1

Introduction to Conservation Laws

- Course DE2 : an introduction to nonlinear first order PDEs

- Hamilton - Jacobi equ

(HJ) $\quad v_t + H(\nabla v) = 0 \quad t \geq 0,\ x \in \mathbb{R}^n$

$H : \mathbb{R}^n \to \mathbb{R}$ convex, coercive : $\lim_{|p| \to \infty} \frac{H(p)}{|p|} = +\infty$

- $n = 1$, v differentiable a.e., $u = v_x$

(1) $\quad u_t + H'(u)\, u_x = 0$

(2) $\quad u_t + H(u)_x = 0$

$[H(u)]_x$

(CL) $\quad u_t + f(u)_x = 0$ — Scalar Conservation Law in 1-space variable

$f : \mathbb{R} \to \mathbb{R},\ x \in \mathbb{R}$

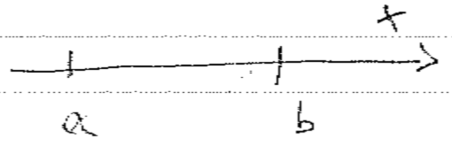
$(\underline{\text{flux}})$

$u(t,x)$ sol of (CL) : conserved quantity

Integrating (CL) over an interval $[a,b]$ (formally) one obtains

$$\frac{d}{dt}\int_a^b u(t,x)\,dx = \int_a^b u_t(t,x)\,dx = -\int_a^b \left[f(u(t,x))\right]_x dx$$

$$= f(u(t,a)) - f(u(t,b))$$



[inflow at a] [outflow at b]

Quantity u neither created, neither destroyed
Total amount of u contained in $[a,b]$ can change only due to the flow of u across endpoints a, b.

Notice : deriving formally (CL) we obtain an eqn as (1):

$$(3) \qquad u_t + a(u)\,u_x = 0 \qquad \left(a(u) = f'(u)\right)$$

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If we consider smooth sol'ns of (CL) then (CL) is equivalent to (3). But we know that in general (CL) admits <u>discontinuous sol'ns</u> (Burgers' eqn $u_t + \left(\frac{u^2}{2}\right)_x = 0, \quad u(0,x) = -x$ has discont. sol. at time $t = 1$). In this case (CL) contains a product of discontinuous function $a(u)$ with a distributional derivative u_x (if $u(t,\cdot)$ contains a discont at $\bar{x}$, then $u_x(t,\cdot)$ contains a Dirac mass at $\bar{x}$) which is not well defined in general.

However, (CL) is an <u>equation</u> <u>in divergence form</u>:

$$(4) \qquad \operatorname{div}(u, f(u)) = 0$$

$$\left(\operatorname{div}(u(t,x), v(t,x)) = u_t + v_x \right)$$

$\Rightarrow$ may consider discont. sol'ns of (4) interpreted in <u>distributional sense</u>

$u(t,x)$ is a weak (distributional) sol. if (CL)
iff:

(5) $$\iint \left[u(t,x)\, \phi_t(t,x) + f(u(t,x))\, \phi_x(t,x) \right] dx\,dt = 0$$

for all test functions $\phi \in C_c^1$.
(we require $u(t,x)$, $f(u(t,x))$ locally integr.)

$\boxed{\text{Example 1 (Traffic flow)}}$

- M. J. Lighthill & G.B Whitham '55
- P. I. Richards '56

$\rho(t,x)$: density of cars in a lane of highway at point x, time t
(# cars per Km)

Assume: - traffic only in one direction,
- no passing allowed
- great number of cars (consider average behaviour)
- no entrance or exit in the piece of line considered

$v = v(\rho)$: average velocity of cars depends only on their density

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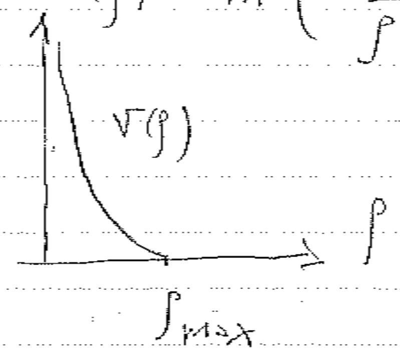
$\frac{dv}{d\rho} < 0$ (speed decrease as concentration increase)

Ex: $v(\rho) = \ln\left(\frac{\rho_{max}}{\rho}\right)$, ρ_{max} : max allowed density



Given two points a, b on the highway, variation of # of cars between a, b is expressed by :

$$\frac{d}{dt}\int_a^b \rho(t,x)\,dx = [\text{inflow at } x=a] - [\text{outflow at } x=b]$$

$$= v(\rho(t,a))\cdot\rho(t,a) - v(\rho(t,b))\cdot\rho(t,b)$$

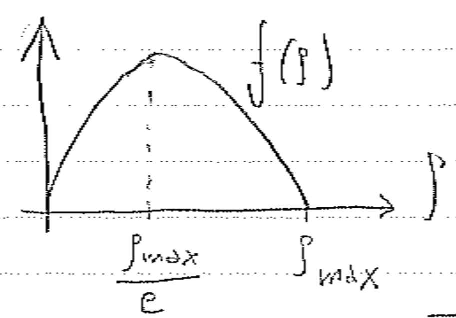
$$= -\int_a^b \frac{\partial}{\partial x}\left[v(\rho(t,x))\,\rho(t,x)\right]dx$$

(# cars is conserved)

[$f(\rho) = \rho\, v(\rho)$

flow of cars at density ρ

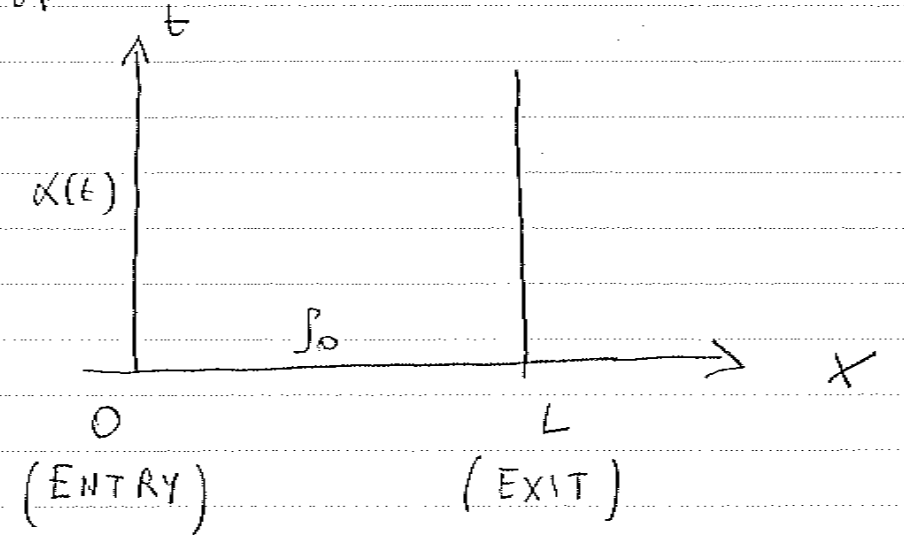
Ex: $f(\rho) = \rho \ln\left(\frac{\rho_{max}}{\rho}\right)$]



Control theory point of view : investigate effect of external controls affecting evolution of the system, for example through boundary cond's.

$\boxed{\text{Control Pb (Traffic flow)}}$

Minimize average time spent by cars between arrival at entry and travelling through an exit



$\alpha = \alpha(t)$: density of cars entering at $x = 0$ (treated as a control)

$\rho_0 = \rho_0(x)$: initial density of cars in $[0, L]$

$F = F(t)$: incoming flow of cars (# of cars arriving at $x=0$ per unit time)

4/ Consider mixed initial-boundary value problem (IBVP)

$$(6)_\alpha \begin{cases} \partial_t \rho + \partial_x [\rho v(\rho)] = 0 & (f(\rho) = \rho v(\rho) \text{ flux}) \\ \rho(0,x) = \rho_0(x) \\ \rho(t,0) \text{ "=" } \alpha(t) \end{cases}$$

$\rho_\alpha(t,x)$: sol. of $(6)_\alpha$

Constraints :

(7) $\quad 0 \leq \alpha(t) \leq \rho_{max} \quad \forall t$

(8) $$\underbrace{\int_0^t f(\alpha(s))\,ds}_{\text{entering flow}} \leq \underbrace{\int_0^t F(s)\,ds}_{\text{incoming flow}}$$

(9) $\exists T > 0$ s.t. $\alpha(t) = F(t) = \rho(t,L) = 0$ (no car arrives at the entry or enter the line after suff. large time)

(10) $$\int_0^T f(\alpha(s))\,ds = \int_0^T F(s)\,ds = \int_0^T f(\rho_\alpha(s,L))\,ds$$

(all cars arrived at the entry eventually pass through the entry and reach the exit)

Admissible boundary controls :

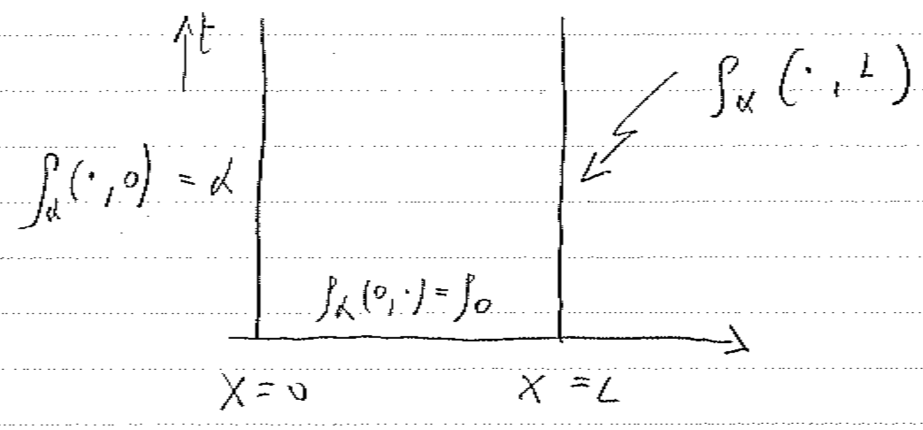
$$\mathcal{U} = \{ \alpha : [0,T] \to \mathbb{R} \,,\ \text{measurable}, \ \text{satisfying } (7), (8), (9), (10) \}$$

Optimal control pb :

$$\min_{\alpha \in \mathcal{U}} \left\{ \underbrace{\frac{\int_0^T t\, f(\rho_\alpha(t,L))\, dt}{\int_0^T F(t)\, dt}}_{\substack{\text{average incoming} \\ \text{time at exit } x=L}} - \underbrace{\frac{\int_0^T t\, F(t)\, dt}{\int_0^T F(t)\, dt}}_{\substack{\text{average incoming} \\ \text{time at entry } x=0}} \right\} \qquad (11)$$

$$\Downarrow$$

$$\min_{\alpha \in \mathcal{U}} \int_0^T t\, f(\rho_\alpha(t,L))\, dt \qquad (12)$$



Need to study structure of sol'ns to $(6)_\alpha$ at $x = L$

5/ **Remarks:**

1) $f(\rho) = \rho\, v(\rho)$ is <u>concave</u>

$$\Downarrow$$

we may provide <u>explicit representation of solutions to</u> (Cauchy pb & IBVP) for (CL) with convex (or concave) flux which extends Hopf - lax formula for (HJ) : <u>Lax-Oleinik formula</u> for (CL).

Relying on L-O formula we may provide <u>characterization</u> of <u>reachable set</u> (attainable configurations at $x = L$)

$$R(L) = \{ \rho_\alpha(\cdot, L) \,;\ \alpha \in \mathcal{U} \}$$

2) Admissible boundary controls satisfying (7) - (10) satisfy <u>convex constraints</u> of the form:

i) $\alpha(t) \in G(t)$,

$G : [0,T] \to \mathcal{P}(\mathbb{R})$ multifunction with convex valued values $G(t)$

ii) $\displaystyle\int_0^t \eta\left(f(\alpha(s))\right) ds \leq c(t)$

$\eta : \mathbb{R} \to \mathbb{R}$ convex, $c : [0,T] \to \mathbb{R}$ continuous

$\Downarrow$ [1) - 2)]

<u>$R(L)$ compact in $L^1([0,T])$</u>

Therefore, since the map

$\rho \in L^1([0,T])$, $\displaystyle\rho \mapsto \int_0^T t\, f(\rho(t))\, dt$, $\|\rho\|_\infty \leq \rho_{max}$

is continuous w.r.t. $L^1([0,T])$ it follows <u>$\exists$ sol. to optimal control problem</u> (12) (and hence to (11))

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Remark 3: if set admissible boundary controls don't satisfies convex constraint one may have a reachable set $R(L)$ which is not closed w.r.t L^1 topology

Remark 4: relying on a comparison principle one can actually construct optimal solution.

i) Reformulate (12)

$$\int_0^T \int_0^t f(\alpha(s))\,ds\,dt = \left(t \int_0^t f(\alpha(s))\,ds \right)\Bigg|_{t=0}^{t=T} +$$

$$= \int_0^T t\, f(\alpha(t))\,dt$$

$$\Rightarrow \int_0^T t\, f(\alpha(t))\,dt = -\int_0^T \int_0^t f(\alpha(s))\,ds\,dt - T\int_0^T f(\alpha(s))\,ds$$

$$\underset{(10)}{=} -\int_0^T \int_0^t f(\alpha(s))\,ds\,dt - T\int_0^T F(s)\,ds$$

$$\Rightarrow \quad \min_{\alpha \in \mathcal{U}} \int_0^T t\, f(\rho_\alpha(t,L))\, dt \quad \text{equivalent to}$$

$$(13) \qquad \max_{\alpha \in \mathcal{U}} \int_0^T \int_0^t f(\rho_\alpha(s,L))\, ds\, dt$$

ii) Rely on <u>comparison principle</u> for sol'ns to (CL) (as for sol'ns to (HJ))

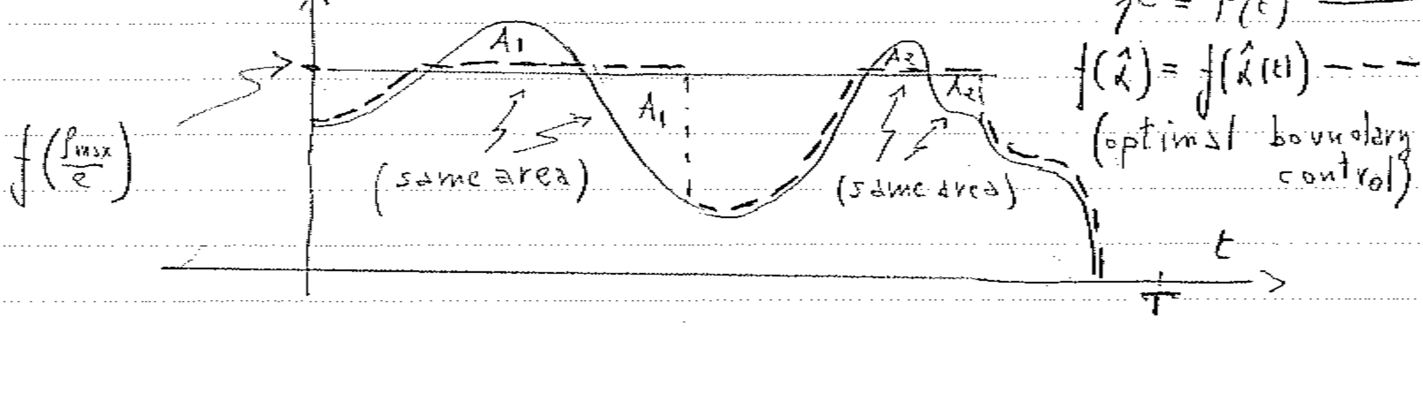
Indeed one can establish both a pointwise comparison principle and a comparison property envolving the integral of solutions

$$\int_0^t f(\alpha(s))\, ds \le \int_0^t f(\beta(s))\, ds \quad \forall t$$

$$(\alpha, \beta \in \mathcal{U})$$

$$\Downarrow$$

$$\int_0^t f(\rho_\alpha(s,L))\, ds \le \int_0^t f(\rho_\beta(s,L))\, ds \quad \forall t$$



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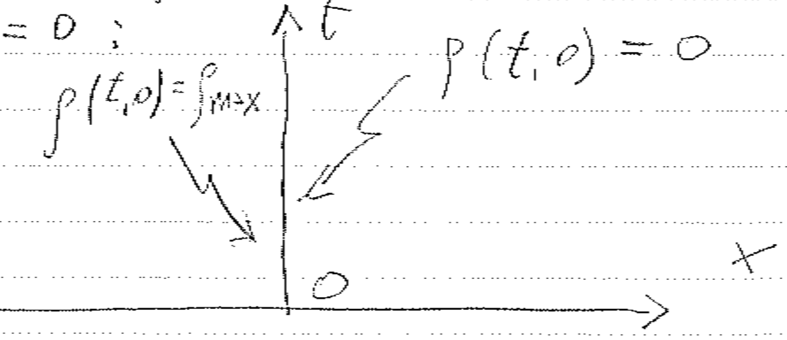
Traffic flow at a junction $\Longleftrightarrow$ Networks

Analyzing traffic flow at a <u>traffic light</u> say located at $x = 0$.

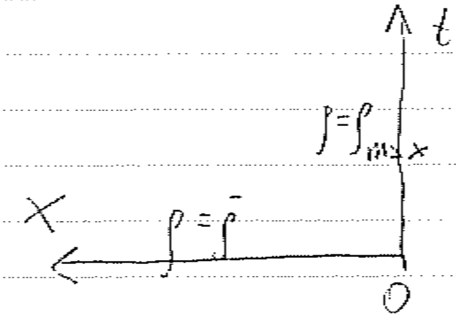
Assume: i) initial datum $\rho_0(x) = \bar{\rho}$ const

(light is green : cars flow through the line with a uniform density)

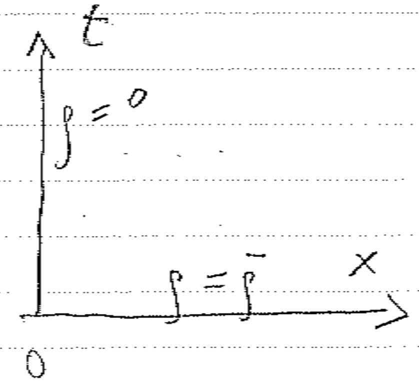
ii) red light starts $\Longleftrightarrow$ corresponds to impose two boundary conditions on the left and on the right of $x = 0$:



$\Rightarrow$ Evolution of traffic density around $x = 0$ as long as light remains red given by the solutions of two IBVP

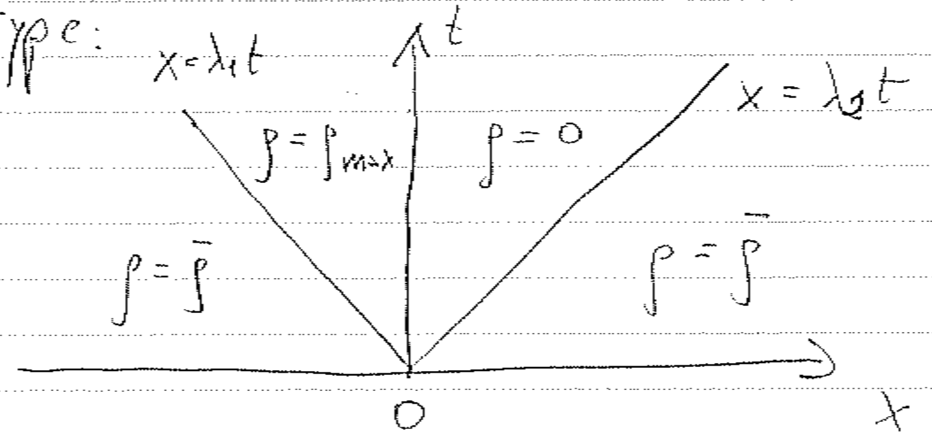


$$(14)\quad \begin{cases} \rho_t + [\rho\, v(\rho)]_x = 0 \\ \rho(0,x) = \bar\rho & t \geq 0,\ x \leq 0 \\ \rho(t,0) = \rho_{max} \end{cases}$$



$$(15)\quad \begin{cases} \rho_t + [\rho\, v(\rho)]_x = 0 \\ \rho(0,x) = \bar\rho & t \geq 0,\ x \geq 0 \\ \rho(t,0) = 0 \end{cases}$$

<u>Claim</u>: we expect sol'ns of (14), (15) of the type:



$$\rho(t,x) = \begin{cases} \bar\rho & x < \lambda_1 t \\ \rho_{max} & \lambda_1 t < x < 0 \\ 0 & 0 < x < \lambda_2 t \\ \bar\rho & \lambda_2 t < x \end{cases} \qquad (16)$$

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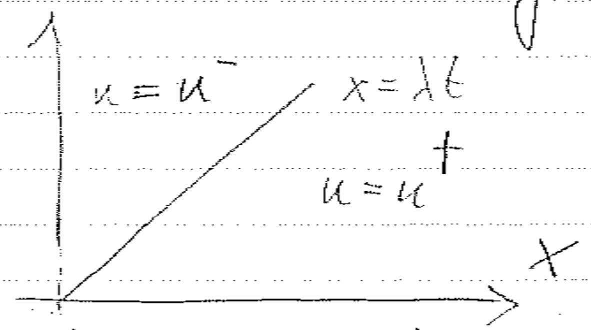
In order to verify that $\rho = \rho(t,x)$ is weak distributional sol. of (14), (15) consider more general situation.

$$(CL) \qquad u_t + f(u)_x = 0$$

<u>Goal</u>: derive conditions which must be satisfied by piecewise constant function

$$(17) \qquad u(t,x) = \begin{cases} u^+ & \text{if } x > \lambda t \\ u^- & \text{if } x < \lambda t \end{cases}$$

for some $u^-, u^+, \lambda \in \mathbb{R}$, to be weak distributional sol. of (CL)



<u>Rankine - Hugoniot conditions</u>

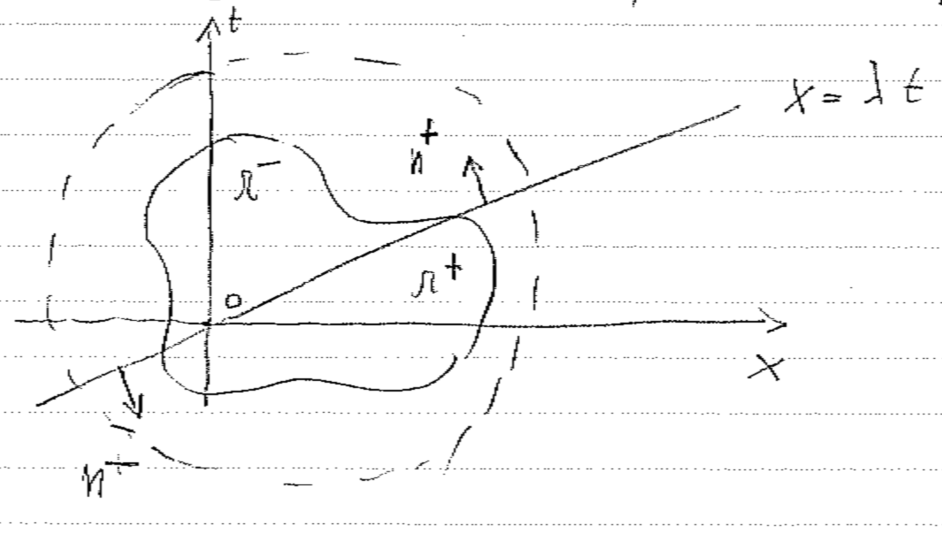
<u>Lemma 1</u>: the function $u = (t,x)$ in (17) is a weak solution of (CL) iff

$$(17) \qquad \lambda(u^+ - u^-) = f(u^+) - f(u^-) \quad \left\{ \begin{array}{l} \text{Rankine-} \\ \text{Hugoniot} \\ \text{equation} \end{array} \right.$$

<u>Proof</u>: let $\phi = \phi(t,x)$ be C^1 with compact support, let Ω be open disc containing $supp(\phi)$, consider

$$\Omega^+ = \{(t,x) \in \Omega ;\ x > \lambda t\}$$

$$\Omega^- = \{(t,x) \in \Omega ;\ x < \lambda t\}$$



Normal vector to line $x = \lambda t$ (arclength)

$$\underline{n}^+ ds = (\lambda, -1) dt, \quad \underline{n}^- ds = (-\lambda, 1)\ dt$$

Consider vector field $\underline{v}(t,x) = \big(u(t,x)\,\phi(t,x),\ f(u(t,x))\,\phi(t,x)\big)$

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Assume $u(t,x)$ defined in (17) is weak sol of (CL)

Then:

$$0 = \iint_{\Omega} \{u \phi_t + f(u) \phi_x\}(t,x)\, dx\, dt \overset{\left(\substack{u \text{ constant on} \\ \text{each domain } \Omega^-, \Omega^+}\right)}{=}$$

$$= \iint_{\Omega^+} \operatorname{div} \underline{v}(t,x)\, dx\, dt + \iint_{\Omega^-} \operatorname{div} \underline{v}(t,x)\, dx\, dt$$

$$= \int_{\partial\Omega^+} \underline{v} \cdot \underline{n}^+\, ds + \int_{\partial\Omega^-} \underline{v} \cdot \underline{n}^-\, ds =$$

(divergence thm on each domain Ω^-, Ω^+)

$$= \int [\lambda u^+ - f(u^+)]\, \phi(t, \lambda t)\, dt +$$

$$\int [-\lambda u^- + f(u^-)]\, \phi(t, \lambda t)\, dt \Longrightarrow$$

$$0 = \int \left[\lambda(u^+ - u^-) - \left(f(u^+) - f(u^-)\right)\right] \phi(t, \lambda t)\, dt$$

$\forall \phi \in C_c^1 \Rightarrow$ (17) holds

Repeating backward the same argument one shows that the function $u(t,x)$ in (17) is a weak sol to (CL).

<u>Example 1</u>: Burgers eqn $f(u) = \frac{u^2}{2}$

$$\lambda = \frac{f(u^+) - f(u^-)}{u^+ - u^-} = \frac{1}{2}\,\frac{(u^+)^2 - (u^-)^2}{u^+ - u^-} =$$

$$= \frac{1}{2}\left(u^+ + u^-\right)$$

$$\Rightarrow u(t,x) = \begin{cases} u^- & \text{if } x < \frac{u^+ + u^-}{2}\cdot t \\ u^+ & \text{if } x > \frac{u^+ + u^-}{2}\cdot t \end{cases}$$

is a weak sol of $u_t + \left(\frac{u^2}{2}\right)_x = 0$

<u>Example 2</u>: Traffic flow at a red light

$$\lambda_2 = \frac{f(\bar{\rho}) - f(0)}{\bar{\rho} - 0} = \frac{\bar{\rho}\, v(\bar{\rho})}{\bar{\rho}} = v(\bar{\rho})$$

$$\lambda_1 = \frac{f(\rho_{max}) - f(\bar{\rho})}{\rho_{max} - \bar{\rho}} = \frac{\rho_{max}}{\rho_{max} - \bar{\rho}}\, v(\rho_{max})$$

$$\lambda_3 = \frac{f(0) - f(\rho_{max})}{0 - \rho_{max}} = 0$$

$$\Longrightarrow \rho(t,x) = \begin{cases} \bar\rho & \text{if } x < \dfrac{\rho_{max} v(\rho_{max})}{\rho_{max} - \bar\rho} t \\ \rho_{max} & \text{if } \dfrac{\rho_{max} v(\rho_{max})}{\rho_{max} - \bar\rho} t < x < 0 \\ 0 & \text{if } 0 < x < v(\bar\rho) t \\ \bar\rho & \text{if } v(\bar\rho) \cdot t < x \end{cases}$$

is weak sol of $\begin{cases} \rho_t + [\rho v(\rho)]_x = 0 \\ \rho(0,x) = \bar\rho \end{cases}$

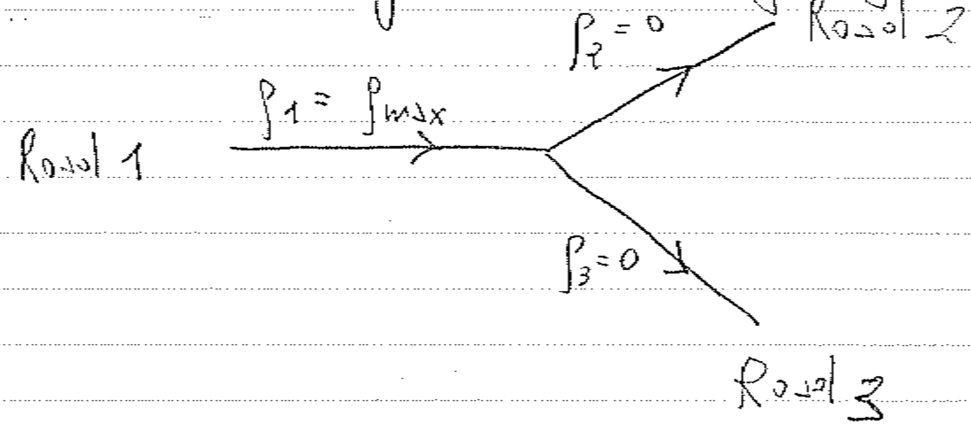
Restriction of $\rho(t,x)$ to $[0,+\infty[\times [0,+\infty[$ is sol of (15)

Restriction of $\rho(t,x)$ to $[0,+\infty[\times]-\infty, 0]$ is sol. of (14)

<u>Notice</u>: discontinuity in the sol. of (14) travelling (in t-x plane) with speed λ_1 corresponds to the formation of a queue in the traffic

Note: we have analyzed behaviour of sol at red light between an incoming and an outgoing road by solving two IBVP

What about a junction between an incoming and two outgoing roads?



$$(\rho_i)_t + (\rho_i\, v(\rho_i))_x = 0$$ ← 3 conservation laws

$i = 1, 2, 3$

Assume light green starts and cars start to pass through. There can be two extreme behaviour:

- all cars flow towards the left outgoing road
- all cars flow towards the right outgoing road

11/ <u>Notice</u> : conservation of cars do not guarantees uniqueness of solutions.

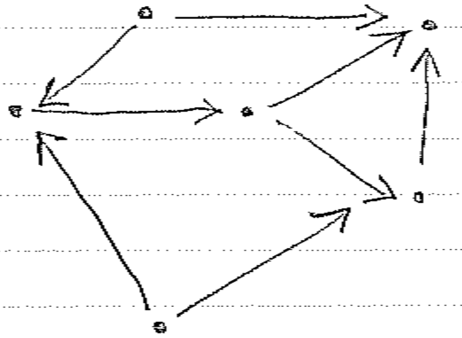
Need to take into account drivers preference and attitude to pass through the junction

<u>Prescribe</u> : i) Traffic <u>distribution coefficients</u> (percentage of cars travelling to each outgoing road)

ii) <u># cars passing</u> the junction is the <u>maximum possible</u> (respecting rule i))

Rules i)-ii) allow to select unique sol'n if # outgoing roads $\geq$ # incoming road.

Analysis of <u>Conservation laws on networks</u> : collection of <u>arcs</u> (edges) connecting <u>nodes</u> (vertices, junction). Along each arc evolution is given by a conservation law, at each node junction condition must be prescribed.



Given departure times of N drivers, and the path along which they (can) travel, describe overall traffic pattern.

Optimization pb : choose departure rate so to minimize overall cost for departing early plus cost for arriving late.

Analysis of CL on networks :

- urban vehicular traffic
- data transmission on internet, telecommunication
- gas pipeline
- air traffic management
- supply chains
- blood circulation - vascular stents

M. Garavello & B. Piccoli, Traffic flow on network, 2006.

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Example 2 (Supply Chain)

- Factory which produces large number of items in many steps can be modelled as a continuous flow.

  ↳ Describe average behavior of production system where items visit a machine more than once (highly re-entrant factory)

  ex: semiconductor manufacturing

  $\rho(t,x)$: density of produced item at time t, at stage $x \in [0,1]$ of production process

  ($x=0$: beginning of production line, $x=1$: end of pr. line)

  $v(\rho(\cdot))$: production velocity depends on total load (total number of produced parts both downstream and upstream in the production line).

↳ Non local velocity :

$$v(\rho(\cdot)) = \phi\left(\int_0^1 \rho(x)\,dx\right)$$

$$\left(\phi(s) = \frac{v_{max}}{1+s}\right)$$

↳ Non local conservation law :

$$\rho_t + \underbrace{\left[\rho\, v(\rho(\cdot))\right]_x}_{} = 0$$

flux depends on all values attained by function ρ

- D. Armbuster, D. E. Marthaler, C. Ringhofer, K. Kempf & T.-C. Jo, 2006

Pedestrian Flow

Assume pedestrian adjust their speed according with local mean density at their position

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$\rho(t, x_1, x_2)$: density of pedestrians

$$\vec{w}(\rho(\cdot)) = \phi(\rho * \eta)\,\vec{v} :$$

$\phi : \mathbb{R} \to \mathbb{R}$

$\eta \in C_c^1(\mathbb{R}^2; [0,1])$ $\quad$ ($\vec{v} : \mathbb{R}^2 \to \mathbb{R}^2$ direction of motion of pedestrian)

$\|\eta\|_{L^1} = 1$

$$\left(\rho * \eta\,(x_1, x_2) = \iint_{\mathbb{R}^2} \rho(y_1, y_2)\, \eta(x_1 - y_1, x_2 - y_2)\, dy_1\, dy_2 \right)$$

average of values attained by ρ in $B_{L^1}((x_1, x_2), 1)$

↳ <u>2D - Non local conservation law :</u>

$$\rho_t + \operatorname{div}_x \left(\rho\, \vec{w}(\rho(\cdot)) \right) = 0$$

Euler equations of gas dynamics (1755)

compressible, non-viscous gas in Eulerian coordinates

$$\begin{cases} \rho_t + (\rho v)_x = 0 & \text{conservation of mass} \\ (\rho v)_t + (\rho v^2 + p)_x = 0 & \text{conservation of momentum} \\ (\rho E)_t + (\rho E v + p v)_x = 0 & \text{conservation of energy} \end{cases}$$

$\rho = \rho(t, x)$: density of gas

$v = v(t, x)$: local velocity of gas

$E = e + \frac{v^2}{2}$: density of energy

internal energy   kinetic energy

$p = p(\rho, e)$: constitutive gas law.

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– State of the art –

1. 1D - Scalar Conservation Laws

$$u_t + [f(u)]_x = 0, \quad x \in \mathbb{R}, u \in \mathbb{R}$$
$$f : \mathbb{R} \to \mathbb{R}$$

- E. Hopf (1950), O.A. Oleinik (1954), P.D. Lax (1954)

(viscous Burgers eqn)

2. MULTID - Scalar Conservation Laws

$$u_t + \mathrm{div}_x [f(u)] = 0, \quad x \in \mathbb{R}^n, u \in \mathbb{R}$$
$$f : \mathbb{R} \to \mathbb{R}^n$$

- S. N. Kruzhkov (1969)

(complete theory Cauchy pb.)

3. 1D - Systems of Conservation Laws

$$u_t + [f(u)]_x = 0, \quad x \in \mathbb{R}, \quad u \in \mathbb{R}^N$$
$$f : \mathbb{R}^N \to \mathbb{R}^N$$

- P.D. Lax (1957), J. Glimm (1965), T.P. Liu (1977, 1981)

A. Bressan & collab. (1995, 2000), A. Bressan & S. Bianchini (200

small BV data)

(Most general well-posedness result on Cauchy pb with

4. Multi D - Systems of Conservation Laws

$$u_t + \sum_{\alpha=1}^{n} \left[f^{\alpha}(u) \right]_{x_\alpha} = 0, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^N$$

$$f^{\alpha} : \mathbb{R}^N \to \mathbb{R}^N$$

Well-posedness results only for particular class of systems (with radial symmetry in the flux)

- L. Ambrosio & C. De Lellis (2003 - 2004)

Problems to investigate:

- Cauchy problem
- Mixed initial-boundary value problem
- Control problems (with boundary or distributed controls acting on the eqn.)
- Conservation Laws on network
- Differential games related to conservation laws

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Classical (smooth) solutions of Cauchy pb & formation of singularities

(18) $$u_t + [f(u)]_x = 0, \quad t \geq 0, \ x \in \mathbb{R}$$

(19) $$u(0,x) = u_0(x), \quad x \in \mathbb{R}$$

$$f \in \mathcal{C}^2(\mathbb{R}), \quad u_0 \in \mathcal{C}^1(\mathbb{R})$$

Pb 1: find maximal value $T > 0$ s.t. (18)-(19) admits a smooth $\mathcal{C}^1$ sol. on $[0,T[\times \mathbb{R}$, i.e. there exists a $\mathcal{C}^1$ function that satisfies (19) and the quasilinear equation:

(20) $$u_t + f'(u)\, u_x = 0 \quad \text{at any } (t,x) \in \,]0,T[\times \mathbb{R}.$$

- Method of characteristics

Characteristic curve for (18):

$t \mapsto x(t,y)$, $t \geq 0$ sol. of

(21) $$\begin{cases} \dot{x} = f'(u(t,x)) \\ x(0) = y \end{cases}$$

Observe that, if $u(t,x)$ smooth sol. of (20), $x(t)$ sol. of (21), then

$$\frac{d}{dt}\, u(t, x(t)) = u_t + u_x \dot{x} = 0$$

$\Rightarrow$ 1. smooth sol. are constant along characteristics

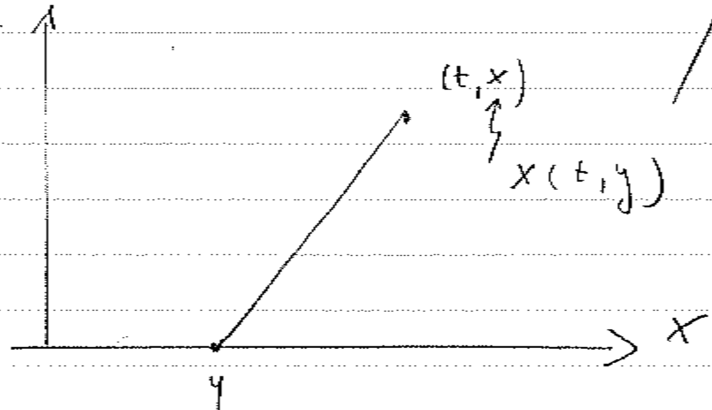
2. slope of characteristics is constant

↳ sol of (21) is:

$$x(t,y) = y + t\, f'(u_0(y)), \quad t \geq 0 \tag{22}$$

$$\left(\begin{aligned} f'(u(t, x(t,y))) &= f'(u(0, x(0,y))) \\ &= f'(u(0,y)) \\ &= f'(u_0(y)) \end{aligned} \right)$$

3. The value of smooth sol'n to (18-19) at point (t,x) can be determined as follows:



i) draw characteristic line from (t,x) until it intersects x-axis at point y s.t.

$$y = x - t\, f'(u_0(y)) = x - t\, f'(u(t,x))$$

16/ ii) Since u constant along characteristics we derive implicit eqn on u:

(23) $$u(t,x) = u_0\left(x - t\, f'(u(t,x))\right)$$

Therefore Pb 1 is equivalent to

<u>Pb 2</u>: <u>find maximal $T>0$ s.t.</u> for any fixed $t \in [0,T]$

(24) $$G(t,x,u) = u - u_0\left(x - t\, f'(u)\right) = 0$$

has unique C^1 sol. $u = u(t,x)$, $x \in \mathbb{R}$

Notice: i) when $t=0$, (24) takes the form

$$G(0,x,u) = u - u_0(x) = 0$$

which has sol. $u(0,x) = u_0(x)$.

ii) when $t>0$, by implicit funct. thm., we need cond that guarantees

$$G_u(t,x,u) \neq 0$$

for all (x,u) sol of (24)

$$G_u(t,x,u) = 1 - u_0'\left(x - t f'(u)\right)\left(-t f''(u)\right)$$

$$= 1 + t\, u_0'\left(x - t f'(u)\right) f''\left(u_0(x - t f'(u))\right)$$

$$u = u_0\left(x - t f'(u)\right)$$

Indeed one may solve (24) without requiring $G_u(t,x,u) > 0$ but we'll see in Remark 2 that the maximal time T s.t. (24) admits C^1 sol. coincides with (25)

Since $G_u(0,x,u) = 1$, it follows that

$$T := \sup\left\{t > 0\,;\; G_u(t,x,u) > 0,\ \forall (x,u) \text{ sol of } (24)\right\}$$

$$= \sup\left\{t > 0\,;\; 1 + t\, u_0'(y)\, f''(u_0(y)) > 0 \quad \forall y \in \mathbb{R}\right\}$$

$$= \sup\left\{t > 0\,;\; 1 + t \inf_{y \in \mathbb{R}} \left\{u_0'(y)\, f''(u_0(y))\right\} > 0\right\}$$

$$(25) = \begin{cases} +\infty & \text{if } \inf_{y \in \mathbb{R}} \left\{u_0'(y)\, f''(u_0(y))\right\} \geq 0 \\[2ex] \dfrac{-1}{\inf_{y \in \mathbb{R}} \left\{\frac{d}{dy} f'(u_0(y))\right\}}, & \text{otherwise} \end{cases}$$

(in this case $\sup\{t > 0\,;\, 1 + t \inf\{\ \} > 0\}$ is $\bar{t}$ s.t. $1 + \bar{t} \inf\{\ \} = 0$)

17/ <u>Remark 1</u> : if f <u>is convex</u> (concave) and <u>initial data $u_0 \in C^1(\mathbb{R})$</u> is <u>monotone increasing</u> (decreasing), the Cauchy pb (18)-(19) admits a <u>global</u> (defined for any time $t>0$) <u>smooth solution.</u>

Ex : Burgers equ, $f(u) = \frac{u^2}{2}$, $u_0(x) = x$

Check that $u(t,x) = \frac{x}{1+t}$, $x \in \mathbb{R}$, $t \geq 0$ is sol. of corresponding Cauchy pb.

<u>Remark 2</u> : one may provide <u>explicit definition of smooth sol. to (18)-(19)</u> as follows :

i) assume that the map

$$y \mapsto x(t,y) := y + t f'(u_0(y)), \quad y \in \mathbb{R} \tag{26}$$

is injective & surjective

ii) let $x \mapsto z(t,x)$, $x \in \mathbb{R}$

be the inverse map of the map in (26) so that $x(t, z(t,x)) = x \quad \forall x \in \mathbb{R}$

iii) Set: (Notice, $z(t,x)$ can be expressed as: $z(t,x) = x - t\, f'(u(t,x))$: cfr. p. 15)

(27) $$\boxed{u(t,x) \doteq u_0\big(z(t,x)\big)}$$

<u>Claim</u>: if i) is verified then the map in (27) provides smooth sol. to (18) - (19)

proof: by (27) we have:

(28) $$u(t, x(t,y)) = u_0\big(z(t, x(t,y))\big) = u_0(y)$$

differentiating (28) in time we find:

$$0 = u_t(t, x(t,y)) + u_x(t, x(t,y))\, \dot{x}(t,y)$$

$$= u_t(t, x(t,y)) + u_x(t, x(t,y))\, f'(u_0(y))$$

(26) $$= u_t(t, x(t,y)) + u_x(t, x(t,y))\, f'(u(t, x(t,y)))$$

(28) Since $y \mapsto x(t,y)$ is surjective we thus find:

$$u_t(t,x) + u_x(t,x) \cdot f'(u(t,x)) = 0$$

proving that the function in (27) is a sol. of (20).

Therefore, Pb 1, Pb 2 are equivalent to:

<u>Pb 3</u>: find maximal $\hat{T} > 0$ s.t. for any fixed $t \in [0, \hat{T}[$ the map in (26) is bijective from $\mathbb{R}$ to $\mathbb{R}$ (hence strictly monotone)

Notice: $\frac{\partial}{\partial y} X(t,y) = 1 + t\, f''(u_0(y))\, u_0'(y)$

$$\stackrel{!}{=} 1 + t \frac{d}{dy} f'(u_0(y))$$

Since $\frac{\partial}{\partial y} x(0,y) = 1$, it follows that

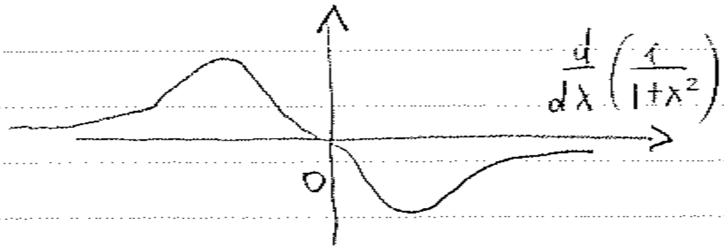
$$(29) \quad \hat{T} = \sup \left\{ t > 0 ;\ \inf_{y \in \mathbb{R}} \left\{ \frac{\partial}{\partial y} x(t,y) \right\} > 0 \right\}$$

and hence we deduce that $\hat{T}$ is precisely the time T given by formula (25).

<u>Example 1</u>: $f(u) = \frac{u^2}{2}$, $u_0(x) = \frac{1}{1+x^2}$

$$f'(u_0(x)) = \frac{1}{1+x^2}, \quad \frac{d}{dx} \frac{1}{1+x^2} = -\frac{2x}{(1+x^2)^2}$$

$$\frac{d^2}{dx^2}\frac{1}{1+x^2} = \frac{6x^2-2}{(1+x^2)^3} = 0 \iff x = \pm\frac{1}{\sqrt{3}}$$



minimum of $\frac{d}{dx}\frac{1}{1+x^2}$ attained at $x = \frac{1}{\sqrt{3}}$

$$\min_{x\in\mathbb{R}} \frac{d}{dx}\frac{1}{1+x^2} = -\frac{\frac{2}{\sqrt{3}}}{\left(\frac{4}{3}\right)^2} = -\frac{3\sqrt{3}}{8}$$

$\Rightarrow T = \frac{8}{3\sqrt{3}}$ is max time of existence of smooth solution.

<u>Remark</u>: if $\inf_{y\in\mathbb{R}}\left\{\frac{d}{dy} f'(u_0(y))\right\} = -\infty$

by (25) the Cauchy pb (18)-(19) doesn't admit a smooth sol. on any strip $[0,T[\times\mathbb{R}$

<u>Example 2</u>: $f(u) = \frac{u^2}{2}$, $u_0(x) = e^{-x}$

$$\inf_{y\in\mathbb{R}}\left\{\frac{d}{dy} f'(u_0(y))\right\} = \inf_{y\in\mathbb{R}}\left\{u_0'(y)\right\} = -\infty.$$

Exercise 1 : for the Burgers equ

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

find maximal time $T > 0$ so that the Cauchy pb (18) - (19) admits smooth C^1 sol on $[0, T[\times \mathbb{R}$ for :

i) $u_0(x) = \arctan x$

ii) $u_0(x) = -\arctan x$

iii) $u_0(x) = \sin x$

iv) $u_0(x) = -\sin x$

v) $u_0(x) = x^3$

vi) $u_0(x) = -x^3$

vii) $u_0(x) = x^2$

Exercise 2 : determine maximal time interval of existence of smooth C^1 sol'ns (as in Ex. 1) for :

i) $f(u) = \cos u$, $u_0(x) = x$

ii) $f(u) = \cos u$, $u_0(x) = \sin x$

iii) $f(u) = \frac{u^3}{3}$, $\quad u_0(x) = \sin x$

iv) $f(u) = u^4$, $\quad u_0(x) = x$

v) $f(u) = u^4$, $\quad u_0(x) = -x$

<u>Exercise 3 :</u> for the Burgers eqn

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

given $T > 0$, determine condition on initial data $u_0 \in \mathcal{C}^1(\mathbb{R})$ which guarantee existence of smooth $\mathcal{C}^1$ sol of Cauchy pb (18)-(19) on time interval $[0, T]$.

Determine such conditions for a general convex (concave) conservation law

$$u_t + [f(u)]_x = 0$$

with $f''(u) \leq K$ $\left(f''(u) \geq -K \right)$, for some $K > 0$.

20/

Formation of Singularities

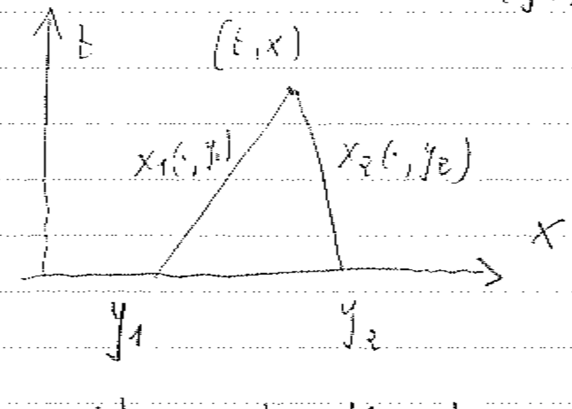
What happens when two (or more) characteristic lines $X(t, y_1)$, $X(t, y_2)$, given by (22) intersect at a point (t, x) ?

The value of the solution of (18)-(19), at (t,x) which should be equal to $u_0(y_1)$, $u_0(y_2)$, is no longer uniquely determined since

$$y_1 + f'(u_0(y_1))t = y_2 + f'(u_0(y_2))\, t \quad , y_1 \neq y_2$$

$$\Downarrow$$

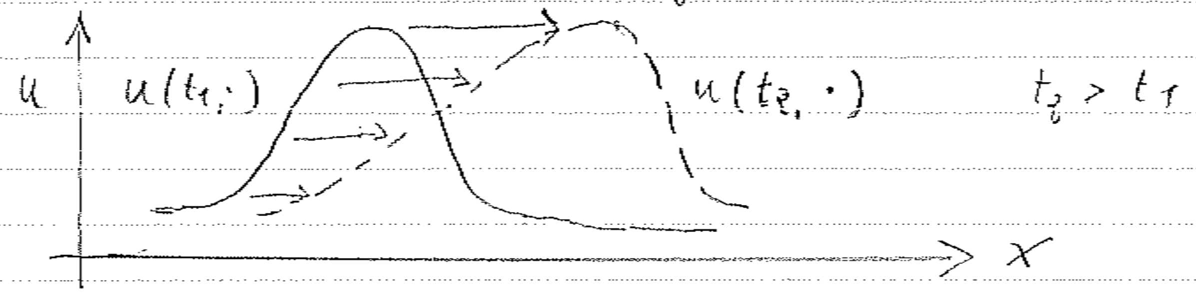
$$u_0(y_1) \neq u_0(y_2) .$$



$u(t, x) = ?$

Let's look to evolution in time of the graph of solution $u(t, \cdot)$. when $t \to T$ maximal time of existence of smooth sol'ns given by (25

Assume $f''(u) \geq 0$ for all u



Recalling implicit def. of solution:

$$(23) \qquad u(t,x) = u_0\big(x - t\, f'(u(t,x))\big)$$

we deduce that points of the solution profile $u(t,\cdot)$ moves horizontally to the right (if $f'(u(t,x)) > 0$) with speed $f'(u(t,x))$ which is bigger if their height (with respect to x-axis) $u(t,x)$ is bigger (since $f''(u) \geq 0$).

This determines a change in the profile of the solution that steepens as t approach the critical time T.

21/ In fact, differentiating w.r.t. x (23) we find

$$u_x(t,x) = u_0'(x - t f'(u(t,x)))\,(1 - t f''(\ )\,u_x(t,x))$$

$$\Downarrow$$

$$u_x(t,x)\left[1 + t f''(u(t,x))\,u_0'(x - t f'(u(t,x)))\right] =$$

$$= u_0'(x - t f'(u(t,x)))$$

$$\Downarrow$$

(30) $$u_x(t,x) = \frac{u_0'(y)}{1 + t f''(u_0(y))\cdot u_0'(y)}, \quad y = x - t f'(u(t,x))$$

By (30) it follows that, if $u_0'(y) < 0$, $\underbrace{\ }_{\text{for some}}$ $y \in \mathbb{R}$ (and $f''(u_0(y)) \geq 0$) one has:

(31) $$\lim_{t \to \hat{t}(y)-} u_x(t, x(t,y)) = -\infty$$

(32) $$\hat{t}(y) = \frac{-1}{u_0'(y)\cdot f''(u_0(y))}$$

This means that if $u_0'(y) < 0$, $f''(u) \geq 0$, then the derivative of the solution (along characteristic emanating from y) blows up at $-\infty$ (gradient catastrophe phenomenon) at the critical time $\hat{t}(y)$ given by (32).

Hence, we recover the property we found in (25) for a general con law (18) with $f \in C^2(\mathbb{R})$:

i) if $\inf_{y\in\mathbb{R}} \{ u_0'(y)\, f''(u_0(y)) \} \geq 0$ then there exists a global smooth sol. of (18)-(19)

ii) if $\inf_{y\in\mathbb{R}} \{ u_0'(y) \cdot f''(u_0(y)) \} < 0$ then there exists a smooth sol. of (18)-(19) on $[0, T[$ with:

$$T = \inf \left\{ \frac{-1}{u_0'(y) \cdot f''(u_0(y))} \; ; \; y \in \mathbb{R},\ u_0'(y) \cdot f''(u_0(y)) < 0 \right\}$$

$$= \frac{-1}{\inf_{y\in\mathbb{R}} \{ u_0'(y) \cdot f''(u_0(y)) \}} .$$

A non-smooth solution u of (18) can form <u>two types of discontinuities (singularities)</u>:

I) a <u>weak discontinuity</u> at a point $(\bar{t}, \bar{x})$ if $x \mapsto u(\bar{t}, x)$ is continuous at $\bar{x}$ but not differentiable at $\bar{x}$

II) a <u>strong discontinuity</u> at a point $(\bar{t}, \bar{x})$ if $x \mapsto u(\bar{t}, x)$ is discontinuous at $\bar{x}$ and $\lim\limits_{x \to \bar{x}\pm} u(\bar{t}, x)$ exist, finite.

(a strong discontinuity is usually called <u>a shock</u>)

<u>Remark 1</u>: if $\inf\limits_{y \in \mathbb{R}} \left\{ u_0'(y) \cdot f''(u_0(y)) \right\} < 0$ and there <u>exists a unique $y_0 \in \mathbb{R}$ where</u> such <u>inf is attained</u> then one has:

i) sol. $u(t,x)$ to (18)-(19) is smooth on $[0, T[\times \mathbb{R}$ with

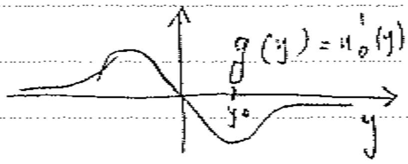
$$T = \frac{-1}{u_0'(y_0) \cdot f''(u_0(y_0))}$$

ii) a weak discontinuity occurs at the point $\left(T,\ y_0 + f'(u_0(y_0))\cdot T\right)$

iii) a strong discontinuity occurs at any $t > T$, at points (t,x) where the characteristics cross.

Example 1 p. 18 continued :

$f''(u) = 1, \quad u_0'(x) = -\frac{2x}{(1+x^2)^2}$



$g(y) = f''(u_0(y))\cdot u_0'(y) = -\frac{2y}{(1+y^2)^2}$ all sins

unique point of minimum at $y_0 = \frac{1}{\sqrt{3}}$

$\Rightarrow$ i) sol. is smooth on $[0, T[\times \mathbb{R}$ with

$$T = -\frac{1}{g\left(\frac{1}{\sqrt{3}}\right)} = \frac{8}{3\sqrt{3}} \qquad \left(\min_y g(y) = u_0'\left(\tfrac{1}{\sqrt{3}}\right)\right)$$

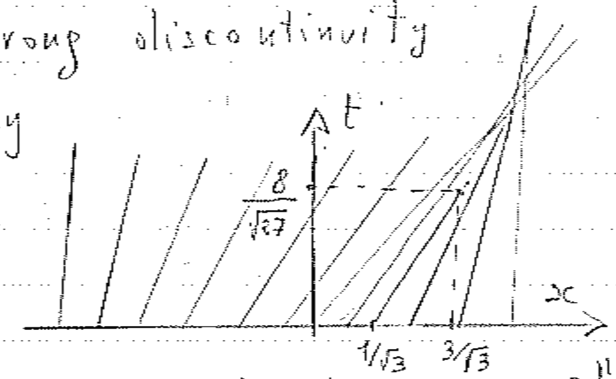
ii) a weak discontinuity occurs at

$u_0\left(\frac{1}{\sqrt{3}}\right) = \frac{3}{4}$

$$\left(\frac{8}{3\sqrt{3}},\ \frac{1}{\sqrt{3}} + \frac{8}{3\sqrt{3}}\,u_0\left(\frac{1}{\sqrt{3}}\right)\right) =$$

$$= \left(\frac{8}{3\sqrt{3}},\ \frac{1}{\sqrt{3}} + \frac{8}{3\sqrt{3}}\cdot\frac{3}{4}\right) = \left(\frac{8}{3\sqrt{3}},\ \frac{3}{\sqrt{3}}\right)$$

23 iii) a strong discontinuity occurs at any $t > \frac{8}{3\sqrt{3}}$



Characteristics:
$$x(t,y) = y + \frac{t}{1+y^2}$$

Remark 2: if $\inf_{y \in \mathbb{R}} \{ u_0'(y) \cdot f''(u_0(y)) \} = I < 0$

and $u_0'(y) \cdot f''(u_0(y)) = I \quad \forall y \in [y^-, y^+]$

for some $y^-, y^+ \in \mathbb{R}$, then:

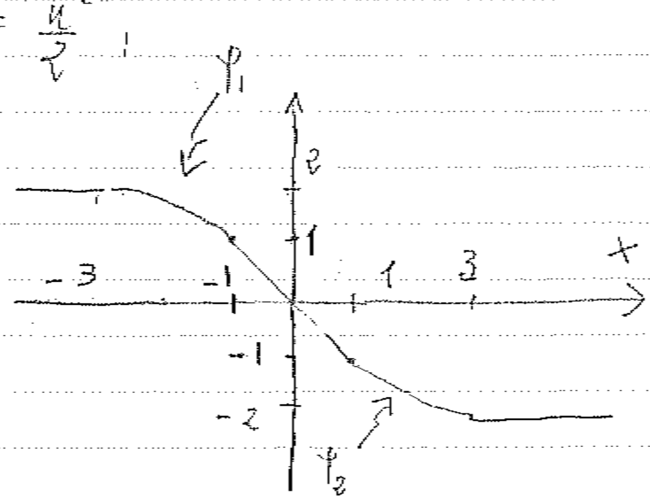
sol $u(t,x)$ to (18)-(19) is smooth on $[0, T[\times \mathbb{R}$

with $\quad T = \frac{-1}{I}$

and a strong discontinuity occurs at any $t \geqslant T$ (characteristics cross at $t \geqslant T$).

Example 3: $f(u) = \frac{u^2}{2}$,

$$u_0(x) = \begin{cases} 2 & \text{if } x \leq -3 \\ \varphi_1(x) & \text{if } -3 < x < -1 \\ -x & \text{if } -1 \leq x \leq 1 \\ \varphi_2(x) & \text{if } 1 < x < 3 \\ -2 & \text{if } x \geqslant 3 \end{cases}$$



$\varphi_i \in C^1(\mathbb{R})$ connect in smooth way the costant values of u_0 for $|x| \geq 3$ with the linear function defined by u_0 for $|x| \leq 1$.

We may also assume that $-1 < \varphi_i'(x) \leq 0$ for $1 < |x| < 3$ so that we have

$$-1 \leq u_0'(x) \leq 0 \quad \forall x \in \mathbb{R}$$ (see Note p. 24)

By (25) there exists a smooth sol. of (18)-(19) on $[0, T[\times \mathbb{R}$, with

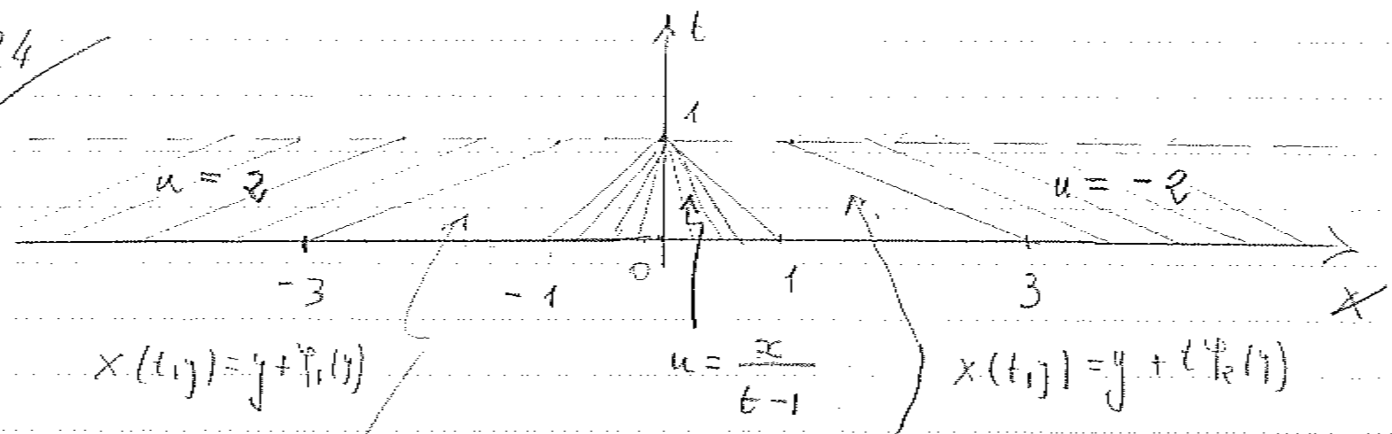
$$T = \frac{-1}{-1} = 1$$

If we constrct the characteristics we find

$$x(t,y) = y + t\, u_0(y) = \begin{cases} y + 2t & \text{if } y \leq -3 \\ y - yt & \text{if } -1 \leq y \leq 1 \\ y - 2t & \text{if } y \geq 3 \end{cases}$$

$$\Downarrow$$

$$u(t,x) = \begin{cases} 2 & \text{if } 0 \leq t \leq 1, \quad x \leq -3 + 2t \\ \dfrac{-x}{1-t} & \text{if } 0 \leq t < 1, \quad |x| \leq 1 - t \\ -2 & \text{if } 0 \leq t \leq 1, \quad x \geq 3 - 2t \end{cases}$$



characteristics (and hence solution) in these regions depend on the choice of φ_i. They surely don't intersect in the strip $[0,1]\times\mathbb{R}$ since $\varphi_i' > -1 \Rightarrow u_0'(y)$ do not attains its minimum at points $|y|>1$. In fact, by (32) we find in particular that the critical time is

$$\hat{t}(y) = \frac{-1}{\varphi_i'(y)} > 1 \quad \text{if} \quad 1 < |y| < 3$$

More generally, through each point $(1,x)$, $x\neq 0$ there passes one and only one characteristic starting at a point $(0,y)$, for some $|y|>1$. Instead $(1,0)$ is the point of intersection of all characteristics emanating from the points $(0,y)$, $|y|\le 1$.

Notice that $\lim\limits_{x\to 0^-} z(1,x) = -1$, $\lim\limits_{x\to 0^+} z(1,x) = 1$

$\Big(z(1,x) = x - u(1,x)$ denoting the inverse map of the characteristic map $x(1,y)$ at (26) $\Big)$

Therefore, it follows also that:

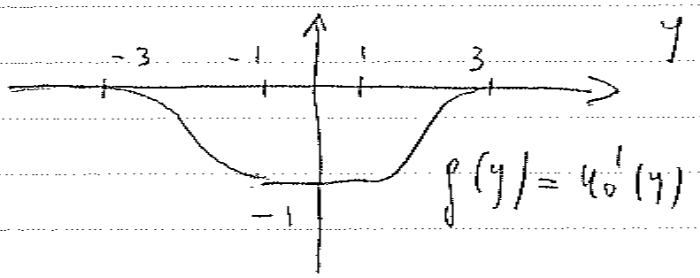
$$\lim_{x\to 0^-} u(1,x) = \lim_{x\to 0^-} u_0(z(1,x)) = \lim_{x\to 0^-} \varphi_1(z(1,x)) = \lim_{y\to -1} \varphi_1(y) = 1$$

$$\lim_{x\to 0^+} u(1,x) = \lim_{x\to 0^+} \varphi_2(z(1,x)) = -1$$

$\Rightarrow$ a strong discontinuity occurs at $(1,0)$ and one can check that a strong discont occurs also for any $t>1$ since characteristics cross for any $t \geq 1$.

<u>Notice</u>: setting $g(y) = u_0'(y) \cdot f''(u_0(y))$, one finds $g(y) = u_0'(y) =$

$$= \begin{cases} 0 & \text{if } |y| \leq 3 \\ -1 & \text{if } |y| \leq 1 \\ \varphi_1'(y) & \text{if } -3 < y < 1 \\ \varphi_2'(y) & \text{if } 1 < y < 3 \end{cases}$$



$$\min_{y\in\mathbb{R}} g(y) = -1$$

25/ <u>Exercise 4</u>: show that if $u_0(x) = -x$, the corresponding solution of the Cauchy problem for Burgers equation, at $t=1$ takes the value $u(1,x) = \begin{cases} +\infty & \text{if } x<0 \\ -\infty & \text{if } x>0 \end{cases}$

<u>Remark 3</u>: we may also analyze the evolution of u_x along characteristics as follows.

- differentiating w.r.t. x quasilinear eq. (20) we find

$$u_{tx} + f''(u)\,(u_x)^2 + f'(u)\,u_{xx} = 0$$

$$\hookrightarrow v = u_x \to \quad v_t + f'(u)\,v_x = -f''(u)\,v^2 \qquad (33)$$

- if we let $x(t) = x(t,y)$ denote a characteristic for (18), i.e. sol of (21) we find by (33)

$$\frac{d}{dt} v(t, x(t)) = -f''\big(u(t,x(t))\big)\cdot\big(v(t,x(t))\big)^2 \qquad (34)$$

- Setting $w(t) = v(t, x(t))$, and observing that $u(t,x(t)) = u(0,y) = u_0(y)$, we deduce from (34) that

$$(35) \qquad \dot{w}(t) = -f''(u_0(y))\cdot w^2(t)$$

which yields (setting $c = f''(u_0(y))$)

$$\frac{1}{w(0)} - \frac{1}{w(t)} = -ct \Rightarrow w(t) = \frac{w(0)}{1+ct\,w(0)} \qquad (36)$$

Thus, recalling

$$w(0) = u_x(0,y) = u_0'(y)$$

we deduce from (36) that the x-derivative of the sol. along the characteristic starting at y satisfies:

$$w(t) = u_x(t,x(t)) = \frac{u_0'(y)}{1+t\,f''(u_0(y))\cdot u_0'(y)}, \quad t \geq 0$$

which, if $u_0'(y) < 0$, yields (cfr. p. 21)

$$\lim_{t \to \hat{t}(y)^-} u_x(t,x(t)) = -\infty, \quad \hat{t}_y = \frac{-1}{f''(u_0(y))\cdot u_0'(y)}$$

and we recover (30), (31).

Weak (distributional) solutions

- In order to prolong sol. to (18) after the formation of a (weak or strong) discontinuity we must adopt a weak concept of solution in distributional sense which allow the presence of discontinuities (in the solution or in its space derivative).

Definition 1 a function $u \in L^\infty(]0,T[\times \mathbb{R})$ is a weak solution of (18) if, for every function $\phi \in C^1(]0,T[\times \mathbb{R})$ with compact support, one has

$$\iint_{]0,T[\times\mathbb{R}} \left[u(t,x)\,\phi_t(t,x) + f(u(t,x))\,\phi_x(t,x) \right] dt\,dx = 0 \qquad (37)$$

Remark 1: $u \in C^1(]0,T[\times \mathbb{R})$ is classical sol. of (18) iff u is a weak sol. of (18) proof of Remark 1:

i) $u(t,x)$ classical sol. $\Rightarrow$ $u(t,x)$ weak sol.

Consider vector field $V(t,x) = \left(u(t,x)\cdot\phi(t,x),\ f(u(t,x))\,\phi(t,x)\right)$

Since ϕ has compact support it follows that there will be some open domain $\Omega \subseteq]0,T[\times\mathbb{R}$ such that $\mathrm{supp}(V) \subseteq \Omega$. Hence, by divergence thm we find

$$\iint_{]0,T[\times\mathbb{R}} \mathrm{div}\, \underline{V}(t,x)\, dt\, dx = \int_{\partial\Omega} \underline{V}\cdot\underline{n}\, ds = 0$$

$$\Downarrow$$

$$0 \overset{(38)}{=} \underbrace{\iint_{]0,T[\times\mathbb{R}} \left\{ u_t + [f(u)]_x \right\} \phi\, dt\, dx}_{\substack{\| \\ \left(0 \text{ since } u(t,x) \text{ classical sol. of } (18)\right)}} + \iint_{]0,T[\times\mathbb{R}} \left\{ u\,\phi_t + f(u)\,\phi_x \right\} dt\, dx$$

Hence the second integral in (38) must be zero as well $\Rightarrow$ $u(t,x)$ is weak sol. of (18)

ii) u weak sol. $\Rightarrow$ u classical solution

Since u is smooth, if u is not classical sol. of (18)

27/ one must have

$$u_t(\bar t, \bar x) + \left[f(u(\bar t, \bar x)) \right]_x > 0 \quad (\text{or} < 0)$$

at some point $(\bar t, \bar x) \in]0, T[\times \mathbb{R}$. By regularity we then find a neighborhood of $(\bar t, \bar x)$, say $B \subseteq]0, T[\times \mathbb{R}$ s.t.

$$(39) \quad u_t(t,x) + \left[f(u(t,x)) \right]_x > 0 \quad \forall (t,x) \in B.$$

Then, consider a non-zero test function $\phi \in C_c^1(]0,T[\times \mathbb{R})$ such that $\phi(t,x) \geq 0 \ \forall (t,x)$, $\mathrm{supp}(\phi) \subseteq B$.

By (39) we find

$$0 < \iint_B \left\{ u_t + [f(u)]_x \right\} \phi \, dt\, dx = \iint_{]0,T] \times \mathbb{R}} \left\{ u_t + [f(u)]_x \right\} \phi \, dt\, dx$$

$$\underset{(38)}{=} - \iint_{]0,T[\times \mathbb{R}} \left\{ u\, \phi_t + f(u)\, \phi_x \right\} dt\, dx = 0$$

(since u is weak sol.)

which yields a contradiction. ☒

Lemma 2 (closure of set of weak sol'ns in L^1_{loc})

Let $\{u_n\}_n$ be a sequence of weak sol'ns of (18).

i) If $u_n \to u$, $f(u_n) \to f(u)$ in L^1_{loc}

then the limit function u is itself a sol. of (18)

ii) The same conclusion holds if

$$u_n \to u \quad \text{in } L^1_{loc}$$

$$u_n(x) \in K \text{ compact } \quad \forall x \in \mathbb{R}, \ \forall n$$

Proof: under assumptions in i) we find

$$\iint_{]0,T[\times\mathbb{R}} \{u\,\phi_t + f(u)\,\phi_x\}\,dx\,dt = \lim_n \iint_{]0,T[\times\mathbb{R}} \{u_n\,\phi_t + f(u_n)\,\phi_x\}\,dx\,dt = 0$$

$\forall \phi \in C^1_c(]0,T[\times\mathbb{R})$, which proves that u is a weak sol.

Concerning ii), by possibly taking a subsequence we may assume that $u_n(t,x) \to u(t,x)$ for a.e. (t,x)

In turn this implies $f(u_n(t,x)) \to f(u(t,x))$ for a.e. (t,x)

The boundedness assumption on u_n by dominated conv. thm. thus implies that $f(u_n) \to f(u)$ in L^1_{loc}.

28/ One then reaches the conclusion of ii) applying i).

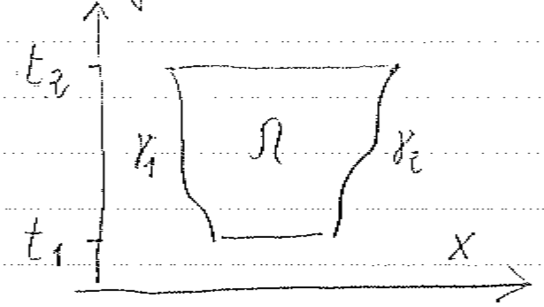
$\square$

<u>Remark 2</u> (<u>consequence of conservative form eqn.</u>):

Consider

$$\Omega = \{ (t,x) \; ; t \in [t_1, t_2], \; \gamma_1(t) < x < \gamma_2(t) \}$$

for some $0 < t_1 < t_2 < T$, $\gamma_i : [t_1, t_2] \to \mathbb{R}$ Lipschitz curves



If $u \in C^1(]0,T[\times \mathbb{R})$ is classical sol. of (18) then, applying div. thm to $\underline{v}(t,x) = (u(t,x), f(u(t,x)))$ one finds

$$0 = \iint_\Omega \{ u_t + [f(u)]_x \} \, dt \, dx = \iint_\Omega \operatorname{div} \underline{v}(t,x) \, dt \, dx = \iint_{\partial\Omega} \underline{v} \cdot \underline{n} \, ds =$$

$$= \int_{\gamma_1(t_2)}^{\gamma_2(t_2)} u(t_2, x) \, dx - \int_{\gamma_1(t_1)}^{\gamma_2(t_1)} u(t_1, x) \, dx + \text{(continues on the back)}$$

$$+\int_{t_1}^{t_2} \left\{ \dot\gamma_1(t)\, u(t,\gamma_1(t)) - f(u(t,\gamma_1(t))) \right\} dt$$

$$-\int_{t_1}^{t_2} \left\{ \dot\gamma_2(t)\, u(t,\gamma_2(t)) - f(u(t,\gamma_2(t))) \right\} dt$$

- This formula tells us that the variation of the quantity of u contained between γ_1 and γ_2 at different times $t_1 < t_2$ is given by the flow of the vector field $\underline{v}$ through the two curves γ_1, γ_2. In the particular case where $\dot\gamma_i = 0$ (γ_i are vertical lines in the x-t plane) the flow of $\underline{v}$ is precisely the flow of $f(u)$.

- This formula holds also <u>for weak solution u</u> provided that:

  i) $t \mapsto u(t,\cdot)$ is continuous with values in L^1_{loc}

  ii) $x \mapsto u(t,x)$ is right continuous $\forall (t,x)$, i.e.

$$u(t,x) = \lim_{y \to x+} u(t,y).$$

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In particular we may require that condition ii) is satisfied by a particular representative of the equivalence class of $u \in L^\infty(]0,T[\times\mathbb{R})$

If such cond's are verified, there holds:

$$\int_{\gamma_1(t_2)}^{\gamma_2(t_2)} u(t_2,x)\,dx - \int_{\gamma_1(t_1)}^{\gamma_2(t_1)} u(t_1,x)\,dx = \tag{40}$$

$$= \int_{t_1}^{t_2} \left\{ \dot\gamma_2(t)\, u(t,\gamma_2(t)) - f(u(t,\gamma_2(t))) \right\} dt - \int_{t_1}^{t_2} \left\{ \dot\gamma_1(t)\, u(t,\gamma_1(t)) - f(u(t,\gamma_1(t))) \right\} dt$$

$\forall\ 0 < t_1 < t_2 < T$, $\gamma_i : [t_1, t_2] \to \mathbb{R}$ Lipsch.

Weak solutions of Cauchy problem

Given an initial data $u_0 \in L^1_{loc}(\mathbb{R})$, we shall adopt the following

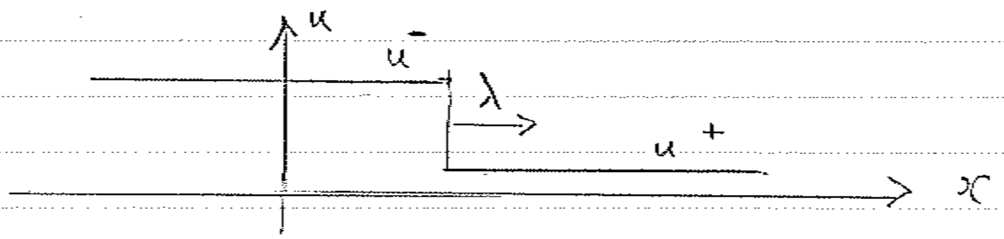
Definition 2 a function $u : [0, T[\times\mathbb{R} \to \mathbb{R}$ is a weak solution of the Cauchy problem (18)-(19) if the map $t \mapsto u(t,\cdot)$ is continuous as function from $[0,T[$ into $L^1_{loc}(\mathbb{R})$, the initial condition (19) holds and u is a weak solution of (18) on the strip $]0,T[\times\mathbb{R}$. (continuity assumptions w.r.t. time implies that, for any given $M>0$, one has $\lim_{t\to 0}\|u(t,\cdot)-u_0\|_{L^1(-M,M)}=0$)

Remark 3 : we have seen (p.8) that a discontinuous map $u(t,x)=\begin{cases}u^- & \text{if } x<\lambda t\\ u^+ & \text{if } x>\lambda t\end{cases}$ (shock discontinuity with speed λ),

is a weak sol. of (19) on $\mathbb{R}^+\times\mathbb{R}$ iff the (RH) cond's are verified :

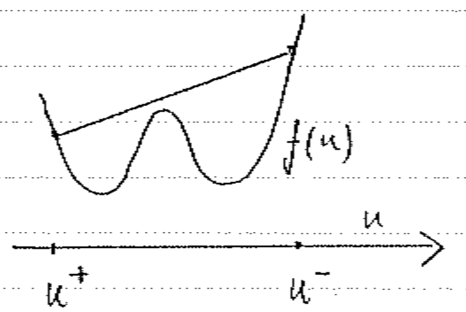
$$\lambda\,(u^+-u^-) = f(u^+)-f(u^-)$$

[speed of shock] $\times$ [jump in the state] = [jump in the flux]



$$(RH) \Rightarrow \quad \lambda = \frac{f(u^+)-f(u^-)}{u^+-u^-} = \frac{1}{u^+-u^-}\int_{u^-}^{u^+} f'(s)\,ds$$

Geometric interpretation :



[speed of shock] =

= [slope of secant through u^-, u^+ to graph(f)]

= [average characteristic speeds between u^-, u^+]

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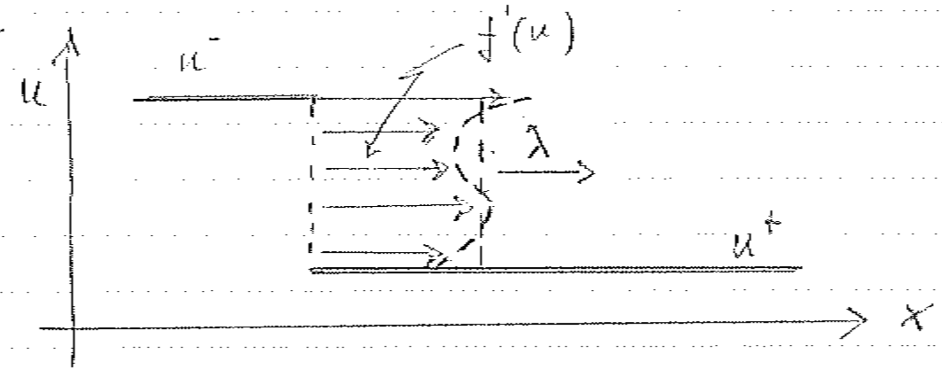
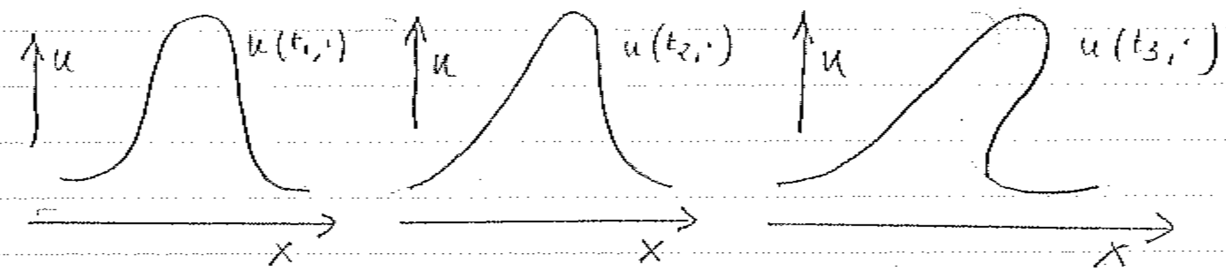
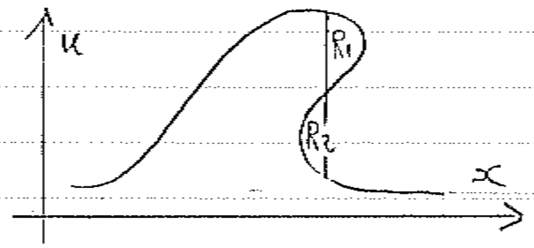


Notice: if we follow evolution of the profile of a solution after the formation of a discontinuity (with the rule that points of the solution $u(t,\cdot)$ moves horizontally with speed $f'(u(t,\cdot))$) we would observe a curve which is the graph of a multivalued function ($t_1 < t_2 < t_3$)



One can then replace the multivalued funtion with a single valued discontinuous function performing a cut in the graph of the multivalued map according with the Equal Area Rule.



(Equal Area Rule)

The area of the two lobes are equal: the area of the cut-off region R_1 is equal to the area of extra region R_2

Equal area rule ensures that total mass of shock solution matches that of the multi-valued solution, as one may expect observing that, for a (smooth) sol. $u(t,x)$ of (18), assuming that $\lim_{x\to+\infty} f(u(t,x)) = \lim_{x\to-\infty} f(u(t,x))$ $\forall t$, one has:

$$\frac{d}{dt}\int_{\mathbb{R}} u(t,x)\,dx = \int_{\mathbb{R}} u_t(t,x)\,dx = -\int_{\mathbb{R}} \left[f(u(t,x))\right]_x dx$$

$$= \lim_{x\to-\infty} f(u(t,x)) - \lim_{x\to+\infty} f(u(t,x)) = 0,$$

which implies that $\int_{\mathbb{R}} u(t,x)\,dt$ is constant in time.

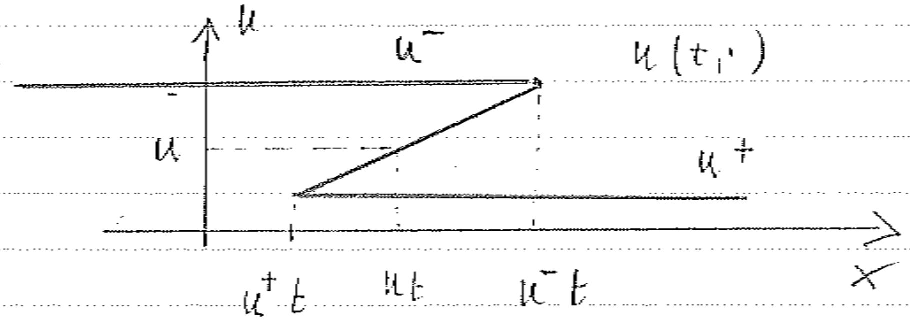
One can show that "the cut of the graph performed by the Equal Area Rule corresponds to considers a discontinuous solution whose discontinuity travels with the speed prescribed by the (RH) conditions

Example 1: consider Cauchy pb

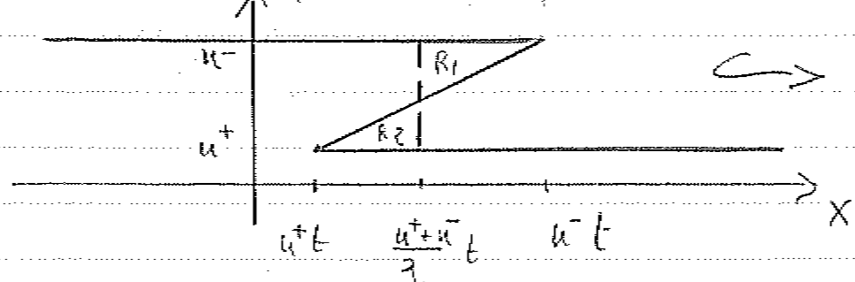
$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

$$u(0,x) = \begin{cases} u^- & \text{if } x<0 \\ u^+ & \text{if } x>0 \end{cases}$$

with $u^- > u^+ > 0$. Observing that any point $(0,u)$, $u \in [u^+, u^-]$ of the vertical line at $x=0$ of the graph $(u(0,\cdot))$ moves on the right with speed u, we find that the graph of the multivalued map $u(t,\cdot)$, $t>0$ would become:



Equal area rule:



$\hookrightarrow \lambda = \frac{u^+ + u^-}{2}$ is precisely speed of a shock for Burgers eqn, connecting u^-, u^+, prescribed by (RH) condition.

GOAL: derive (RH) cond's for discontinuous weak solutions more general than the piecewise constant function (17) with a single discontinuity. To this end, we introduce the following

Definition 3 (approximate jump):
We say that a function $u = u(t,x)$ $(u \in L^1_{loc})$ with values in $\mathbb{R}$, has an approximate jump discontinuity at a point $(\bar{t}, \bar{x})$ if there exist $u^-, u^+, \lambda \in \mathbb{R}$ such that, setting

$$(41) \qquad U(t,x) \doteq \begin{cases} u^- & \text{if } x < \lambda t \\ u^+ & \text{if } x > \lambda t \end{cases}$$

there holds

$$(42) \qquad \lim_{r \to 0+} \frac{1}{r^2} \iint_{[-r,r]^2} \left| u(\bar{t}+t, \bar{x}+x) - U(t,x) \right| dt\, dx = 0$$

Moreover, we say that u is approximately continuous at $(\bar{t}, \bar{x})$ if (42) holds for a function U defined as in (41) with $u^- = u^+$ and arbitrary λ.
(if u is approximat. continuous at $(\bar{t},\bar{x})$, and $u(\bar{t},\bar{x}) = u^- = u^+$, then $(\bar{t},\bar{x})$ is a Lebesgue point of u).

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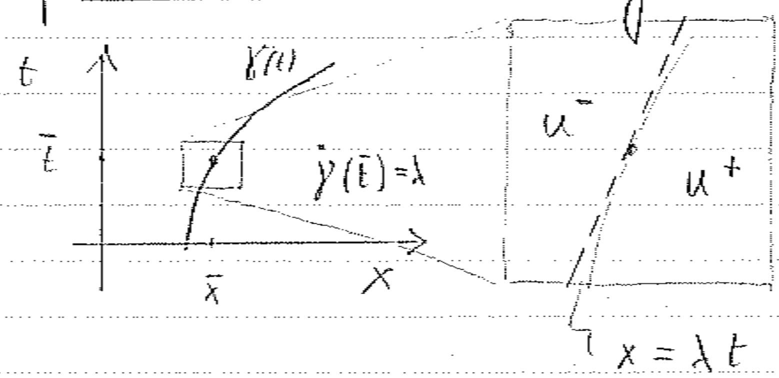
If condition (41) is verified we say that u^-, u^+ are the left and right (approximate) limits of u at $(\bar t, \bar x)$, and λ is the jump speed.

Remark 4: definition 3 depends only on the L^∞ equivalence class of u since the limit (42) is unaffected if the values of u are modified on a set $N \subseteq \mathbb{R}^2$ of Lebesgue measure zero. Moreover, the triple (u^-, u^+, λ) so that (41)-(42) is verified, if it exist, is unique

Remark 5: limit (42) says that looking $u(t,x)$ through a microscope, i.e. rescaling the variables t, x in a neighborhood of the point $(\bar t, \bar x)$, the function $u(t,x)$ becomes arbitrarly close (in an integral sense) to the piecewise constant function $U(t,x)$ in (41)



$\gamma(t)$: curve of discontinuity of u

Example 2 (piecewise continuous function):

Let $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}$ be two continuous functions

$\gamma : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function, and consider the function:

$$u(t,x) \doteq \begin{cases} g_1(t,x) & \text{if } x < \gamma(t) \\ g_2(t,x) & \text{if } x \geq \gamma(t) \end{cases}$$

- If $g_1(\bar{t},\bar{x}) = g_2(\bar{t},\bar{x})$ at some point $(\bar{t},\bar{x})$ with $\bar{x} = \gamma(\bar{t})$, then u is continuous at $(\bar{t},\bar{x})$, and hence also approximately continuous at $(\bar{t},\bar{x})$.
- If $g_1(\bar{t},\bar{x}) \neq g_2(\bar{t},\bar{x})$, at some point $(\bar{t},\bar{x})$, with $\bar{x} = \gamma(\bar{t})$ and γ is differentiable at $\bar{t}$, then u has an approximate jump discontinuity at $(\bar{t},\bar{x})$, with the function U in (41) defined in connection with $\underline{u^- = g_1(\bar{t},\bar{x})}$, $\underline{u^+ = g_2(\bar{t},\bar{x})}$, $\underline{\lambda = \dot{\gamma}(\bar{t})}$.

We next show that (RH) condition is satisfied at any point of approximate jump discontinuity of a weak solution of (18)

33/ **Proposition 1** (Rankine-Hugoniot equation)

Let u be a weak solution of (18) having an approximate jump discontinuity at a point $(\bar{t}, \bar{x})$. In other words, assume that (41)-(43) hold for some $u^-, u^+, \lambda \in \mathbb{R}$. Then the (RH) equation (17) is verified.

Proof: 1. for any fixed $\vartheta > 0$ sufficiently small, one can easily check that the rescaled function

$$(43) \qquad u^{\vartheta}(t, x) = u(\bar{t} + \vartheta t, \bar{x} + \vartheta x)$$

is also a weak solution of (18).

In fact, if u is classical sol. of (18) by a direct computation we find

$$u^{\vartheta}_t + f'(u^{\vartheta})\, u^{\vartheta}_x = \vartheta \left[u_t + f'(u)\, u_x \right] = 0$$

Instead if u is a weak sol., for any given $\phi \in C^1_c(]0,T[\times \mathbb{R})$ we find:

$$\int_0^T \int_{-\infty}^{+\infty} \left[u^\vartheta(t,x)\, \phi_t(t,x) + f(u^\vartheta(t,x))\, \phi_x(t,x) \right] dt\, dx =$$

$$\left(\begin{array}{l} (\tau, z) = \Psi(t,x) = (\bar{t} + \vartheta t,\ \bar{x} + \vartheta x) \\ |\det(J\Psi)| = \left| \det \begin{pmatrix} \vartheta & 0 \\ 0 & \vartheta \end{pmatrix} \right| = \vartheta^2 \\ d\tau\, dz = \vartheta^2\, dt\, dx \end{array} \right)$$

$$= \int_{\bar{t}}^{\bar{t}+\vartheta T} \int_{-\infty}^{+\infty} \left[u(\tau,z)\, \phi_t\left(\frac{\tau - \bar{t}}{\vartheta}, \frac{z - \bar{x}}{\vartheta}\right) + f(u(\tau,z))\, \phi_x\left(\frac{\tau-\bar{t}}{\vartheta}, \frac{z-\bar{x}}{\vartheta}\right) \right] \frac{1}{\vartheta^2}\, d\tau\, dz$$

Assuming $\vartheta < 1 - \frac{T}{\bar{t}}$, setting $\tilde{\phi}(\tau, z) = \phi\left(\frac{\tau-\bar{t}}{\vartheta}, \frac{z-\bar{x}}{\vartheta}\right)$, we deduce that the second double integral above vanishes since $\tilde{\phi} \in C^1_c(]0,T[\times\mathbb{R})$ as well, and u is a weak solution. Hence, also the first double integral above vanishes, proving that u^ϑ is a weak solution.

2. We claim that, as $\vartheta \to 0$, one has

$$u^\vartheta \longrightarrow U \quad \text{in} \quad L^1_{loc}(\mathbb{R})$$

$$U(t,x) = \begin{cases} u^- & \text{if } x < \lambda t \\ u^+ & \text{if } x > \lambda t \end{cases}$$

In fact, for any given $R>0$, one has

$$\lim_{\vartheta\to 0} \iint_{[-R,R]^2} \left| u^{\vartheta}(t,x) - \mathcal{V}(t,x)\right| dt\,dx \underset{(\tau = \vartheta t,\ z = \vartheta x)}{=}$$

$$= \lim_{\vartheta\to 0} \frac{1}{\vartheta^2} \iint_{[-\vartheta R,\vartheta R]^2} \left| u(\bar t+\tau, \bar x + z) - \mathcal{V}\left(\frac{\tau}{\vartheta}, \frac{z}{\vartheta}\right)\right| d\tau\, dz \underset{\left(\mathcal{V}\left(\frac{\tau}{\vartheta}, \frac{z}{\vartheta}\right) = \mathcal{V}(\tau, \vartheta)\right)}{=}$$

$$= R^2 \lim_{r\to 0} \frac{1}{r^2} \iint_{[-r,r]^2} \left| u(\bar t + \tau, \bar x + z) - \mathcal{V}(\tau, z)\right| d\tau\, dz \underset{(\text{by } (42))}{=} 0$$

Notice that, since $u \in L^\infty(]0,T[\times\mathbb{R})$, u^ϑ are uniformly bounded. Hence, applying Lemma 1 at p.27 we deduce that $\mathcal{V}$ is a weak sol. of (18).

3. Since $\mathcal{V}$ is weak sol. of (18), by Lemma 1 at p.8 we deduce that (RH) equation is verified. ∎

Remark 6: by Prop 1 we know that (RH) holds at any point of approximate jump discontinuity of a weak solution of (18). What about the other points? The next result shows that, if we know that the solution is a function of bounded variation, then the set of such points is $\mathcal{H}^1$-neglegible.

We recall that a function $u = u(t,x)$ has locally bounded variation on $\Omega \subseteq \mathbb{R}^2$ open, if $u \in L^1_{loc}(\Omega)$ and its distributional derivative Du is a Radon measure on Ω, i.e., for every compact set $K \subseteq \Omega$ there exists $c_K > 0$ such that

$$\left| \iint_\Omega u \phi_t \, dt\, dx \right| + \left| \iint_\Omega u \phi_x \, dt\, dx \right| \leq c_K \|\phi\|_{\mathcal{C}^0}$$

for any $\phi \in \mathcal{C}^1_c$ with $\operatorname{supp}(\phi) \subseteq K$.

We denote by $BV_{loc}(\Omega)$ the class of functions $u : \Omega \to \mathbb{R}$ that have locally bounded variation on Ω.

35/ A useful description of the structure of BV functions is provided by the following

Theorem 1 (Fine properties of BV functions)

Let $u : \Omega \to \mathbb{R}$, $\Omega \subseteq \mathbb{R}^2$, $u \in BV_{loc}(\Omega)$.
Then Ω is the union of three, pairwise disjoint subsets $\mathcal{C}$, $\mathcal{J}$, $\mathcal{I}$ with the following properties:

i) $\mathcal{C}$ is the set of points of approximate continuity of u (u can be normalized so that $\mathcal{C}$ coincide with the set of Lebesgue points of u)

ii) J is the set of points of approximate jump discontinuity of u. Moreover J is countably rectifiable, i.e. J is essentially covered by the countable union of C^1 curves $\{\Gamma_i\}_i$ up to a set with zero one-dimensional Hausdorff measure
$\mathcal{H}^1(J \setminus \bigcup_i \Gamma_i) = 0$.

iii) $\mathcal{H}^1(\mathcal{I}) = 0$ ($\mathcal{I}$: set of irregular points)

References:

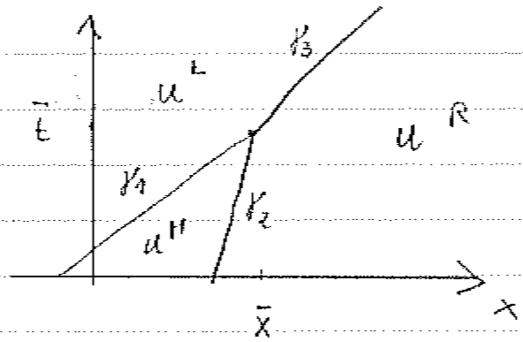
Dafermos, Hyperbolic conservation laws in continuum physics, Thm 1.7.4

Evans & Gariepy, Measure theory and fine properties of functions, Thm. 1.3 in Section 5.9

(see also course of Theory of Functions 2, Novaga & Vittone)

Notice 1: we will see that if $u_0 \in BV(\mathbb{R})$, then the Cauchy problem for (18), with initial data u_0, admits a weak (admissible) solution $u \in BV([0,+\infty[\times \mathbb{R})$. Moreover, one can show that if f is uniformly convex, $u_0 \in L^\infty$, then the Cauchy pb for (18), with initial data u_0, admits a weak (admissible) solution $u \in BV(]0,+\infty[\times \mathbb{R})$.

Notice 2: consider the piecewise constant function:



(γ_i : C^1 curves)

$$u(t,x) = \begin{cases} u^L & \text{if } x < \gamma_1(t),\ t \le \bar{t} \quad \text{or} \quad x < \gamma_3(t),\ t \ge \bar{t} \\ u^M & \text{if } \gamma_1(t) < x < \gamma_2(t),\quad t < \bar{t} \\ u^R & \text{if } x > \gamma_2(t),\ t \le \bar{t}, \quad \text{or } x > \gamma_3(t),\ t \ge \bar{t} \end{cases}$$

then $J = \bigcup_i \Gamma_i$, $\Gamma_i = \{(t, \gamma_i(t)),\ 0 < t < \bar{t}\}$ $i = 1,2$

$\Gamma_3 = \{(t, \gamma_3(t)),\ t > \bar{t}\}$

$\mathcal{I} = \{(\bar{t}, \bar{x})\}$, $\mathcal{C} = \mathbb{R}^+ \times \mathbb{R} \setminus (J \cup \mathcal{I})$.

We provide now a necessary and sufficient condition for a piecewise Lipschitz function to be a weak solution of (18).

<u>Definition 4</u> (Piecewise Lipschitz regularity)
We say that a <u>measurable</u>, <u>bounded</u> function <u>$u : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$</u> enjoys the <u>piecewise Lipschitz (PL) regularity</u> property if there exist finite number of points $P_i = (t_i, x_i)$, and finitely many disjoint Lipschitz continuous maps $\gamma_j :]a_j, b_j[\to \mathbb{R}$ such that the following hold.

a) every point $P = (\bar{t}, \bar{x})$ different from P_i, and not lying on the curves $\Gamma_j = \text{graph}\{\gamma_j\}$, has a neighborhood where the function u is Lipschitz continuous

b) every point $Q = (\bar{t}, \gamma_j(\bar{t}))$, on each curve γ_j, has a neighborhood V such that the restriction

of u to the subsets $V^+ = V \cap \{x > \gamma_j(t)\}$,
$V^- = V \cap \{x < \gamma_j(t)\}$ are both Lipschitz continuous

<u>Notice</u> : by property b) of Def 4, there exist the left and right limits of $u(t,\cdot)$ along the curves γ_j, which shall be denoted

$$u_j^-(t) \doteq \lim_{x \to \gamma_j(t)-} u(t,x), \qquad u_j^+(t) \doteq \lim_{x \to \gamma_j(t)+} u(t,x),$$

and the functions $u_j^\pm$ are continuous on $]a_j, b_j[$.

<u>Theorem 2</u> (<u>characterization of PL weak solutions</u>)

Let $u : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$, ($\Omega$ open) be a (PL) function according with Def. 4. Then, the following are equivalent:

(i) u is a weak solution of (18)

(ii) u satisfies the quasilinear equation

$$(44) \qquad u_t + f'(u)\, u_x = 0 \qquad \text{for a.e. } (t,x)$$

II. Moreover, for every jump curve γ_j one has:

$$(45) \qquad \dot{\gamma}_j(t)\left(u_j^+(t) - u_j^-(t)\right) = f(u_j^+(t)) - f(u_j^-(t))$$

for a.e. $t \in]a_j, b_j[$.

37/ Notice: since the maps $u_j^{\pm}$ are continuous if equation (45) is satisfied γ_j turns out to be a C^1 curve.

Proof of Thm ?:

As preliminary we observe that since u is a PL function, by Rademacher's theorem, the function u and $f(u)$ are differentiable almost everywhere (see Bressan, Hyperbolic Systems of Conservation Law Thm 2.8). Similarly, every curve γ_j is a.e. differentiable.

1. i) $\Rightarrow$ ii) Assume that u is a weak sol. of (18). Let $\Omega' \subseteq \Omega$ be any open set where u is Lipschitz continuous and consider $\phi \in C_1^c(\Omega')$. Then, applying the divergence theorem to the vector field $\underline{v} = (u\phi,\ f(u)\phi)$ we find (as in (38)):

$$\underset{(u \text{ weak sol.})}{0} = \iint_{\Omega'} \{u\phi_t + f(u)\phi_x\}\, dt\, dx = -\iint_{\Omega'} \{u_t + [f(u)]_x\}\, \phi\, dt\, dx$$

$$\Rightarrow \iint_{\Omega'} \{u_t + [f(u)]_x\} \phi \, dt\, dx = 0 \qquad \forall \phi \in C_c^1(\Omega')$$

With similar arguments to the proof of Remark 1-ii) p 26-27, we then deduce that (44) is verified a.e. in Ω'.

Next, observe that if the curve γ_j is differentiable at some time $\bar{t}$, then recalling Example 2 p. 32, and by property b) of (PL) functions, we deduce that the point $Q = (\bar{t}, \gamma_j(\bar{t}))$ is a point of approximate jump discontinuity for u. Applying Proposition 1 we thus conclude that the (RH) equation (45) is satisfied at $\bar{t}$.

2. ii) $\Rightarrow$ i) Assume now that u satisfies (44)-(45). Let $\phi \in C_c^1(\Omega)$ and assume for the moment that every point P_i of Def. 4 lies outside the support of ϕ. Then, with the same arguments of the proof of Lemma at p. 8, applying the

38 divergence theorem to the vector field $\underline{v} = (u\phi, f(u)\phi)$ on any bounded subregion of $\Omega \setminus \bigcup_j \Gamma_j$ where $\underline{v}$ is Lipschitz continuous, we find

$$\iint_\Omega \{u\phi_t + f(u)\phi_x\} dt\,dx = -\iint_\Omega \{u_t + [f(u)]_x\} \phi\, dt\,dx +$$

$$+ \sum_j \int_{a_j}^{b_j} \{\dot{\gamma}_j (u_j^+ - u_j^-) - (f(u_j^+) - f(u_j^-))\} \phi(t, \gamma_j(t))\, dt$$

$$= 0 .$$

(by (44) - (45))

In the case where some point P_i of Def 4 belongs to the support of ϕ, we cannot repeat the above argument since the vector field $\underline{v}$ is no more guaranteed to be Lipschitz continuous on bounded subregions of $\Omega \setminus \bigcup_j \Gamma_j$. However, we may replace ϕ with another test function $\phi^\varepsilon \in \mathcal{C}_1^c(\Omega)$ having the properties:

i) each point P_i lies outside the support of ϕ^ε

ii) $\| \nabla(\phi - \phi^\varepsilon) \|_{L^1} \to 0$ as $\varepsilon \to 0$

Observing that

$$\left| \iint_{\Omega} \left\{ u(\phi - \phi^{\varepsilon})_t + f(u)(\phi - \phi^{\varepsilon})_x \right\} dt\, dx \right| \leq \tag{46}$$
$$\leq \left(\|u\|_{L^\infty} + \|f(u)\|_{L^\infty} \right) \|\nabla(\phi - \phi^{\varepsilon})\|_{L^1}$$

we then conclude:

$$\iint_{\Omega} \left\{ u\phi_t + f(u)\phi_x \right\} dt\, dx =$$

$$= \lim_{\varepsilon \to 0} \Bigg[\underbrace{\iint_{\Omega} \left\{ u\phi^{\varepsilon}_t + f(u)\phi^{\varepsilon}_x \right\} dt\, dx}_{\substack{\| \\ 0 \left(\text{by above arguments which can be applied thanks to property i)} \right)}} + \iint_{\Omega} \left\{ u(\phi - \phi^{\varepsilon})_t + f(u)(\phi - \phi^{\varepsilon})_x \right\} dt\, dx \Bigg] =$$

$$= \lim_{\varepsilon \to 0} \iint_{\Omega} \left\{ u(\phi - \phi^{\varepsilon})_t + f(u)(\phi - \phi^{\varepsilon})_x \right\} dt\, dx \underset{\text{by (46)}}{=} 0.$$

For a precise definition of the modified test function ϕ^{ε} see: [Bressan, Hyperbolic System of Conservation Laws, proof of Thm. 4.2].

∎

<u>Remark 1</u> : the characterization of (PL) weak solutions provided by Thm 2 shows that we can verify if a (PL) function is a weak sol. checking if u satisfies the quasilinear equation (44) in the region of continuity, and controlling the (RH) conditions (45) along the strong discontinuities of u.
<u>No particular condition is required to be satisfied in the points of weak discontinuity.</u>

<u>Remark 2 :</u> (<u>no invariance of weak solutions under coordinate transformations</u>)
Assume $u = u(t,x)$ be a C^1 solution of (18) and hence of the quasilinear eq. (44) and consider the change of variable given by

$$v = h(u)$$

for some bijective $h \in C^1(\mathbb{R})$. We may compute

$$v_t = h'(u)\, u_t = h'(u)\left(-f'(u)\, u_x\right)$$
$$= h'\left(h^{-1}(v)\right)\left(-f'\left(h^{-1}(v)\right)\right)\left(\left(h^{-1}\right)'(v)\cdot v_x\right)$$
$$= -f'\left(h^{-1}(v)\right) v_x$$

Then, setting $g(v) \doteq \int_0^v f'\left(h^{-1}(s)\right) ds$, we deduce that the function $v(t,x) = h(u(t,x))$ is a C^1 solution of:

$$v_t + \left[g(v)\right]_x = 0 \qquad \left\{ h^{-1}(v) = \operatorname{sgn}(v)\, |v|^{1/3} \right.$$

<u>Example:</u> $f(u) = \frac{u^2}{2}$, $v = h(u) = u^3$, $g(v) = \frac{3}{4}|v|^{4/3}$

$$u(t,x) \ C^1 \text{ sol. of } \ u_t + \left(\frac{u^2}{2}\right)_x = 0$$

$$\left(\begin{array}{l} v_t = 3u^2 u_t = -3u^3 u_x = -u\, v_x \\ v_x = 3u^2 u_x \qquad = -\operatorname{sgn}(v)\,|v|^{1/3} v_x = -\left(\frac{3}{4}|v|^{4/3}\right)_x \end{array}\right) \quad \Downarrow$$

$$v(t,x) = u^3(t,x) \ \text{ sol. of } \ v_t + \left(\frac{3}{4}|v|^{4/3}\right)_x = 0 \qquad (47)$$

This equivalence is no more true for weak solutions with strong discontinuities (shock) due to the non equivalence of the (RH) conditions. We have already seen that (RH) cond's for Burgers yield the shock speed $\lambda_1 = \frac{u^+ + u^-}{2}$ (cfr. p. 9)

However, the (RH) conditions applied to (47) prescribe the shock speed :

$$\lambda_2 = \frac{\frac{3}{4}(v^+)^{4/3} - \frac{3}{4}(v^-)^{4/3}}{v^+ - v^-} = \frac{3}{4}\,\frac{(u^+)^4 - (u^-)^4}{(u^+)^3 - (u^-)^3}$$

(assume $v^\pm \geq 0$)

which is different from λ_1.

<u>Exercise 4</u>: construct some weak non classical (not C^1) solution of the equations :

i) $u_t - (u^3)_x = 0$

ii) $u_t + (\cos u)_x = 0$

iii) $u_t - (e^u)_x = 0$

iv) $u_t + (e^u)_x = 0$

v) $u_t + \left(\frac{1}{u}\right)_x = 0$

Admissibility Conditions

If we consider a Cauchy problem for (18) the concept of weak distributional solution is not sufficient to single out a unique solution whenever a strong discontinuity appears in the solution. Indeed, for a given initial datum, we can find an entire family of weak solutions of the same Cauchy problem.

Example 1: consider Burgers' equation with initial data:

$$(48) \qquad u_0(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

For any fixed $\alpha \in [0,1]$, define the piecewise constant function

$$u_\alpha(t,x) = \begin{cases} 0 & \text{if } x < \frac{\alpha}{2} t \\ \alpha & \text{if } \frac{\alpha}{2} t \leq x < \frac{\alpha+1}{2} t \\ 1 & \text{if } x \geq \frac{\alpha+1}{2} t \end{cases}$$

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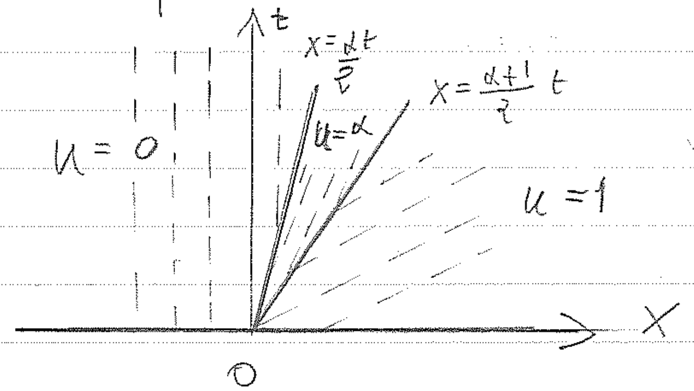
One can easily check that the (RH) conditions hold along the two lines of discontinuity

$$\gamma_1(t) = \frac{\alpha}{2} t, \qquad \gamma_2(t) = \frac{\alpha+1}{2} t.$$

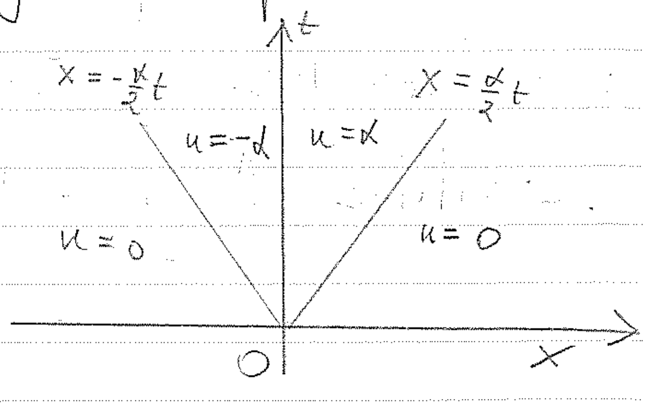
Therefore, by Thm 2 we deduce that each u_α is a weak sol. of the Cauchy problem for Burgers equation with initial data (48).



Example 2 : consider again Burgers equation with zero initial datum : $u_0(x) \equiv 0$.

For any fixed $\alpha \geq 0$ define the piecewise constant function

$$u_\alpha(t,x) = \begin{cases} 0 & \text{if } x < -\frac{\alpha}{2}t \\ -\alpha & \text{if } -\frac{\alpha}{2}t \leq x < 0 \\ \alpha & \text{if } 0 \leq x < \frac{\alpha}{2}t \\ 0 & \text{if } x \geq \frac{\alpha}{2}t \end{cases}$$



As in Example 1, we can easily check that every u_α is a weak solution of the Burgers equation with zero initial data since (RH) conditions are satisfied along the three lines of discontinuities

$$\gamma_1(t) = -\frac{\alpha}{2}t, \quad \gamma_2(t) = 0, \quad \gamma_3(t) = \frac{\alpha}{2}t, \quad t > 0 .$$

Exercise 5: is it possible to construct a non zero weak solution of Burgers eqn with zero initial data that has only two lines of discontinuity?

Pb: we need to supplement the notion of weak solution with "further admissibility" conditions possibly motivated by physical considerations, which restrict the class of admissible discontinuities that can be present in a weak solution, in order to achieve uniqueness and continuous dependence on the initial data, of the solutions.

1. Vanishing Viscosity

We say that a weak solution $u : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ of (18) is admissible in the vanishing viscosity sense if there exists a sequence of smooth solutions of the viscous parabolic approximation

$$(49) \qquad u^\varepsilon_t + f'(u^\varepsilon)\, u^\varepsilon_x = \varepsilon\, u^\varepsilon_{xx}, \qquad (t,x) \in \Omega$$

so that $u^\varepsilon \to u$ in $L^1_{loc}(\Omega)$ as $\varepsilon \to 0$.

Drawback of this approach: very difficult to provide a-priori estimates on solutions to (49) that allow to prove the convergence when $\varepsilon \to 0$, and to characterize the corresponding limit in the case of systems ($u \in \mathbb{R}^N$).

However, one can deduce from the vanishing viscosity condition other conditions that can be more easily verified in practice.

2. Entropy conditions.

Motivated by the second principle of thermodynamics valid for the Euler equations of gas we introduce the concept of entropy which characterize irreversible processes (Kinetic energy dissipates when a shock appears: a part of it is transformed into heat)

Definition 5 (entropy - entropy flux)

We say that a pair of $\mathcal{C}^1$ (or locally Lipschitz) functions $(\eta, q) : \mathbb{R} \to \mathbb{R}$ is an entropy - entropy flux pair for equation (18) if

$$q'(u) = \eta'(u)\, f'(u) \tag{50}$$

at every u where η, q, f are differentiable.

Remark 1 : if $u = u(t,x)$ is a $\mathcal{C}^1$ solution of (18) then u satisfies also the equation

$$[\eta(u)]_t + [q(u)]_x = 0 \tag{51}$$

In fact, one finds:

$[\eta(u)]_t + [q(u)]_x = \eta'(u)\, u_t + q'(u)\, u_x =$

$$= \eta'(u)\left(-f'(u)\,u_x\right) + \eta'(u)\,f'(u)\,u_x = 0$$

(by (18), (50)) proving (51).

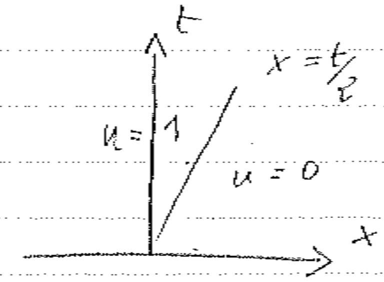
On the other hand, one can easily verify that if $u = u(t,x)$ is a discontinuous weak solution of (18) then (50) is not verified in distributional sense.

<u>Example</u>: a pair of entropy - entropy flux for Burgers equation is given by

$$\eta(u) = u^3, \qquad q(u) = \frac{3}{4}u^4$$

The function

$$u(t,x) = \begin{cases} 1 & \text{if } x < t/2 \\ 0 & \text{if } x \geq t/2 \end{cases}$$



is a weak solution of Burgers equation since (RH) condition is verified along the discontinuity. However, u is not a distributional solution of (51) since it should hold $q(1) - q(0) = \frac{1}{2}(\eta(1) - \eta(0))$, while we have

$$q(1) - q(0) = \frac{3}{4} \neq \frac{1}{2} = \frac{1}{2}\left(\eta(1) - \eta(0)\right).$$

Remark 2 : if $u^\varepsilon = u^\varepsilon(t,x)$ is a C^1 solution of (49) multiplying both sides of (49) by $\eta'(u^\varepsilon_t)$ we deduce that u^ε is a solution of

(52) $$[\eta(u^\varepsilon)]_t + [q(u^\varepsilon)]_x = \varepsilon \left\{ [\eta(u^\varepsilon)]_{xx} - \eta''(u^\varepsilon)(u^\varepsilon_x)^2 \right\}$$

If we assume now that $\eta \in C^2$ and convex, i.e. that $\eta''(u) \geq 0 \ \forall u$, we deduce from (52) that

(53) $$[\eta(u^\varepsilon)]_t + [q(u^\varepsilon)]_x \leq \varepsilon [\eta(u^\varepsilon)]_{xx}$$

Multiplying (53) by a nonnegative function $\phi \in C^1_c$ and integrating by parts we then deduce from (53) the integral inequality

(54) $$\iint_\Omega \{\eta(u^\varepsilon)\phi_t + q(u^\varepsilon)\phi_x\} \, dt\, dx \geq -\varepsilon \iint_\Omega \eta(u^\varepsilon)\phi_{xx} \, dt\, dx$$

Next, assume that:

$$u^\varepsilon \to u \text{ in } L^1_{loc}(\Omega), \quad u^\varepsilon \text{ uniformly bounded}$$

Then, taking the limit in (54) as $\varepsilon \to 0$ we find:

$$\iint_{\Omega} \{ \eta(u)\, \phi_t + q(u)\, \phi_x \} \, dt\, dx \geq 0 \qquad (55)$$

whenever $\phi \in \mathcal{C}^1_c(\Omega)$, $\phi(t,x) \geq 0 \quad \forall (t,x) \in \Omega$.

We are thus lead to state the following

Definition 6:

We say that a weak solution $u = u(t,x)$ of (18) is entropy admissible if it satisfies the inequality

$$(56) \quad [\eta(u)]_t + [q(u)]_x \leq 0$$

in distributional sense (i.e. satisfies the integral inequality (55) for every pair (η, q) where η is a convex entropy and q the corresponding entropy flux). and for any $\phi \in \mathcal{C}^1_c$, $\phi \geq 0$

Remark 3: Any $\mathcal{C}^1$ (or locally Lipschitz) convex function $\eta : \mathbb{R} \to \mathbb{R}$ provides a convex entropy for (18), with entropy flux q defined by

$$q(u) = \int_{\bar{u}}^{u} \eta'(s) \cdot f'(s)\, ds \,, \text{ for any fixed } \bar{u} \in \mathbb{R}.$$

A particular class of entropy - entropy flux pairs which is quite useful in analyzing the behaviour of entropy admissible weak solutions is given by the <u>Kruzkhov's entropies</u>

(57) $\underline{\eta_k(u) = |u-k|}$, $\underline{q_k(u) = \mathrm{sgn}(u-k)\left(f(u)-f(k)\right)}$

One can easily check that, for any fixed $k \in \mathbb{R}$, η_k, q_k are locally Lipschitz functions that satisfy (50) at every $u \neq k$. The important fact is that the distributional inequalities (56) associated to such class of entropy - entropy flux pairs are equivalent to the inequalities (56) associated to the family of "all" entropy - entropy flux pairs for (18). Moreover, one can show that the entropy inequalities (55) for the Kruzkhov's entropies imply in particular also the integral equalities (37). These properties are the content of the next

45/ Proposition 2

A function $u \in L^\infty(]0,T[\times\mathbb{R})$ is an entropy admissible weak solution of (18) if and only if, for every $K\in\mathbb{R}$ there holds:

$$(58)\quad \iint_{]0,T[\times\mathbb{R}} \Big\{ |u(t,x)-K|\,\phi_t(t,x) + \operatorname{sgn}\big(u(t,x)-K\big)\big(f(u(t,x))-f(K)\big)\phi_x(t,x)\Big\}\,dt\,dx \geqslant 0$$

for all $\phi\in C^1_c(]0,T[\times\mathbb{R})$, $\phi\geqslant 0$.

Proof:

1. We show that if $u\in L^\infty(]0,T[\times\mathbb{R})$ satisfies (58) for all $K\in\mathbb{R}$, $\phi\in C^1_c(]0,T[\times\mathbb{R})$, $\phi\geqslant 0$ then u is a weak solution of (18).

i) Since $u\in L^\infty$ there exists $M>0$ s.t. $|u(t,x)|\leq M$ for a.e. (t,x). Therefore, applying (58) with $K=M$, we have $|u(t,x)-M| = M-u(t,x)$, $\operatorname{sgn}(u(t,x)-M)=-$ and we find

$$\iint_{]0,T[\times\mathbb{R}} \{u\phi_t + f(u)\phi_x\}\,dt\,dx \leqslant M\int_{-\infty}^{\infty}\int_0^T \phi_t(t,x)\,dt\,dx + f(M)\int_0^T\int_{-\infty}^{\infty}\phi_x(t,x)\,dx\,dt$$

$$= 0 \left(\begin{array}{l}\text{since } \phi \text{ has compact}\\ \text{support} \subseteq\,]0,T[\times\mathbb{R}\end{array}\right)$$

Similarly, applying (58) with $K=-M$ we find

$$\iint_{]0,T[\times\mathbb{R}} \{u\phi_t + f(u)\phi_x\}\,dt\,dx \geq -M\int_{-\infty}^{+\infty}\int_0^T \phi_t(t,x)\,dt\,dx + f(-M)\int_0^T\int_{-\infty}^{+\infty}\phi_x(t,x)\,dx\,dt$$

$$= 0 .$$

Hence, combining the two integral inequalities we derive:

(59) $$\iint_{]0,T[\times\mathbb{R}} \{u\phi_t + f(u)\phi_x\}\,dt\,dx = 0 \quad \forall\, \phi\in\mathcal{C}^1_c(]0,T[\times\mathbb{R}),\ \phi\geq 0 .$$

ii) Observe that, relying on (59) we can show that the integral equality (37) indeed holds for every $\phi\in \mathrm{Lip}_c(]0,T[\times\mathbb{R})$, $\phi\geq 0$ (Lipschitz functions with compact support)

In fact, for any such function ϕ one can construct a sequence $\{\phi_n\}_n$, $\phi_n\in\mathcal{C}^1_c(]0,T[\times\mathbb{R})$, $\phi_n\geq 0$, s.t.

$$(\phi_n)_t \longrightarrow \phi_t \qquad (\phi_n)_x \longrightarrow \phi_x \quad \text{in } L^1(]0,T[\times\mathbb{R})$$

as $n\to\infty$.

Then, since the equality (37) holds for any ϕ_n, passing to the limit as $n\to\infty$ in (37) we deduce that

46/ the same integral equality holds for the test function ϕ.

[Sketch of the construction of the sequence ϕ_n:

Given $\phi \in Lip_c(\Omega)$, $\Omega \subseteq \mathbb{R}^2$ open, consider a mollifier $\Psi \in \mathcal{C}^1_c(\mathbb{R}^2)$, $\Psi \geq 0$ with the properties:

a) $\iint_{\mathbb{R}^2} \varepsilon^{-2} \Psi\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) dt\, dx = \iint_{\mathbb{R}^2} \Psi(t,x)\, dt\, dx = 1$

b) $supp(\Psi_\varepsilon) \subseteq B(0, r/2)$, $\Psi_\varepsilon(t,x) \doteq \varepsilon^{-2} \Psi\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)$

$r = dist\left(supp(\phi), \partial\Omega\right)$

Then define:

$\phi_\varepsilon(t,x) = \phi * \Psi_\varepsilon (t,x)$

and observe that $\phi_\varepsilon \in \mathcal{C}^1_c(\mathbb{R}^2)$,

$(\phi_\varepsilon)_t \to \phi_t \qquad (\phi_\varepsilon)_x \to \phi_x$ in $L^1(\Omega)$ as $\varepsilon \to 0$,

$supp(\phi_\varepsilon) \subseteq \overline{supp(\phi) + B(0, r/2)} \subseteq \Omega$.]

iii) Given any $\phi \in \mathcal{C}^1_c(]0,T[\times\mathbb{R})$ we can write $\phi = \phi^+ - \phi^-$, where $\phi^+ = \max\{\phi, 0\}$, $\phi^- = -\min\{\phi, 0\}$

Notice that $\phi^+, \phi^- \in Lip_c(]0,T[\times\mathbb{R})$, $\phi^+, \phi^- \geq 0$.

Therefore, because of ii) the integral equality (37) holds for both test function ϕ^+ and ϕ^-. Hence, also the difference of the corresponding integrals vanishes, which proves that (37) is verified also by ϕ.

2. We show that if $u \in L^\infty(]0,T[\times\mathbb{R})$ satisfies (58) for all $K \in \mathbb{R}$, $\phi \in \mathcal{C}^1_c(]0,T[\times\mathbb{R})$, $\phi \geq 0$, then u satisfies (55) for every pair of convex entropy-entropy flux for (18).

i) Observe that, since u is a weak solution by point 1, we deduce that

$$(60) \quad \iint_{]0,T[\times\mathbb{R}} \left\{ (u(t,x)-K)\,\phi_t(t,x) + \big(f(u(t,x)) - f(K)\big)\,\phi_x(t,x) \right\} dt\,dx \geq 0$$

for all $K \in \mathbb{R}$, $\phi \in \mathcal{C}^1_c(]0,T[\times\mathbb{R})$, $\phi \geq 0$.

Then, combining (58) with (60) we derive:

$$(61) \quad \iint_{]0,T[\times\mathbb{R}} \left\{ [u(t,x)-K]^+ \phi_t(t,x) + \left(\frac{1+\mathrm{sgn}(u(t,x)-K)}{2}\right)\big(f(u(t,x)) - f(K)\big)\,\phi_x(t,x) \right\} dt\,dx \geq 0$$

for all $K \in \mathbb{R}$, $\phi \in \mathcal{C}^1_c(]0,T[\times\mathbb{R})$, $\phi \geq 0$, 47/ where $\lfloor u \rfloor^+ = \max\{u, 0\} = \frac{u + |u|}{2}$ denotes the positive part of u.

Therefore, relying on (61), we deduce that the integral inequality (55) is verified for every pair (η, q), with η convex piecewise affine entropy of the form:

(62) $\eta(u) = a_0 + a_1 u + \sum_i c_i \lfloor u - K_i \rfloor^+, \quad a_0, a_1, c_i \in \mathbb{R}$
$c_i > 0,$

with corresponding flux:

(63) $q(u) = a_2 + f(u) + \frac{1}{2} \sum_i (1 + \mathrm{sgn}(u - K_i))\, c_i \,(f(u) - f(K_i)), \quad a_2 \in \mathbb{R}$

ii) For any given convex entropy $\eta \in C^1(\mathbb{R})$ for (18), and for any $M > 0$, one can construct a sequence of convex, piecewise affine entropies η_n of the form (62) such that

$\eta_n \longrightarrow \eta, \quad \eta_n' \longrightarrow \eta'$ uniformly on $[-M, M]$

(at points where η_n' is differentiable)

Then, letting q, q_n be the entropy fluxes associated to η, η_n defined by (see p. 44)

$$q(u) = \int_0^u \eta'(s)\, f'(s)\, ds\,, \qquad q_n(u) = \int_0^u \eta_n'(s)\, f'(s)\, ds$$

we deduce:

$$|q(u) - q_n(u)| \leq \int_{-M}^{M} C_M \, |\eta_n'(s) - \eta'(s)|\, ds \longrightarrow 0$$

uniformly on $[-M, M] \setminus N$

$$C_M = \sup_{s \in [-M, M]} \{ |f'(s)| \}$$

(N: set of points of non differentiability of η_n)

Therefore, since the integral inequality (55) holds for any (η_n, q_n) because of $i)$, we can pass to the limit in the integrals as $n \to \infty$ and deduce that the same integral inequality (55) is verified by (η, q). This concludes the proof of the Proposition. ∎

[Sketch of construction of the sequence of piecewise affine entropies η_n:

Given $\eta \in \mathcal{C}^1(\mathbb{R})$ convex, we construct η_n interpolating η linearly on a sufficient fine grid.

48/

Set $K_i = -M + i\frac{2M}{n}$, $i = 0, \dots, n$.

$\theta_0 \doteq \eta(-M) + \theta_1 M$, $\theta_1 \doteq \eta'(-M) - 1$

$$c_i = \frac{\eta(K_{i+1}) - \eta(K_i)}{K_{i+1} - K_i} - \theta_1 = \frac{n}{2M}\left(\eta(K_{i+1}) - \eta(K_i)\right) - \theta_1$$

$i = 0, \dots, (n-1)$.

Observe that by convexity of η one has

$c_i \geq 1 \quad \forall i$

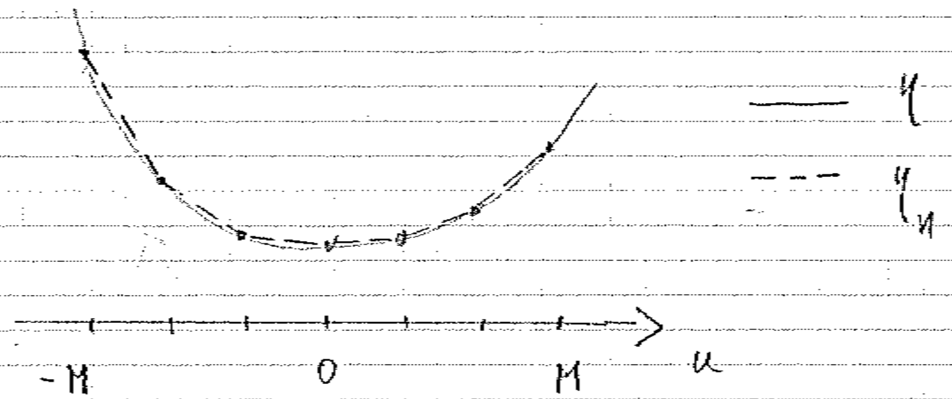
Then, defining

$$\eta_n(u) \doteq \eta(K_i) + \frac{\eta(K_{i+1}) - \eta(K_i)}{K_{i+1} - K_i}(u - K_i)$$

for $u \in [K_i, K_{i+1}]$, $i = 0, \dots, n-1$,

one can check that there holds:

$$\eta_n(u) = \theta_0 + \theta_1 u + \sum_{i=0}^{n-1} c_i \left[u - K_i\right]^+, \quad u \in [-M, M]$$



Observe that since $\psi \in C^1$, ψ' is uniformly continuous on $[-M, M]$. Hence, for every fixed $\varepsilon > 0$ there exists $n = n(\varepsilon)$ s.t.

$$L_n \doteq \sup \left\{ \left| \frac{\psi(\kappa_{i+1}) - \psi(\kappa_i)}{\kappa_{i+1} - \kappa_i} - \psi'(u) \right| ; u \in [\kappa_i, \kappa_{i+1}],\ i = 0, \dots, n-1 \right\} < \varepsilon$$

Therefore it follows:

a) $|\psi_n(u) - \psi(u)| \leq L_n |u - \kappa_i| \leq \frac{2M}{n} \varepsilon \quad \forall u \in [\kappa_i, \kappa_{i+1}]$, $i = 0, \dots, n-1$

b) $|\psi_n'(u) - \psi'(u)| \leq L_n < \varepsilon \quad \forall u \in]\kappa_i, \kappa_{i+1}[$, $i = 0, \dots, n-1$.

a) & b) prove the uniform convergence of ψ_n and ψ_n' on $[-M, M] \setminus \mathcal{N}$, with

$$\mathcal{N} = \left\{ -M + i \frac{2M}{n} ; i = 0, \dots, n,\ n \in \mathbb{N} \right\}. \Big]$$

4j/ Motivated by Proposition 2 we can thus employ an equivalent definition of weak admissible solution to Definition 6.

Definition 6 bis :

a function $u \in L^\infty(]0,T[\times\mathbb{R})$ is an entropy admissible weak solution of (18) if, for every $k \in \mathbb{R}$, there holds (58) for all $\phi \in C^1_c(]0,T[\times\mathbb{R})$, $\phi \geq 0$.

Remark 4: a function $u \in L^\infty(]0,T[\times\mathbb{R})$ is an entropy admissible weak sol. of (18) according with Definition 6 if and only if it is an entropy admissible weak sol. of (18) according with Definition 6 bis.
by Proposition 2

Similarly we provide the next definition for the Cauchy problem.

<u>Definition 7</u>: given $u_0 \in L^1_{loc}(\mathbb{R})$ we say that $u \in L^\infty(]0,T[\times\mathbb{R})$ is an <u>entropy admissible weak solution of the Cauchy problem</u> (18)-(19) if:

i) for any costant $K \in \mathbb{R}$ and for every $\phi \in C^1_c(]-\infty,T[\times\mathbb{R})$, $\phi \geq 0$, there holds:

$$\iint_{[0,T]\times\mathbb{R}} \Big\{ |u(t,x)-K|\,\phi_t(t,x) + \mathrm{sgn}(u(t,x)-K)\big(f(u(t,x))-f(K)\big)\,\phi_x(t,x)\Big\}\,dt\,dx + \int_{\mathbb{R}} |u_0(x)-K|\,\phi(0,x)\,dx \geq 0 \tag{64}$$

ii) the map $t \mapsto u(t,\cdot)$ is continuous as a function from $[0,T[$ into $L^1_{loc}(\mathbb{R})$.

<u>Remark 5</u>: relying on the conservative form of the equation (18) one can show that we can always choose a normalized representative of any weak solution u of (18) (and hence, in particular, of any entropy admissible weak solution) so that the map $t \to u(t,\cdot)$ becomes continuous as a function with values in $L^\infty(\mathbb{R})$ with the weak-$*$ convergence (see [Dafermos, Hyperbolic conservation

laws in continuum physics, Lemma 1.3.3].

Moreover, one can show that, if $u \in L^\infty([0,T[\times\mathbb{R})$ is a weak solution of (18) that satisfies (64) then, for every given entropy η, the map $t \to u(t,\cdot)$ turns out to be continuous from $[0,T[\setminus N$ into $L^\infty(\mathbb{R})$ with weak-$*$ topology, where N is an at most countable set, with $0 \notin N$.

If we consider in particular a uniformly convex C^2 entropy η ($\eta''(u) \geq c > 0 \quad \forall u$) we then deduce that the map $t \mapsto u(t,\cdot)$ is continuous as a function from $[0,T[\setminus N$ into $L^1_{loc}(\mathbb{R})$. (see [Dafermos, Hyperbolic conservation laws in continuum physics, Theorem 4.5.1]). Therefore, we almost recover property ii) of Definition 7. Indeed, it has been recently proved that, for linearly nondegenerate conservation laws (i.e. when $\{u : f'(u) = c\}$ has zero Lebesgue measure $\forall c \in \mathbb{R}$), the set N is empty so that for such conservation laws we may employ the definition of entropy admissible weak solution as an L^∞ function that satisfies (64) for any $K \in \mathbb{R}$ and for every $\phi \in \mathcal{C}_c^1$ (see [De Lellis, Otto, Westdickenberg, 2003]), without requiring condition ii).

[Sketch of the proof of the L^1_{loc} continuity of $t \mapsto u(t,\cdot)$

Assume that $t \mapsto u(t,\cdot)$, $t \mapsto \eta(u(t,\cdot))$ are continuous at $t=\tau$ as maps with values in $L^\infty(\mathbb{R})$ with weak-* topology, $\eta \in \mathcal{C}^2$, $\eta''(u) \geq c > 0$.

i) Fix $R>0$ and write

$$I(t) \doteq \int_{-R}^{R} \Big(\eta(u(t,x)) - \eta(u(\tau,x))\Big)\,dx = I_1(t) + I_2(t) \doteq$$

$$= \underbrace{\int_{-R}^{R} \eta'(u(\tau,x))\Big(u(t,x) - u(\tau,x)\Big)\,dx}_{I_1} +$$

$$+ \underbrace{\int_{-R}^{R} \eta''(z(x))\Big(u(t,x) - u(\tau,x)\Big)^2 dx}_{I_2}, \quad \text{for some}$$

51/ $Z(x) \in \left[u(\tau,x),\ u(t,x)\right]$ or $Z(x) \in \left[u(t,x),\ u(\tau,x)\right]$.

Observe that by the continuity of $u(t,\cdot)$, $\eta(u(t,\cdot))$ at $t=\tau$, interpreting $\eta'(u(\tau,x))$ as a test function with respect to L^∞ weak-$*$ topology $(\eta'(u(\tau,\cdot)) \in L^1)$, we deduce:

$$\lim_{t\to\tau} I(t) = \lim_{t\to\tau} I_1(t) = 0$$

Therefore, it follows

$$0 \leq \lim_{t\to\tau} \int_{-R}^{R} \left(u(t,x) - u(\tau,x)\right)^2 dx \leq$$

$$\leq \frac{1}{c} \lim_{t\to\tau} \int_{-R}^{R} \eta''(Z(x)) \left(u(t,x) - u(\tau,x)\right)^2 dx = 0.$$

This implies that $\lim_{t\to\tau} \| u(t,\cdot) - u(\tau,\cdot) \|_{L^2} = 0$,

which, together with the fact that $u \in L^\infty([0,T[\times \mathbb{R})$ yields $\lim_{t\to\tau} \| u(t,x) - u(\tau,x) \|_{L^1} = 0$.].

Remark 6: if u is an entropy admissible weak solution of (18)-(19) on $[0,T[\times \mathbb{R}$, then u is an entropy admissible weak sol. of (18) on $]0,T[\times \mathbb{R}$

A classical theorem of Kruzhkov provides an <u>estimates between</u> any <u>two entropy-admissible weak solutions</u> which yields the uniqueness of the entropy admissible weak solution of a Cauchy problem with initial data $u_0 \in L^\infty(\mathbb{R})$.

<u>**Theorem 3**</u> : Given $f \in Lip_{loc}(\mathbb{R})$, let $u, v \in L^\infty([0,T[\times\mathbb{R})$ be two entropy admissible weak solutions of the Cauchy problem (18)-(19) with initial data $u_0, v_0 \in L^\infty(\mathbb{R})$, respectively. Let $L, M > 0$ be such that

(65) $\quad |u(t,x)| \le M, \quad |v(t,x)| \le M \quad$ for a.e. (t,x)

(66) $\quad |f(u) - f(v)| \le L\,|u - v| \quad \forall\, u, v \in [-M, M]$.

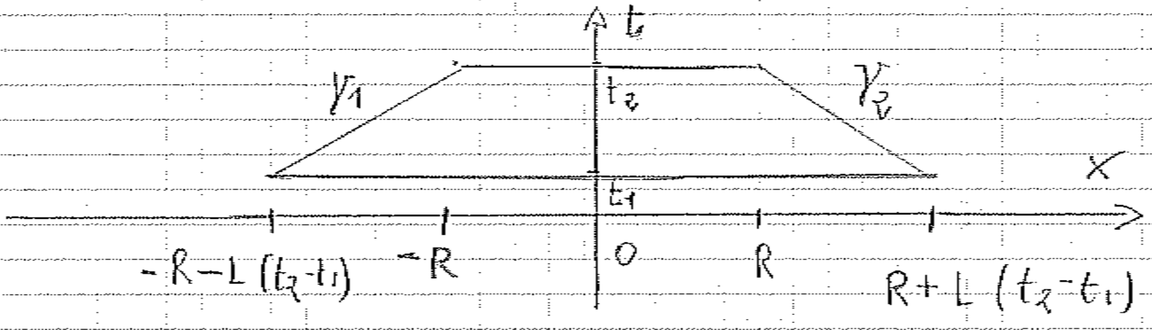
Then, for every $0 \le t_1 < t_2 < T$, and for any $R > 0$ one has

(67) $$\int_{|x| \le R} |u(t_2,x) - v(t_2,x)|\,dx \le \int_{|x| \le R + L(t_2 - t_1)} |u(t_1,x) - v(t_1,x)|\,dx$$

Sketch of the proof:

i) Consider the trapezoid

$$\Omega = \{ (t,x) : t \in [t_1, t_2], \ |x| \le R + L(t_2 - t) \}$$



Let (η, q) be a Kruzkhov entropy - entropy flux pair for (18) of the form (57). By Remark 6 u satisfies formally the inequality (56) on the domain Ω. Then, applying the divergence theorem to the vector field $\underline{V}(t,x) = \big(\eta(u(t,x)), q(u(t,x)) \big)$, we find (with a similar computation to the one in Remark 2 p. 28)

$$0 \geqslant \iint_\Omega \{ [\eta(u)]_t + [q(u)]_x \} \, dt\, dx \Longrightarrow$$ (continues on the back)

$$(68)\quad 0 \geq \int_{-R}^{R} \eta(u(t_2,x))\,dx - \int_{-R-L(t_2-t_1)}^{R+L(t_2-t_1)} \eta(u(t_1,x))\,dx \; +$$

$$+ \int_{t_1}^{t_2} \Big\{ L\,\eta(u(t,\gamma_1(t))) - q(u(t,\gamma_1(t))) \Big\} dt \; +$$

$$+ \int_{t_1}^{t_2} \Big\{ L\,\eta(u(t,\gamma_2(t))) - q(u(t,\gamma_2(t))) \Big\} dt \,,$$

where $\gamma_1(t) = -R - L(t_2-t)$, $\gamma_2(t) = R + L(t_2-t)$.

Observe now that, because of (65)-(66) we have $L\,\eta(u) - q(u) = L|u-k| - \mathrm{sgn}(u-k)(f(u)-f(k)) \geq 0$.

Therefore, from (68) we derive, for any $k \in \mathbb{R}$,

$$(69)\quad \int_{|x|\leq R} |u(t_2,x) - k|\,dx \leq \int_{|x|\leq R+L(t_2-t_1)} |u(t_1,x) - k|\,dx$$

ii) Notice that (69) yields the estimate (67) in the special case where $v(t,x) \equiv k$ is a constant function.

In order to show that (69) remains valid if we replace K with the other entropy solution $v(t,x)$ we need to extend the integral inequality (58). We shall adopt a doubling the variable argument. Consider the two solutions u and v as functions of distinct independent variables

$$u = u(t,x) \qquad v = v(s,y)$$

then write the corresponding entropy integral inequality (58) with a test function that depends on four variables: $\phi = \phi(t,x,s,y)$, $\phi \geq 0$. Next substitute $K = v(s,y)$ in the integral inequality for u and then integrate w.r.t. s,y. Similarly, substitute $K = u(t,x)$ in the integral inequality for v and then integrate w.r.t. t,x. Summing up the two results we obtain an integral inequality on $\mathbb{R}^4$ for a test function ϕ depending on four variables.

iii) Introduce a suitable sequence of test functions $\phi_n = \phi_n(t,x,s,y) \geq 0$ that concentrate most of the mass along the diagonal where $t = s$ and $x = y$. Taking the limit over the corresponding integral inequality for ϕ_n we obtain, for any $t > 0$, the inequality

$$(70) \quad \iint_{]0,T[\times\mathbb{R}} \Big\{ |u(t,x) - v(t,x)|\, \psi_t(t,x) + \operatorname{sgn}\big(u(t,x) - v(t,x)\big)\Big(f(u(t,x)) - f(v(t,x))\Big)\psi_x(t,x) \Big\}\, dt\, dx \geq 0$$

for every $\psi \in C^1_c(]0,T[\times\mathbb{R})$, $\psi \geq 0$, which provides the desired extension of the integral inequality (58)

iv) Use (70) in connection with test functions ψ_n which approximate the characteristic function of Ω so that one has:

$$(\psi_n)_t\, dt\, dx \xrightarrow{n\to\infty} \delta_{\{t_1\}}(t)\, dx - \delta_{\{t_2\}}(t)\, dx - \underset{= L}{|\dot{\gamma}_1(t)|}\, \delta_{\{\gamma_1(t)\}}(x)\, dt - \underset{= L}{|\dot{\gamma}_2(t)|}\, \delta_{\{\gamma_2(t)\}}(x)\, dt$$

$$(\psi_n)_x\, dt\, dx \xrightarrow{n\to\infty} \delta_{\{\gamma_1(t)\}}(x)\, dt - \delta_{\{\gamma_2(t)\}}(x)\, dt$$

where $\delta_{\{x_0\}}(x)$, $\delta_{\{t_0\}}(t)$ denote the Dirac masses concentrated at x_0, t_0, respectively.

Taking the limit in the integral inequality (70) with $\varphi = \varphi_n$, as $n \to \infty$, and relying on the continuity of the maps $t \mapsto u(t,\cdot)$, $t \mapsto v(t,\cdot)$ as functions with values in L^1_{loc} we thus find:

(71) $$\int_{|x| \le R + L(t_2 - t_1)} |u(t_1,x) - v(t_1,x)|\, dx - \int_{|x| \le R} |u(t_2,x) - v(t_2,x)|\, dx +$$

$$+ \int_{t_1}^{t_2} \Big\{ \big[f(u(t,\gamma_1(t))) - f(v(t,\gamma_1(t))) \big] \operatorname{sgn}\big(u(t,\gamma_1(t)) - v(t,\gamma_1(t))\big) +$$

$$- L \,\big| u(t,\gamma_1(t)) - v(t,\gamma_1(t)) \big| \Big\}\, dt \; +$$

$$+ \int_{t_1}^{t_2} \Big\{ -\big[f(u(t,\gamma_2(t))) - f(v(t,\gamma_2(t))) \big] \operatorname{sgn}\big(u(t,\gamma_2(t)) - v(t,\gamma_2(t))\big) +$$

$$- L \,\big| u(t,\gamma_2(t)) - v(t,\gamma_2(t)) \big| \Big\}\, dt \ge 0 .$$

Observe now that, thanks to (66), we have:

$$\pm \big[f(u) - f(v) \big] \operatorname{sgn}(u-v) - L\,|u-v| \le 0 \quad \forall u, v.$$

Therefore, from (71) we deduce that also the difference of the first two integrals is non negative, which yields (67) for $0<t_1<t_2<T$. By continuity, (67) remains true for $0\le t_1<t_2<T$.
For details of the proof of Theorem 3 see [Bressan: Hyperbolic Systems of Conservation Laws, proof of Thm. 6.2].

<u>Corollary 1</u> (<u>Uniqueness of entropy weak solutions</u>)

Given $f\in Lip_{loc}(\mathbb{R})$, let $u,v\in L^\infty([0,T[\times\mathbb{R})$ be two entropy admissible weak solutions of the Cauchy problem (18)-(19), with initial data $u_0, v_0\in L^\infty(\mathbb{R})$, respectively, such that $\| u_0 - v_0\|_{L^1}<\infty$.
Then, for every $t\ge 0$ there holds

$$(72)\quad \| u(t,\cdot) - v(t,\cdot)\|_{L^1(\mathbb{R})} \le \| u_0 - v_0\|_{L^1(\mathbb{R})}$$

Moreover, for any $u_0\in L^\infty(\mathbb{R})$, the Cauchy problem (18)-(19) admits at most one entropy admissible weak solution.

55/

Proof of Corollary 1:

Since $u, v \in L^\infty$, there exist $M, L > 0$ such that (65)-(66) hold. Thus applying Theorem 3 we deduce that, for every $t > 0$ and for any $R > 0$, there holds

$$(73) \quad \int_{|x| \le R} |u(t,x) - v(t,x)|\,dx \le \int_{|x| \le R + Lt} |u_0(x) - v_0(x)|\,dx$$

Taking the limit in (73) as $R \to +\infty$ we obtain (72), which in turn yields the uniqueness of entropy admissible weak solutions. ☒

<u>Remark 6</u>: Theorem 3 shows that in order to achieve uniqueness of solutions of the Cauchy problem for a conservation law we need to supplement the equation with infinitely many entropy inequalities (associated to Kruzhkov's entropies). However, it has been shown

that, in the case of a convex conservation law (as Burgers' equation), it is sufficient to require that a single entropy inequality is satisfied. Namely, if f is strictly convex and u is a weak solution of (18)-(19) which satisfies the entropy inequality (55) for a single strictly convex entropy η, then u is the entropy admissible weak solution of the Cauchy problem (18)-(19) (according with Definition 7).

[Panov (1994), De Lellis, Otto, Westdickenberg (2004)]

- We wish to show now that, for a piecewise Lipschitz weak solution, the entropy decrease conditions (55) or (58) turn out to be equivalent to impose additional restrictions only along the discontinuity lines.

<u>Theorem 4</u> (<u>characterization</u> of <u>PL entropy weak solutions</u>) Let $u :]0,T[\times \mathbb{R} \to \mathbb{R}$ be a (PL) function according with Definition 4. Then, the following are equivalent:

(i) u is an entropy admissible weak solution of (18) (according with Definition 6 or 6 bis)

(ii) u satisfies the quasilinear equation

(74) $$u_t + f'(u)\, u_x = 0 \qquad \text{for a.e. } (t,x)$$

Moreover, for every jump curve γ_j, and for every entropy - entropy flux Kruzkhov pair (η_k, q_k), $\eta_k(u) = |u-k|$, $q_k(u) = \mathrm{sgn}(u-k)\big(f(u) - f(k)\big)$ there holds

(75) $$\dot{\gamma}_j(t)\left(\eta_k(u_j^+(t)) - \eta_k(u_j^-(t))\right) \geq q_k(u_j^+(t)) - q_k(u_j^-(t))$$

for all $t \in]a_j, b_j[$.

Proof : (recall that, by Radamacher Thm, u, $f(u)$ $\eta(u)$, $q(u)$ are all functions a.e. differentiable)

1. (i) => (ii)

Assume that u is an entropy admissible weak sol. of (18). Then by Definition 6 u is in particular a weak solution of (18). Thus, applying Theorem 2, we deduce that u satisfies the quasilinear equation (74), which, in turn, implies that u satisfies the equation

$$(76) \quad [\eta(u)]_t + [q(u)]_x = 0 \qquad \text{for a.e. } (t,x)$$

$$\Big(\text{since } [\eta(u)]_t + [q(u)]_x = \eta'(u)\, u_t + q'(u)\, u_x = \eta'(u)\big[u_t + f'(u)\, u_x\big] = 0. \Big)$$

for every entropy - entropy flux pair (η, q).

On the other hand, applying the divergence theorem to the vector field $\underline{v} = \big(\eta(u)\phi,\ q(u)\phi\big)$, $\phi \in \mathcal{C}^1_c$, we deduce as in the proof of Thm 2 that

$$(77) \quad \iint_{]0,T[\times\mathbb{R}} \big\{\eta(u)\phi_t + q(u)\phi_x\big\}\, dt\, dx = -\iint_{]0,T[\times\mathbb{R}} \big\{[\eta(u)]_t + [q(u)]_x\big\}\phi\, dt\, dx$$
$$+ \sum_j \int_{a_j}^{b_j} \big\{\dot\gamma_j\big(\eta(u_j^+) - \eta(u_j^-)\big) - \big(q(u_j^+) - q(u_j^-)\big)\big\}\, \phi(t, \gamma_j(t))\, dt.$$

57/ Assume in particular that (η, q) is a Kruzkhov entropy - entropy flux pair. Then, since u is entropy admissible the integral inequality (58) is satisfied $\left(\text{i.e. } \iint \{ \eta(u)\phi_t + q(u)\phi_x \} \, dt\,dx \geq 0 \right)$ and from (76) - (77) we deduce

$$(78) \quad \sum_j \int_{a_j}^{b_j} \left\{ \dot{\gamma}_j \left(\eta(u_j^+) - \eta(u_j^-) \right) - \left(q(u_j^+) - q(u_j^-) \right) \right\} \phi(t, \gamma_j(t))\, dt \geq 0$$

Since (78) holds for every $\phi \in C_c^1(]0,T[\times\mathbb{R})$, $\phi \geq 0$, considering in particular test functions ϕ with support that has non empty intersection with a single jump curve γ_j, we deduce from (78) the pointwise inequality (75).

2. (ii) $\Rightarrow$ (i)

Assume that u satisfies (74). Then we deduce as above that (76) is verified. On the other hand (77) continues to hold and hence, relying on (75) we deduce

$$\iint_{]0,T[\times\mathbb{R}} \{ \eta(u)\phi_t + q(u)\phi_x \} \, dt\,dx \geq 0 \quad \forall \phi \in C_c^1, \ \phi \geq 0.$$

Therefore u satisfies the integral inequality (58) for every $K \in \mathbb{R}$, which shows that u is an entropy admissible weak solution according with Definition 6 bis (and hence according also with Definition 6). ∎

Remark 7 : $u \in \mathcal{C}^1(]0,T[\times\mathbb{R})$ is a classical solution of (18) if and only if u is an entropy admissible weak solution of (18). In fact, if $u \in \mathcal{C}^1$ then u is a (PL) function and the quasilinear equation (74) is satisfied for a.e. (t,x) iff it is satisfied for all (t,x), while the set of jump curves is empty.

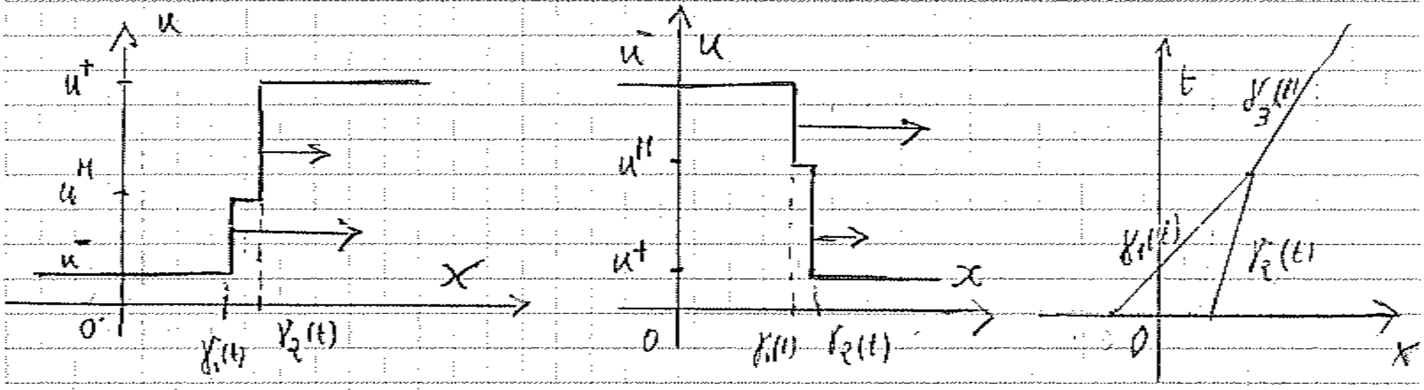
3. Stability conditions

We wish to derive now simple geometric conditions which allow to check the pointwise inequality (75) stated in Theorem 4, that can be obtained purely from stability considerations, without any reference to physical models (as we have done for the entropy conditions).

Let

$$U(t,x) = \begin{cases} u^- & \text{if } x < \lambda t \\ u^+ & \text{if } x > \lambda t \end{cases}$$

be a piecewise constant weak solution of (18) and consider a slightly perturbed solution $\hat{U}$ where the original shock joining the states u^-, u^+ is splitted in two separated smaller shocks, say located at $\gamma_1(t) < \gamma_2(t)$, respectively, that join the states u^-, u^+ with an intermediate state u^M.



To ensure that the perturbed solution remains close in L^1 to the original solution possessing a single shock, i.e. to guarantee that the L^1 distance between the original solution and the perturbed one does not increase in time we need:

[speed of jump behind] $\geq$ [speed of jump ahead]

By the (RH) condition this is equivalent to require

$$(79) \qquad \frac{f(u^H)-f(u^-)}{u^H-u^-} \geq \frac{f(u^+)-f(u^H)}{u^+-u^H}$$

for all $u^H = \alpha u^+ + (1-\alpha)u^-$, $\alpha \in]0,1[$.

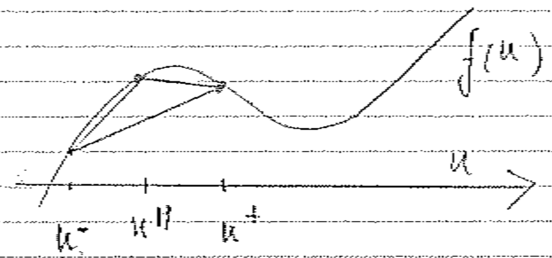
The condition (79) is equivalent to the following stability condition that holds for any $\alpha \in]0,1[$:

$$(80) \quad \begin{cases} f(\alpha u^+ + (1-\alpha)u^-) \geq \alpha f(u^+) + (1-\alpha) f(u^-) & \text{if } u^- < u^+ \\ f(\alpha u^+ + (1-\alpha)u^-) \leq \alpha f(u^+) + (1-\alpha) f(u^-) & \text{if } u^- > u^+ \end{cases}$$

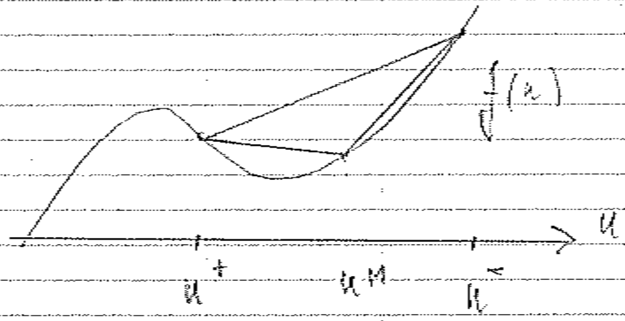
59

which have a simple geometric interpretation

- if $u^- < u^+$ then the shock is stable if the graph of f remains above the secant joining $(u^-, f(u^-))$, $(u^+, f(u^+))$, on the whole interval $[u^-, u^+]$



- if $u^+ < u^-$ then the shock is stable if the graph of f remains below the secant joining $(u^-, f(u^-))$, $(u^+, f(u^+))$, on the whole interval $[u^+, u^-]$

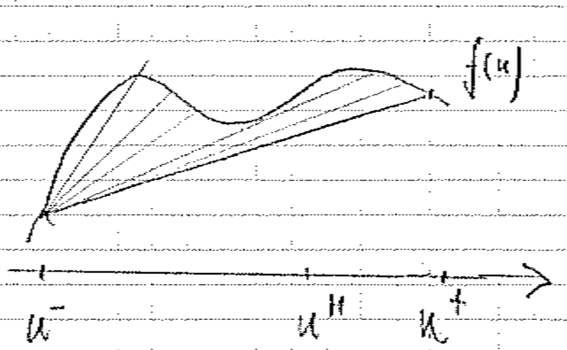


Observe that condition (79) (or (80)) are equivalent to:

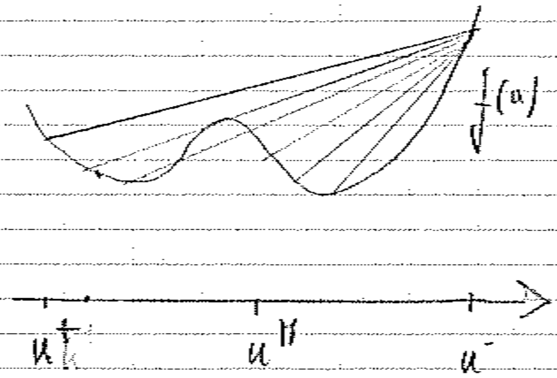
$$(81) \quad \frac{f(u^*) - f(u^-)}{u^* - u^-} \geq \frac{f(u^+) - f(u^-)}{u^+ - u^-}$$

for all $u^* = \alpha u^+ + (1-\alpha) u^-$, $\alpha \in]0, 1[$

which say that the speed of the original shock should be not greater than the speed of any intermediate shock joining the left state u^- with an intermediate state u''.



$(u^- < u^+)$



$(u^+ < u^-)$

We wish to show now that the stability condition (79) (or (80), or (81)) is equivalent to require that the piece-wise constant function (41) is an entropy admissible weak solution.

<u>Proposition 3</u>

The function $$U(t,x) = \begin{cases} u^- & \text{if } x < \lambda t \\ u^+ & \text{if } x > \lambda t \end{cases}$$

is an entropy admissible weak solution of (18) if and only if the Rankine-Hugoniot equation (17) is verified and the inequality (80) holds $\forall \alpha \in]0,1[$.

Proof of Proposition 3:

1. Since U is certainly a (PL) function, we know by Theorem 4 that U is an entropy admissible weak solution if and only if there holds

$$(82)\quad \lambda\left(\eta_k(u^+)-\eta_k(u^-)\right) \geq q_k(u^+)-q_k(u^-) \quad \forall k \in \mathbb{R},$$

where (η_k, q_k) denotes a Kruzhkov entropy-entropy flux pair. Therefore, in order to prove the proposition, it will be sufficient to show that the inequality (82) is verified if and only if the (RH) equation is verified together with the stability condition (80) for any $\alpha \in]0,1[$.

2. Recalling (57) we rewrite (82) as

$$(83)\quad \lambda\left[|u^+-k|-|u^--k|\right] \geq \left[\left(f(u^+)-f(k)\right)\operatorname{sgn}(u^+-k)-\left(f(u^-)-f(k)\right)\operatorname{sgn}(u^--k)\right]$$

- Observe that the inequalities (83), for $k \leq \min\{u^-, u^+\}$ and for $k \geq \max\{u^-, u^+\}$, are equivalent, respectively, to the two inequalities:

$\lambda(u^+-u^-) \geq f(u^+)-f(u^-)$ and $\lambda(u^+-u^-) \leq f(u^+)-f(u^-)$

which, in turn, together are equivalent to the (RH) equation (17)

- On the other hand, when $\min\{u^-, u^+\} < k < \max\{u^-, u^+\}$, the inequality (83) takes the form

$$\lambda (u^+ + u^- - 2k)\,\mathrm{sgn}(u^+ - u^-) \geq \left[f(u^+) + f(u^-) - 2f(k)\right] \mathrm{sgn}(u^+ - u^-).$$

Multiplying both sides of this inequality by $|u^+ - u^-|$ and using the (RH) condition we deduce the equivalent inequality

$$\left(f(u^+) - f(u^-)\right)\left(u^+ + u^- - 2k\right) \geq (u^+ - u^-)\left(f(u^+) + f(u^-) - 2f(k)\right)$$

Writing $k = \alpha u^+ + (1-\alpha) u^-$, $\alpha \in]0,1[$, we then rewrite such an inequality as

$$\left(f(u^+) - f(u^-)\right)(1 - 2\alpha)\,(u^+ - u^-) \geq (u^+ - u^-)\left[f(u^+) + f(u^-) - 2f(\alpha u^+ + (1-\alpha) u^-)\right]$$

which yields:

(84) $$-\left[2\alpha f(u^+) + (2 - 2\alpha) f(u^-)\right](u^+ - u^-) \geq -2(u^+ - u^-)\, f(\alpha u^+ + (1-\alpha) u^-),$$

$\alpha \in]0,1[$. Observing that (84) is equivalent to the stability condition (80), we conclude the proof. $\square$

Relying on Proposition 3 we can now provide a characterization of (PL) entropy weak solutions in terms of the stability inequalities (79) along the jump curves.

61/ **Theorem 5** (characterization of PL entropy weak solutions with stability inequalities).

In the same setting of Theorem 4 the following are equivalent:

(i) u is an entropy admissible weak sol. of (18)

(ii) u satisfies the quasilinear equation (74) and, for every jump curve $\gamma_j :]a_j, b_j[\to \mathbb{R}$, there hold the (RH) conditions (45) together with the stability condition

$$(85) \quad \frac{f(u_j^M(t)) - f(u_j^-(t))}{u_j^M(t) - u_j^-(t)} \geq \frac{f(u_j^+(t)) - f(u^M(t))}{u_j^+(t) - u^M(t)} \quad t \in]a_j, b_j[$$

for all $u_j^M(t) = \alpha\, u_j^+(t) + (1-\alpha)\, u_j^-(t), \ \alpha \in]0,1[$.

Proof: the Thm is an immediate consequence of Theorem 4 and Proposition 3.

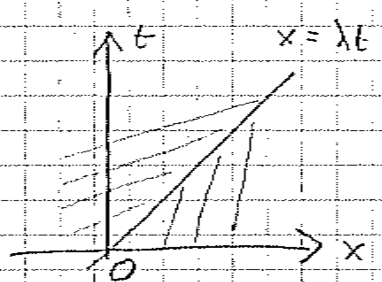
Remark 8: if we take the limit in (79) as $u^M \to u^+$ and then as $u^M \to u^-$ we obtain the condition

$$(86) \qquad f'(u^-) \geq \frac{f(u^+) - f(u^-)}{u^+ - u^-} \geq f'(u^+)$$

which can be seen as another type of admissible condition:

<u>Definition 8</u>

We say that a <u>weak solution</u> of (18) is <u>admissible in the sense of Lax</u> if at every point $(\bar{t}, \bar{x})$ of approximate jump discontinuity, with left and right states u^-, u^+, and speed λ, there holds the <u>Lax condition</u>

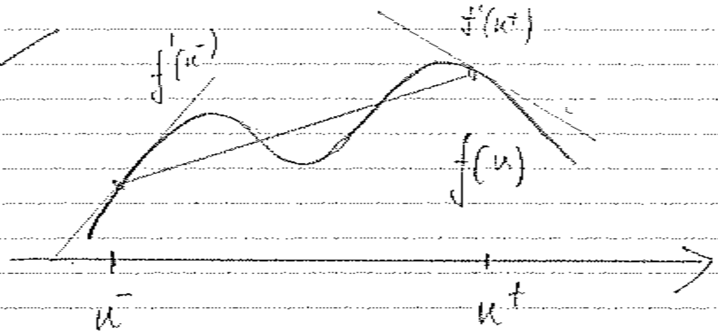


(87) $$f'(u^-) \geq \lambda \geq f'(u^+).$$

<u>Remark 9</u> if the flux f is convex ($f''(u) \geq 0 \ \forall u$) then the stability condition (79) is equivalent to the Lax condition (87). Moreover, if f is strictly convex ($f''(u) > 0 \ \forall u$), then the Lax condition is equivalent to the condition:

(88) $$u^- > u^+.$$

In the case of a general (non convex) flux f the stability condition (79) imply the Lax condition but the viceversa is not true.



(f non convex)

$$\lambda = \frac{f(u^+) - f(u^-)}{u^+ - u^-}$$

satisfies Lax condition (87) but the stability condition (73) is not verified.

Therefore, for a conservation law with a general (non convex) flux, any entropy admissible weak solution is also an admissible weak solution in the sense of Lax. The viceversa is not true.

Instead, in the case of a conservation law with a convex flux f, the two notions of weak solutions (entropy admissible and Lax admissible) turn out to be equivalent for the class of (PL) functions.

If f is strictly convex, a weak solution that is a (PL) function is entropy admissible if and only if along each jump curve the left and right limits satisfy the condition (88).

Summarizing, we have:

$$
\begin{tikzcd}
\text{Vanishing viscosity} \arrow[r, Rightarrow] & \text{Entropy conditions} \arrow[d, Leftrightarrow] & \\
& \text{Stability conditions} + RH \arrow[d, Rightarrow, shift right=2] & \\
& \text{Lax conditions} \arrow[u, Rightarrow, shift right=2, "\substack{\text{if } f \\ \text{convex}}"'] &
\end{tikzcd}
$$

63/ <u>Conservation Laws & HJ equations</u>

Before facing the problem of providing an existence result of admissible weak solutions we wish to compare the notion of admissible weak solutions of conservation laws with the notion of viscosity solutions of Hamilton-Jacobi equations.

Namely, consider the Hamilton-Jacobi equation

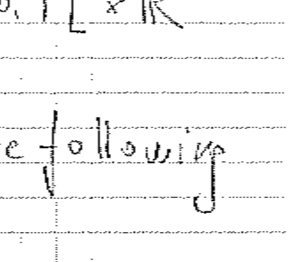
$$(89) \qquad v_t + H(v_x) = 0, \qquad t \in]0,T[\times \mathbb{R}$$

$\Omega \subseteq \mathbb{R}^2$ open, $H \in C^1(\mathbb{R})$. We recall the following

<u>Definition 9</u>: a function $v \in C(]0,T[\times\mathbb{R})$ is a viscosity solution of (89) if, for any $\phi \in C^1(]0,T[\times\mathbb{R})$ the following hold:

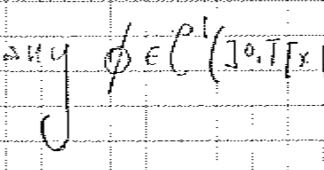
i) at every point $(t_0,x_0) \in]0,T[\times\mathbb{R}$ where $v-\phi$ has a local maximum one has

$$\phi_t(t_0,x_0) + H(\phi_x(t_0,x_0)) \leq 0$$



ii) at every point $(t_0,x_0) \in]0,T[\times\mathbb{R}$ where $v-\phi$ has a local minimum one has

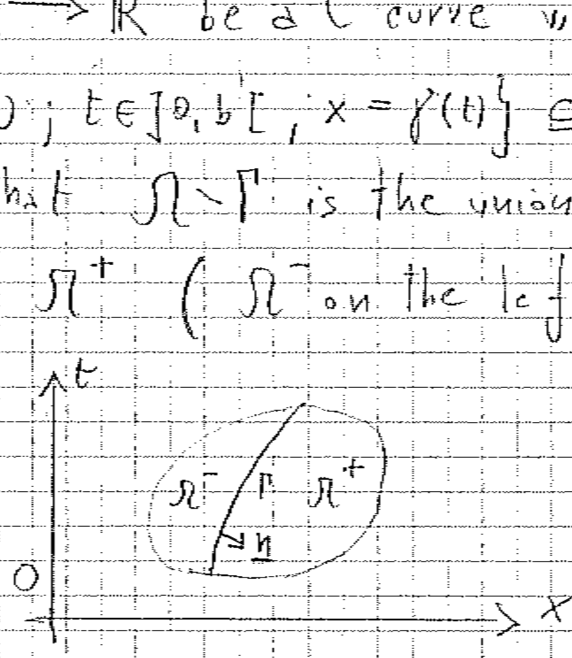
$$\phi_t(t_0,x_0) + H(\phi_x(t_0,x_0)) \geq 0$$



(first part of course)
We have seen that if $v \in C^1(]0,T[\times\mathbb{R})$ is a classical solution of (89) (i.e. satisfies (89) at every point (t,x)), then v is a viscosity solution of (89). We wish to provide now a characterization of a piecewise C^1 viscosity solution of (89). Namely, consider an open, connected domain $\Omega \subseteq]0,T[\times\mathbb{R}$, let $\gamma:]a,b[\longrightarrow \mathbb{R}$ be a C^1 curve with support

$$\Gamma = \{(t,x)\ ;\ t \in]a,b[\ ,\ x = \gamma(t)\} \subseteq \Omega,$$

and such that $\Omega \setminus \Gamma$ is the union of two disjoint domains Ω^-, Ω^+ (Ω^- on the left of Ω^+ in the x-t plane).



Consider a function $v \in C(\Omega)$ and, letting v^-, v^+ denote the restrictions of v to $\Omega^- \cup \Gamma$ and to $\Omega^+ \cup \Gamma$, respectively, assume that both functions v^-, v^+ are of class C^1 on their domains.

Observe that, since v^+, v^- coincide along Γ (being v continuous) and are C^1 on $\Omega^+\cup\Gamma$, $\Omega^-\cup\Gamma$, respectively, it follows that the tangential component of the derivatives of v^-, v^+ along Γ is the same. Denote it by $D^T v^\pm(t,x) \doteq (D_t^T v^\pm(t,x), D_x^T v^\pm(t,x))$, $(t,x) \in \Gamma$.

Thus we have:

(90)
$$\underbrace{D^{T} v^{\pm}(t,\gamma(t))}_{\left(\begin{array}{l}\text{Tangential component}\\ \text{of derivative of } v^{-} (\text{or } v^{+})\\ \text{along } \Gamma\end{array}\right)} = \nabla v^{-}(t,\gamma(t)) \cdot \frac{1}{\sqrt{1+(\dot\gamma(t))^2}} (1, \dot\gamma(t))$$
$$= \nabla v^{+}(t,\gamma(t)) \cdot \frac{1}{\sqrt{1+(\dot\gamma(t))^2}} (1, \dot\gamma(t)),$$

which, in turn, implies:

$$(91) \quad v^{-}_t(t,\gamma(t)) + \dot\gamma(t)\, v^{-}_x(t,\gamma(t)) = v^{+}_t(t,\gamma(t)) + \dot\gamma(t)\, v^{+}_x(t,\gamma(t))$$

On the other hand, the normal component of the derivatives of v^- and v^+ along Γ will be in general different. Letting $\underline{n}(t_0,x_0)$ be the normal versor to Γ at (t_0,x_0), pointing toward Ω^+, and denoting such normal components as $D^{\perp} v^{-}(t_0,x_0)$, $D^{\perp} v^{+}(t_0,x_0)$, we have

$$\left.\begin{array}{l} D^{\perp} v^{-}(t,\gamma(t)) = \nabla v^{-}(t,\gamma(t)) \cdot \underline{n}(t,\gamma(t)) \\ D^{\perp} v^{+}(t,\gamma(t)) = \nabla v^{+}(t,\gamma(t)) \cdot \underline{n}(t,\gamma(t)) \end{array}\right\} \begin{array}{l}\text{Normal components}\\ \text{of derivatives of}\\ v^{-} \text{ and } v^{+} \text{ along } \Gamma\end{array}$$

Employing these notations we state the next result which provides a characterization of piecewise C^1 viscosity solutions of (89) whose proof can be found in [Crandall, Lions, Evans, Some properties of viscosity solutions of Hamilton-Jacobi equation, 1984].

Theorem 6 (characterization of piecewise C^1 viscosity solutions of HJ)

Let $v \in C(\Omega)$, $\Omega \subseteq]0,T[\times \mathbb{R}$ be an open connected domain divided in two open subsets Ω^-, Ω^+ by a C^1 curve Γ as above, parametrized by $(t, \gamma(t))$, $\gamma \in C^1(]a,b[)$, and assume that $v^- \in C^1(\Omega^- \cup \Gamma)$, $v^+ \in C^1(\Omega^+ \cup \Gamma)$, v^-, v^+ denoting the corresponding restrictions of v to $\Omega^- \cup \Gamma$, $\Omega^+ \cup \Gamma$, respectively.

Then, the following are equivalent:

(i) v is a viscosity solution of (89)

(ii) v^- and v^+ are classical solutions of (89) on Ω^- and Ω^+, respectively and at any point $(t_0, x_0) \in \Gamma$, letting $\underline{n}(t_0,x_0) = (n_t(t_0,x_0), n_x(t_0,x_0))$ denote the unit normal to Γ pointing toward Ω^+, there holds

a) if $D^\perp v^-(t_0,\gamma(t_0)) \leq D^\perp v^+(t_0,\gamma(t_0))$ then one has

$$\frac{D^T v^\pm(t_0,\gamma(t_0))}{\sqrt{1+(\dot\gamma(t_0))^2}} + \lambda n_t(t_0,\gamma(t_0)) + H\left(\frac{\dot\gamma(t_0)}{\sqrt{1+(\dot\gamma(t_0))^2}} D^T v^\pm(t_0,\gamma(t_0)) + \lambda n_x(t_0,\gamma(t_0))\right) \geq 0 \tag{92}$$

$$\forall \lambda \in \left[D^\perp v^-(t_0,\gamma(t_0)),\ D^\perp v^+(t_0,\gamma(t_0))\right]$$

b) if $D^{\perp} v^{-}(t_0, \gamma_0(t_0)) \geq D^{\perp} v^{+}(t_0, \gamma_0(t_0))$, then one has

$$(93) \qquad \frac{D^{T} v^{\pm}(t_0, \gamma(t_0))}{\sqrt{1+(\dot\gamma(t_0))^2}} + \lambda n_t(t_0, \gamma(t_0)) + H\left(\frac{\dot\gamma(t_0)}{\sqrt{1+(\dot\gamma(t_0))^2}} D^{T} v^{\pm}(t_0, \gamma(t_0)) + \lambda n_x(t_0, \gamma(t_0))\right) \leq$$

$$\forall \lambda \in \left[D^{\perp} v^{+}(t_0, \gamma(t_0)),\ D^{\perp} v^{-}(t_0, \gamma(t_0))\right].$$

Relying on Theorem 6 we shall establish now:

<u>Proposition 4</u>: Let $v \in \mathcal{C}(\Omega)$ be a viscosity solution of (89) satisfying the assumptions of Theorem 6. Then, the function $u : \Omega \to \mathbb{R}$ defined a.e. by

$$(94) \qquad u(t,x) = v_x(t,x) \quad \text{if } (t,x) \in \Omega \setminus \Gamma$$

is an entropy admissible weak solution of

$$(95) \qquad u_t + \left[H(u)\right]_x = 0 .$$

<u>Proof</u>:

1. Observe that the function u defined in (94) is a (PL) function (according with Definition 4) because of the regularity assumptions on v. Moreover, since by Theorem 6-(ii), we know that v^-, v^+ are classical solutions

of (89) on Ω^- and Ω^+, respectively, it follows that u is a classical solution of (95) (equivalently of the quasilinear equation $u_t + H'(u)\, u_x = 0$) on $\Omega \setminus \Gamma$.
Therefore, relying on Theorem 5, we deduce that in order to prove that u is an entropy admissible weak solution of (95) it will be sufficient to show that along the curve Γ the (RH) condition (45) holds and that the stability condition (85) is verified.

2. Notice that, since by Theorem 6-(ii) v^-, v^+ are solutions of (89) on $\Omega^- \cup \Gamma$ and $\Omega^+ \cup \Gamma$, respectively, using (91) we deduce

$$H\big(v^-_x(t,\gamma(t))\big) - \dot\gamma(t)\, v^-_x(t,\gamma(t)) =$$

$$= -\Big(v^-_t(t,\gamma(t)) + \dot\gamma(t)\, v^-_x(t,\gamma(t))\Big) =$$

$$= -\Big(v^+_t(t,\gamma(t)) + \dot\gamma(t)\, v^+_t(t,\gamma(t))\Big) =$$

$$= H\big(v^+_x(t,\gamma(t))\big) - \dot\gamma(t)\, v^+_x(t,\gamma(t))$$

which yields:

$$\dot\gamma(t)\Big(v^+_x(t,\gamma(t)) - v^-_x(t,\gamma(t))\Big) = H\big(v^+_x(t,\gamma(t))\big) - H\big(v^-_x(t,\gamma(t))\big). \tag{96}$$

Observing that

(97) $u^-(t) \doteq \lim_{x \to \gamma(t)-} u(t,x) = v_x^-(t,\gamma(t))$, $\quad u^+(t) \doteq \lim_{x \to \gamma(t)+} u(t,x) = v_x^+(t,\gamma(t))$

we thus recover from (96) the (RH) condition for u at any point on Γ, with left and right states $u^-(t), u^+(t)$.

of Theorem 6-(ii)

3. Assume that either of cases a), b) is verified. To fix the ideas assume that a) holds, i.e. that

(98) $$D^\perp v^-(t_0,x_0) \leq D^\perp v^+(t_0,x_0), \qquad t_0 \in]a,b[, \; x_0 = \gamma(t_0),$$

Observe that the unit vector $\underline{n}$ to Γ at $(t_0,\gamma(t_0))$ is given by:

(99) $$\underline{n}(t_0,\gamma(t_0)) = \frac{1}{\sqrt{1+(\dot\gamma(t_0))^2}}\left(-\dot\gamma(t_0), 1\right),$$

so that, relying again on the fact that v^-, v^+ are solutions of (8?) we find:

(100) $$\begin{cases} D^\perp v^-(t_0,\gamma(t_0)) = \left[-\dot\gamma(t_0)\, v_t^-(t_0,\gamma(t_0)) + v_x^-(t_0,\gamma(t_0))\right]\dfrac{1}{\sqrt{1+(\dot\gamma(t_0))^2}} \\[2ex] \qquad = \left[\dot\gamma(t_0)\, H\left(v_x^-(t_0,\gamma(t_0))\right) + v_x^-(t_0,\gamma(t_0))\right]\dfrac{1}{\sqrt{1+(\dot\gamma(t_0))^2}} \\[2ex] D^\perp v^+(t_0,\gamma(t_0)) = \left[-\dot\gamma(t_0)\, v_t^+(t_0,\gamma(t_0)) + v_x^+(t_0,\gamma(t_0))\right]\dfrac{1}{\sqrt{1+(\dot\gamma(t_0))^2}} \\[2ex] \qquad = \left[\dot\gamma(t_0)\, H\left(v_x^+(t_0,\gamma(t_0))\right) + v_x^+(t_0,\gamma(t_0))\right]\dfrac{1}{\sqrt{1+(\dot\gamma(t_0))^2}} \end{cases}$$

Therefore we deduce that (98) holds if and only if

$$\left[H\left(v_x^-(t_0,\gamma(t_0))\right)-H\left(v_x^+(t_0,\gamma(t_0))\right)\right]\dot\gamma(t_0)+\left(v_x^-(t_0,\gamma(t_0))-v_x^+(t_0,\gamma(t_0))\right)\leq 0$$

$$\Big\Updownarrow \text{ using (96)}$$

$$\left(v_x^-(t_0,\gamma(t_0))-v_x^+(t_0,\gamma(t_0))\right)\left(\left(\dot\gamma(t_0)\right)^2+1\right)\leq 0$$

$$\Big\Updownarrow$$

$$(101)\quad v_x^-(t_0,\gamma(t_0))\leq v_x^+(t_0,\gamma(t_0)) \overset{(97)}{\Longleftrightarrow} u^-(t_0)\leq u^+(t_0)$$

Next, observe that the tangential component of the derivative of v^- (or of v^+) along Γ takes the form (cfr. (90)):

$$D^T v^\pm(t_0,\gamma(t_0))=\left[v_t^\pm(t_0,\gamma(t_0))+\dot\gamma(t_0)\,v_x^\pm(t_0,\gamma(t_0))\right]\frac{1}{\sqrt{1+(\dot\gamma(t_0))^2}}=$$

$$=\left[-H\left(v_x^\pm(t_0,\gamma(t_0))\right)+\dot\gamma(t_0)\,v_x^\pm(t_0,\gamma(t_0))\right]\frac{1}{\sqrt{1+(\dot\gamma(t_0))^2}}.$$

Thus, since we are assuming to fall in case a) of Theorem 6-(ii), then (91) must be verified, which taking into account (95), and recalling the expression (99) of $\underline{n}$, can be rewritten as:

67/

$$(102)\quad\begin{cases} \dfrac{\left[-H(v_x^-)+\dot{\gamma}\, v_x^-\right]}{1+(\dot{\gamma})^2} - \dfrac{\lambda\,\dot{\gamma}}{\sqrt{1+(\dot{\gamma})^2}} + H\left(\dfrac{\dot{\gamma}}{1+(\dot{\gamma})^2}\left[-H(v_x^-)+\dot{\gamma}\, v_x^-\right] + \dfrac{\lambda}{\sqrt{1+(\dot{\gamma})^2}}\right) \geqslant \\ \forall \lambda \in \left[D^\perp v^-, D^\perp v^+\right] \end{cases}$$

(here all functions are evaluated at $(t_0, \gamma(t_0))$, or at t_0)

Writing $\lambda = \alpha D^\perp v^+ + (1-\alpha) D^\perp v^-$, $\alpha \in [0,1]$, and relying on (100) we deduce that (102) is equivalent to:

$$(103)\quad\begin{cases} \dfrac{1}{1+(\dot{\gamma})^2}\left[\alpha\dot{\gamma}(v_x^- - v_x^+) - \alpha(\dot{\gamma})^2 H(v_x^+) - H(v_x^-) - (1-\alpha)(\dot{\gamma})^2 H(v_x^-)\right] + \\ + H\left[\dfrac{1}{1+(\dot{\gamma})^2}\left[\alpha v_x^+ + (\dot{\gamma})^2 v_x^- + (1-\alpha) v_x^- + \alpha\dot{\gamma}\left(H(v_x^+) - H(v_x^-)\right)\right]\right] \geqslant 0 \\ \forall \alpha \in [0,1] \end{cases}$$

In turn, relying on (96), we find that (103) is equivalent to:

$$-\left[\alpha H(v_x^+) + (1-\alpha) H(v_x^-)\right] + H\left(\alpha v_x^+ + (1-\alpha) v_x^-\right) \geqslant 0$$

$$\forall \alpha \in [0,1]$$

which, recalling (97), can be rewritten as:

$$H\left(\alpha u^{+}(t_0)+(1-\alpha)u^{-}(t_0)\right) \geqslant \alpha H\left(u^{+}(t_0)\right)+(1-\alpha) H\left(u^{-}(t_0)\right) \tag{104}$$

$\forall\, \alpha \in [0,1]$. (because of (101).)

Notice that (104) is precisely the condition (80) saying that the graph of H remains above the secant joining $\left(u^{-}(t_0), H(u^{-}(t_0))\right)$, $\left(u^{+}(t_0), H(u^{+}(t_0))\right)$, on the interval $[u^{-}(t_0), u^{+}(t_0)]$. Since condition (80) is equivalent to the stability condition (85), the proof is completed.

$\blacksquare$