Introduzione alle equazioni alle derivate parziali,
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Removable singularities of harmonic functions.

Definition. *Let Ω be a open set in $\mathbb{R}^n$ and let $x_0 \in \Omega$. Let $u : \Omega \setminus \{x_0\} \to \mathbb{R}$ be a harmonic function (in $\Omega \setminus \{x_0\}$). Then x_0 is a removable singularity of u if there exists a harmonic function $\tilde{u}$ in Ω such that $u = \tilde{u}$ in $\Omega \setminus \{x_0\}$.*

Theorem 1. *Let u be a harmonic function in $\Omega \setminus \{x_0\}$ and assume that $u = o(\Gamma(x - x_0)$ as $x \to x_0$ where Γ is the fundamental solution of the Laplacian (with singularity at 0) that is*

$$\lim_{x\to x_0} u(x)|x - x_0|^{n-2} = 0 \quad \text{if } n \geq 3 \qquad \lim_{x\to x_0} \frac{u(x)}{\log|x - x_0|} = 0 \quad \text{if } n = 2.$$

Then x_0 is a removable singularity for u.

Remark. The condition is sharp, in fact x_0 is not removable for $\Gamma(x - x_0)$.
Note that if u is bounded, then necessarily the condition in the theorem is satisfied.

Remark. *This theorem is the analogous of Riemann theorem on removable singularities for holomorfic functions.*

Proof. Let $r > 0$ such that $B(x_0, r) \subset \Omega$. Let $\tilde{u}$ be the solution of the Dirichlet problem

$$\begin{cases} -\Delta\tilde{u} = 0 & |x - x_0| < r \\ \tilde{u}(x) = u(x) & |x - x_0| = r. \end{cases}$$

Then $\tilde{u}$ is bounded in $B(x_0, r)$ (since by Maximum principle $|\tilde{u}| \leq \max_{|x-x_0|=r} u(x)$) and harmonic in $B(x_0, r)$. To conclude the proof it is enough to show that $\tilde{u} = u$ in $B(x_0, r)$.

Let $w = u - \tilde{u}$. Then w is harmonic in $B(x_0, r) \setminus \{x_0\}$ and moreover (check it!)

$$\lim_{x\to x_0} \frac{w(x)}{r^{2-n} - |x - x_0|^{2-n}} = 0 \quad \text{if } n \geq 3 \qquad \lim_{x\to x_0} \frac{w(x)}{\log r - \log|x - x_0|} = 0 \quad \text{if } n = 2.$$

So for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for all x such that $|x - x_0| \leq \delta$,

$$|w(x)| \leq \varepsilon(r^{2-n} - |x - x_0|^{2-n}) \quad \text{if } n \geq 3 \qquad |w(x)| \leq \varepsilon(\log r - \log|x - x_0|) \quad \text{if } n = 2.$$

Observe that $\varepsilon(r^{2-n} - |x - x_0|^{2-n})$ and $\varepsilon(\log r - \log|x - x_0|)$ are harmonic functions (resp. when $n \geq 3$ and $n = 2$) and are 0 on the set $|x - x_0| = r$. So, weak Maximum principle (applied in the set $\delta \leq |x - x_0| \leq r$) gives that

$$|w(x)| \leq \varepsilon(r^{2-n} - |x - x_0|^{2-n}) \quad \text{if } n \geq 3 \qquad |w(x)| \leq \varepsilon(\log r - \log|x - x_0|) \quad \text{if } n = 2$$

for every x such that $|x - x_0| \leq r$.

We conclude by the arbitrariness of ε that $w = 0$. □