard diagonalization procedure, we can extract from u_R a subsequence (which we will continue to denote as u_R) converging locally uniformly in $\mathbb{R}^n \setminus \overline{\Omega}$ to a function $u \in \mathcal{C}(\mathbb{R}^n \setminus \overline{\Omega})$. We can extend continuously u up to $\partial\Omega$ putting $u = g$.

Finally, u is harmonic in $\mathbb{R}^n \setminus \overline{\Omega}$. In fact, fix $x \in \mathbb{R}^n \setminus \overline{\Omega}$ and a open neighbourhood U of x in $\mathbb{R}^n \setminus \overline{\Omega}$, then there exists R_0 such that for all $R > R_0$, $U \subset B(0, R)$. In U, every function u_R with $R > R_0$ is harmonic, and then also u is harmonic in U since it is the uniform limit of harmonic functions.

In conclusion, $u \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \overline{\Omega}) \cap \mathcal{C}(\mathbb{R}^n \setminus \Omega)$ solves

$$\begin{cases} -\Delta u = 0 & x \in \mathbb{R}^n \setminus \overline{\Omega} \\ u(x) = g(x) & x \in \partial\Omega. \end{cases}$$

Behaviour at ∞. It remains to check that the function u we have constructed vanishes at infinity. Consider r_0 as before and $R > 0$. Observe that

$$w_{r,R} = \frac{r^{n-2}}{|x|^{n-2}} \frac{R^{n-2} - |x|^{n-2}}{R^{n-2} - r^{n-2}} \tag{3}$$

is the solution of the Dirichlet problem

$$\begin{cases} -\Delta w_{r,R} = 0 & r < |x| < R \\ w_{r,R}(x) = 1 & |x| = r \\ w_{r,R}(x) = 0 & |x| = R. \end{cases} \tag{4}$$

(Exercise: check it!).

By weak Maximum principle (applied in the set $r \leq |x| \leq R$), we get that

$$|u_R(x)| \leq \|g\|_\infty w_{r,R}(x) = \|g\|_\infty \frac{r^{n-2}}{|x|^{n-2}} \frac{R^{n-2} - |x|^{n-2}}{R^{n-2} - r^{n-2}} \qquad r \leq |x| \leq R.$$

Passing to the limit $R \to +\infty$ in this inequality (along the converging subsequence u_R obtained in the previous step), we get

$$|u(x)| \leq \|g\|_\infty \frac{r^{n-2}}{|x|^{n-2}} \qquad \forall\ |x| \geq r$$

which concludes the proof. □

Remark. The solution u of (1) with boundary data $g \equiv 1$ is called (conductor) potential of Ω and the capacity of Ω is defined as

$$cap(\Omega) = \lim_{|x| \to +\infty} \frac{u(x)|x|^{n-2}}{n(n-2)\omega_n}.$$

Capacity can also be characterized as follows

$$cap(\Omega) = \inf_{u \in \mathcal{A}} \int_{\mathbb{R}^n \setminus \Omega} |Du|^2 dx$$

where $\mathcal{A} = \{u \in \mathcal{C}^1(\mathbb{R}^n \setminus \Omega) \mid u(x) = 1 \text{ if } x \in \partial\Omega \text{ and } \lim_{|x| \to +\infty} u(x) = 0\}$.

Note that if $u \in \mathcal{A}$ is a minimum and Ω has $\mathcal{C}^1$ boundary, then

$$cap(\Omega) = \int_{\mathbb{R}^n \setminus \Omega} |Du|^2 dx = -\int_{\partial\Omega} \frac{\partial u}{\partial n} dS$$

where n is the exterior normal to Ω (check it!)

The definition of capacity can be extended to domains with non smooth boundaries (by approximation arguments) and is useful to characterize the Δ-regular points of the boundary (see the Wiener criterion, e.g. [Gilbarg, Trudinger]).

We consider now the case of $\mathbb{R}^2$.

Theorem 2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary and $g \in \mathcal{C}(\partial\Omega)$. Then there exists a unique solution $u \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \overline{\Omega}) \cap \mathcal{C}(\mathbb{R}^n \setminus \Omega)$ of the exterion Dirichlet problem*

$$\begin{cases} -\Delta u = 0 & x \in \mathbb{R}^n \setminus \overline{\Omega} \\ u(x) = g(x) & x \in \partial\Omega \\ \lim_{|x|\to+\infty} \frac{u(x)}{\log|x|} = 0. \end{cases} \tag{5}$$

Moreover u is bounded.

Remark. For $n = 2$ we weaken the condition to vanish at infinity, by requiring just the condition to grow less than logarithmically. In particular this implies that there exists a unique bounded solution to (5). In $\mathbb{R}^n$ with $n \geq 3$, we cannot expect a similar result to hold. Observe e.g. that both $u(x) = r^{n-2}|x|^{2-n}$ and $v(x) \equiv 1$ are bounded solutions of the Dirichlet problem in the exterior of the ball of radius r

$$\begin{cases} -\Delta u = 0 & |x| > r \\ u(x) = 1 & |x| = r. \end{cases}$$

Proof. As for the existence of a solution to

$$\begin{cases} -\Delta u = 0 & x \in \mathbb{R}^n \setminus \overline{\Omega} \\ u(x) = g(x) & x \in \partial\Omega. \end{cases}$$

the proof is exactly the same as in the proof of Theorem 1. Moreover $|u(x)| \leq \|g\|_\infty$ for every $x \in \mathbb{R}^n \setminus \Omega$. So u is bounded.

So, it remains to prove the uniqueness. Assume u, v are two solutions to (5). Then $w = u - v$ is harmonic in $\mathbb{R}^n \setminus \overline{\Omega}$, $w = 0$ on $\partial\Omega$ and $\lim_{|x|\to+\infty} \frac{w(x)}{\log|x|} = 0$.

Take $x_0 \in \Omega$ and $r > 0$ such that $B(x_0, r) \subset\subset \Omega$. Observe that also

$$\lim_{|x|\to+\infty} \frac{w(x)}{\log|x - x_0| - \log r} = 0.$$

Exploiting this last property we get that for every $\varepsilon > 0$ there exists $M > 0$ such that $\Omega \subset B(0, M)$ and

$$|w(x)| \leq \varepsilon \log\left(\frac{|x - x_0|}{r}\right) \qquad \forall |x| \geq M.$$

Note that the function $\varepsilon \log\left(\frac{|x-x_0|}{r}\right)$ is harmonic in $\mathbb{R}^2 \setminus \Omega$ and moreover on $\partial\Omega$, $\varepsilon \log\left(\frac{|x-x_0|}{r}\right) > 0$ (by our choice of r). Then, by weak Maximum principle (applied to the set $B(0, M) \setminus \Omega$) we get that

$$|w(x)| \leq \varepsilon \log\left(\frac{|x - x_0|}{r}\right) \qquad \forall x \in \mathbb{R}^2 \setminus \Omega.$$

We conclude by the arbitrariness of ε that $w = 0$. □