Introduzione alle equazioni alle derivate parziali,
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Maximum principle for parabolic operators

Let $\Omega \subset \mathbb{R}^n$ be a open set , $T > 0$ and L be the following linear elliptic operator in $\Omega_T = \Omega \times (0, T)$

$$k(x,t)u_t(x,t)+Lu(x,t) := k(x,t)u_t(x,t)-\text{tr } a(x,t)D^2_x u(x,t)+b(x,t)\cdot D_x u(x,t) \qquad (x,t) \in \Omega\times(0,T),$$

where $D^2_x u(x,t) = (u_{x_i x_j}(x,t))_{i,j=1,\dots,n}$ and $D_x u(x,t) = (u_{x_i}(x,t))_{i=1,\dots,n}$ are the hessian and the gradient with respect to the x coordinates.

We assume the following general conditions on the coefficients of L.

Assumption 1. $a : \Omega_T \to S_n$ is a bounded continuous function, where S_n is the space of symmetric $n \times n$ matrices).
$b : \Omega_T \to \mathbb{R}^n$ is a bounded continuous function.
$k : \Omega_T \to \mathbb{R}$ is a bounded continuous function.

Moreover we assume that $ku_t + L(u)$ is a parabolic operator according to this definition.

Definition. The operator u_t+Lu is parabolic if there exists $k_0 > 0$ such that for every $(x,t) \in \Omega_T$, $k(x,t) \geq k_0 > 0$, and for every $(x,t) \in \Omega_T$ $a(x,t)$ is a $n \times n$ symmetric positive semidefinite matrix (i.e. all the eigenvalues of $a(x)$ are real and nonnegative).

Moreover we consider the following function.

Assumption 2. $c : \Omega \to R$ is a bounded function .

Remark. Note that we are not asking that c is a nonnegative (neither continuous) function.

For parabolic problem, there is a relevant part of the boundary, called the parabolic boundary.

Definition. [parabolic boundary] Let $\Omega_T = \Omega \times (0,T)$. Then the parabolic boundary is $\partial^\star\Omega_T = \partial\Omega \times [0,T] \cup \overline{\Omega} \times \{0\}$.

The previous assumptions will hold throughout this part.

Weak maximum principle for parabolic operators

In this section we will consider parabolic operators of the form $k(x,t)u_t + Lu + c(x,t)u$ where $(x,t) \in \Omega_T$ which satisfy, besides the standing assumptions, also the following.

Assumption 3. For all $(x,t) \in \Omega_T$ such that $c(x,t) = 0$ there exist $\mu > 0$ and $\delta > 0$ such that

$$a_{11}(y,s) > \mu \qquad \forall (y,s) \in B((x,t),\delta). \tag{1}$$

We assume the solutions to the parabolic problem are classical, in the sense that belong to the following set

$$\mathcal{C}^{2,1}(\Omega_T) = \{u : \Omega_T \to \mathbb{R} \mid u(\cdot,t) \in \mathcal{C}^2(\Omega) \;\; u(x,\cdot) \in \mathcal{C}^1(0,T) \;\; \forall x \in \Omega, t \in (0,T)\}.$$

Theorem 1 (Weak maximum principle)**.** *Let Ω be a bounded open set and $u \in \mathcal{C}^{2,1}(\Omega_T) \cap \mathcal{C}(\overline{\Omega_T})$ such that $k(x,t)u_t + Lu + c(x,t)u \leq 0$, where L, k and c are as above. Assume moreover that $c \geq 0$.*

- *If $c \equiv 0$, then $\max_{\overline{\Omega}_T} u = \max_{\partial^\star\Omega_T} u$,*
- *if $c \not\equiv 0$, then $\max_{\overline{\Omega}_T} u \leq \max_{\partial^\star\Omega_T} u^+$, where $u^+(y) := \max(u(y), 0)$.*

Proof. Let $c \equiv 0$. A parabolic operator is in particular a degenerate elliptic operator. So under our assumptions, weak maximum principle holds. This implies that $\max_{\overline{\Omega}_T} u = \max_{\partial\Omega_T} u$. Assume by contradiction that $u(y,s) < \max_{\Omega\times\{T\}} u$ for every $(y,s) \in \Omega_T \cap \partial^\star\Omega_T$. Take $0 < \varepsilon << T$ and define $v_\varepsilon(x,t) = u(x,t) - \varepsilon t$. So $v_\varepsilon \to u$ uniformly in $\overline{\Omega}_T$ as $\varepsilon \to 0$. Let $(x_\varepsilon, t_\varepsilon)$ such that $v_\varepsilon(x_\varepsilon, t_\varepsilon) = \max_{\overline{\Omega}\times[0,T-\varepsilon]} v_\varepsilon$. Then, by uniform convergence, $(x_\varepsilon, t_\varepsilon)$ converge, up to a subsequence, as $\varepsilon \to 0$, to a point (x,t) such that $u(x,t) = \max_{\overline{\Omega}\times[0,T]} u$. By our assumption, necessarily $(x,t) \in \Omega \times \{T\}$.

We compute $(v_\varepsilon)_t = u_t - \varepsilon$, $D_x v_\varepsilon = D_x u_\varepsilon$ and $D_x^2 v_\varepsilon = D_x^2 u$. So

$$k(x,t)(v_\varepsilon)_t + Lv_\varepsilon(x,t) = k(x,t)u_t + Lu(x,t) - \varepsilon k(x,t) \le -\varepsilon k_0 < 0. \tag{2}$$

Moreover, since $(x_\varepsilon, t_\varepsilon) \to (x,t) \in \Omega \times \{T\}$ and $t_\varepsilon \le T_\varepsilon$, we have that for ε sufficiently small x_ε is in the interior of Ω. This implies $D_x v_\varepsilon(x_\varepsilon, t_\varepsilon) = 0$ and $D_x^2 v_\varepsilon(x_\varepsilon, t_\varepsilon) \le 0$. Moreover by maximality $(v_\varepsilon)_t(x_\varepsilon, t_\varepsilon) \ge 0$. So, using the fact that the operator is parabolic,

$$k(x_\varepsilon, t_\varepsilon)(v_\varepsilon)_t(x_\varepsilon, t_\varepsilon) + Lv_\varepsilon(x_\varepsilon, t_\varepsilon) \ge 0. \tag{3}$$

But this is in contradiction with (2).

If $c \not\equiv 0$, then the same arguments apply. We assume by contradiction that $u(y,s) < \max_{\Omega\times\{T\}} u$ for every $(y,s) \in \Omega_T \cap \partial^\star\Omega_T$ and that $\max_{\Omega\times\{T\}} u > 0$. In place of (2) we get

$$k(x,t)(v_\varepsilon)_t + Lv_\varepsilon(x,t) + c(x,t)v_\varepsilon(x,t) = k(x,t)u_t + Lu(x,t) + c(x,t)u(x,t) - \varepsilon k(x,t) - \varepsilon t c(x,t) \le -\varepsilon k_0 < 0$$

and in place of (3)

$$k(x_\varepsilon, t_\varepsilon)(v_\varepsilon)_t(x_\varepsilon, t_\varepsilon) + Lv_\varepsilon(x_\varepsilon, t_\varepsilon) + c(x_\varepsilon, t_\varepsilon)v_\varepsilon \ge c(x_\varepsilon, t_\varepsilon)v_\varepsilon(x_\varepsilon, t_\varepsilon) \ge 0$$

since $v_\varepsilon(x_\varepsilon, t_\varepsilon) \to u(x,t) > 0$. □

Remark. It is possible to state also the weak minimum principle (exercise).

The main consequence of the weak maximum principle is the comparison principle, in which it is only needed to assume that c is bounded (not necessarily nonnegative).

Corollary 1 (Weak comparison principle)**.** *Let $u, v \in \mathcal{C}^{2,1}(\Omega_T) \cap \mathcal{C}(\overline{\Omega_T})$ such that $ku_t + Lu + c(x)u \le 0$, and $kv_t + Lv + cv \ge 0$ in Ω, where L and c satisfies the same assumptions as above.*

If $u \le v$ in $\partial^\star\Omega_T$, then $u \le v$ in $\overline{\Omega_T}$.

Proof. Let $w = u - v$, then $kw_t + Lw + cw \le 0$ in Ω_T and $w \le 0$ on $\partial^\star\Omega$. If $c(x,t) < 0$ at some point, define $v(x,t) = e^{-\frac{\inf c}{k_0}t} w(x,t)$.

We get

$$0 \ge e^{-\frac{\inf c}{k_0}t}\left(kw_t + Lw + cw\right) = kv_t + Lv + \left(c - \frac{\inf c}{k_0}k\right)v.$$

Recalling that $k(x,t) \ge k_0 > 0$ for every x, t and that $\inf c < 0$, we obtain that

$$c(x,t) - \frac{\inf c}{k_0}k(x,t) \ge c(x,t) - \inf c \ge 0 \qquad \forall x, t.$$

So v is a subsolution of the parabolic operator $kv_t + Lv + \tilde{c}v$ where the coefficient $\tilde{c}$ is bounded and nonnegative.

So by the weak maximum principle $\max_{\overline{\Omega}_T} v \le 0$, then also $\max_{\overline{\Omega}_T} w \le 0$, which gives the conclusion. □

The comparison principle implies as usual a uniqueness result (which can be stated for unbounded intervals of time).

Corollary 2 (Uniqueness for the Cauchy-Dirichlet problem). *Let Ω be a bounded open set, then the Cauchy-Dirichlet problem*

$$(D)\begin{cases} ku_t + Lu + c(x,t)u = f(x,t) & (x,t) \in \Omega \times (0,+\infty) \\ u(x,t) = g(x,t) & x \in \partial\Omega \ t \in (0,+\infty) \\ u(x,0) = u_0(x) & x \in \overline{\Omega} \end{cases}$$

admits at most one solution $u \in \mathcal{C}^{2,1}(\Omega \times (0,+\infty) \cap \mathcal{C}(\overline{\Omega} \times [0,+\infty))$.

Proof. If u_1, u_2 are two solutions, then $w = u_1 - u_2$ satisfies $kw_t + Lw + cw = 0$ in $\Omega \times (0,T)$ for every $T > 0$ and $w = 0$ on $\partial^\star\Omega_T$. By the weak maximum and minimum principle $\max_{\overline{\Omega}\times[0,T]} |w| = 0$, which gives the conclusion, by the arbitrariness of T. □

Proposition 1 (Continuous dependance estimates). *Let $f \in \mathcal{C}(\overline{\Omega_T})$, $u_0 \in \mathcal{C}(\overline{\Omega})$ and $g \in \mathcal{C}(\partial\Omega \times (0,T)$ such that $g(x,0) = u_0(x)$.*

Let $u \in \mathcal{C}^{2,1}(\Omega_T) \cap \mathcal{C}(\overline{\Omega_T})$ the solution to the Cauchy-Dirichlet problem

$$(D)\begin{cases} ku_t + Lu + c(x,t)u = f(x,t) & (x,t) \in \Omega \times (0,+\infty) \\ u(x,t) = g(x,t) & x \in \partial\Omega \ t \in (0,+\infty) \\ u(x,0) = u_0(x) & x \in \overline{\Omega}. \end{cases}$$

Then

$$\max_{\overline{\Omega}_T} |u| \leq \|u_0\|_\infty + \|g\|_\infty + \frac{\|f\|_\infty}{k_0} T.$$

Proof. Define $w(x,t) = u(x,t) - \|u_0\|_\infty - \|g\|_\infty - \frac{\|f\|_\infty}{k_0} t$. Then

$$k(x,t)w_t + Lw + c(x,t)w = k(x,t)u_t - k(x,t)\frac{\|f\|_\infty}{k_0} + Lu + c(x,t)u - c(x,t)(\|u_0\|_\infty + \|g\|_\infty + \frac{\|f\|_\infty}{k_0} t) \leq$$

$$\leq f(x,t) - \|f\|_\infty \leq 0$$

since $c(x,t) \geq 0$ and $k(x,t)\frac{\|f\|_\infty}{k_0} \geq \|f\|_\infty$. Moreover $w(x,t) \leq 0$ for $(x,t) \in \partial^\star\Omega_T$. So, by weak maximum principle we have that

$$u(x,t) \leq \|u_0\|_\infty + \|g\|_\infty + \frac{\|f\|_\infty}{k_0} t \leq \|u_0\|_\infty + \|g\|_\infty + \frac{\|f\|_\infty}{k_0} T$$

for every $(x,t) \in \overline{\Omega}_T$. The other inequality is obtained similarly using weak minimum principle. □

Strong maximum principle for parabolic operators

In this section we will consider uniformly parabolic operators, according to the following definition.

Definition. Let $ku_t + Lu$ be a parabolic operator. Then it is uniformly elliptic in Ω_T if there exists $\lambda > 0$ such that

$$\xi^t a(x,t)\xi \geq \lambda|\xi|^2 \qquad \forall (x,t) \in \Omega_T \ \forall \xi \in \mathbb{R}^n.$$

Remark. Note that a uniformly parabolic operator is a degenerate elliptic operator (not uniformly elliptic!)

Also for parabolic operators, there is a strong maximum principle, that we are not going to prove (the proof is based on Harnack inequality for uniformly parabolic operators and can be found in Evans, PDEs).

Theorem 2 (Strong maximum principle). *Let Ω be a connected set and $u \in \mathcal{C}^{2,1}(\Omega_T) \cap \mathcal{C}(\overline{\Omega_T})$ such that $k(x,t)u_t + Lu + c(x)u \leq 0$, where L, k and c are as above. Assume moreover that $c \geq 0$.*

- *If* $c \equiv 0$*, and there exists* $(x,t) \in \Omega_T$ *such that* $M = \max_{\overline{\Omega}_T} u = u(x,t)$*, then* $u(y,s) \equiv M$ *for all* $y \in \overline{\Omega}$ *and all* $s \in [0,t]$*;*
- *if* $c \not\equiv 0$*, and there exists* $(x,t) \in \Omega_T$ *such that* $M = \max_{\overline{\Omega}_T} u = u(x,t) \geq 0$*, then* $u(y,s) \equiv M$ *for all* $y \in \overline{\Omega}$ *and all* $s \in [0,t]$*.*

Remark. We can state as follows this maximum principle: if u attains a maximum (a nonnegative maximum if $c \not\equiv 0$) at some interior point, then u is contant at all earlier times