Part 2 - Modeling traffic flow

- engineering models
- microscopic models
- kinetic models
- macroscopic models

D. Helbing, A. Hennecke, and V. Shvetsov, Micro- and macro-simulation of freeway traffic. *Math. Computer Modelling* **35** (2002).

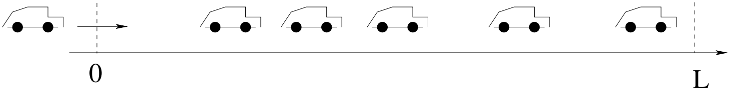
N. Bellomo, M. Delitala, V. Coscia, On the mathematical theory of vehicular traffic flow I. Fluid dynamic and kinetic modelling. *Math. Models Appl. Sci.* **12** (2002).

M. Garavello and B. Piccoli, *Traffic Flow on Networks. Conservation Laws Models.* AIMS Series on Applied Mathematics, Springfield, Mo., 2006.

A delay model (T. Friesz et al., 1993)

$X(t) \;=\;$ number of cars on a road at time t

If a new car enters at time t, it will exit at time $t + D(X(t))$



$D(X) =$ delay $=$ total time needed to travel along the road

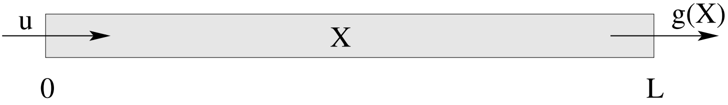
depends only on the total number of cars at the time of entrance

An ODE model (D. Merchant and G. Nemhauser, 1978)

$X(t)$ = total number of cars on a road at time t

$u(t)$ = incoming flux $\qquad$ g(X(t)) = outgoing flux

$$\dot{X}(t) = u(t) - g(X(t)) \qquad \text{conservation equation}$$



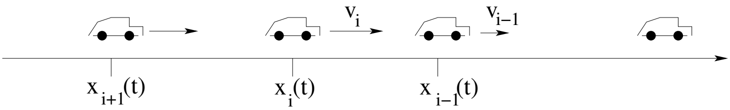
$$L = \text{ length of road,} \qquad \rho \approx \frac{X}{L} = \text{ density of cars}$$

$$g(X) = \rho\, v(\rho) = \frac{X}{L} \cdot v\left(\frac{X}{L}\right)$$

Models favored by engineers:

- simple to use, do not require knowledge of PDEs (or even ODEs)
- easy to compute, also on a large network of roads
- become accurate when the road is partitioned into short subintervals

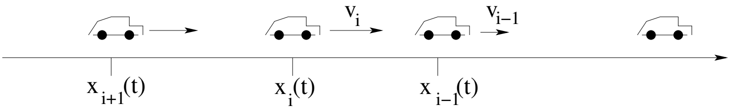
Microscopic models



$$\begin{aligned} x_i(t) &= \text{position of the } i\text{-th car} \\ v_i(t) &= \text{velocity of the } i\text{-th car} \end{aligned} \qquad i = 1, \ldots, N$$

Goal: describe the position and velocity of each car, writing a large system of ODEs

Car following models



Acceleration of i-th car depends on:

- its speed: v_i
- speed of car in front: v_{i-1}
- distance from car in front: $x_{i-1} - x_i$

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = a(v_i,\ v_{i-1},\ x_{i-1} - x_i) \end{cases} \qquad i = 1, \ldots, N$$

Microscopic intelligent driver model (Helbing & al., 2002)

i-th driver $\begin{cases} \text{accelerates, up to the maximum speed } \bar{v} \\ \text{decelerates, to keep a safe distance from the car in front} \end{cases}$

$\bar{v}$ = maximum speed allowed on the road $\qquad v_i \in [0, \bar{v}]$

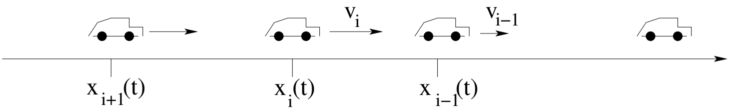
a = maximum acceleration

$$\dot{v}_i = a \cdot \left[1 - \left(\frac{v_i}{\bar{v}}\right)^{\delta}\right] - a \cdot \left(\frac{s^*(v_i,\ \Delta v_i)}{s_i}\right)^2$$

$$s_i = x_{i-1} - x_i = \text{actual gap from vehicle in front}$$

$$s_i^* = \text{desired gap}$$

Desired gap from the vehicle in front



$$s_i^* \;=\; \sigma_0 + \sigma_1\sqrt{\frac{v_i}{\bar v}} + Tv_i + \frac{v_i\,\Delta v_i}{2\sqrt{a\,b}} \;=\; \text{desired gap}$$

$\Delta v_i \;=\; v_i - v_{i-1} \;=\;$ speed difference with car in front

$\sigma_0 =$ jam distance (bumper to bumper)

$\sigma_1 =$ velocity adjustment of jam distance

$T =$ safe time headway

$b =$ comfortable deceleration

Equilibrium traffic

Assume: all cars have the same speed, constant in time.
Choose $\sigma_0 = \sigma_1 = 0$, $\delta = 1$

$$\dot{v}_i \;=\; a \cdot \left[1 - \frac{v_i}{\bar{v}}\right] - a \cdot \left(\frac{s^*(v_i,\; \Delta v_i)}{s_i}\right)^2 \;=\; 0$$

Equilibrium gap from vehicle in front

$$s_e(v) \;=\; s^*(v, 0) \cdot \left[1 - \frac{v_i}{\bar{v}}\right]^{-1/2}$$

Equilibrium velocity: $$v_e(s) \;=\; \frac{s^2}{2\bar{v}T^2} \left(-1 + \sqrt{\frac{4T^2\bar{v}^2}{s^2}}\right)$$

$$\Longrightarrow \qquad v_e \;=\; V_e(\rho) \qquad\qquad \rho \approx s^{-1} \;=\; \text{macroscopic density}$$

Statistical (kinetic) description

$f = f(t, x, V)$ statistical distribution of position and velocity of vehicles

$f(t, x, V)\, dxdV =$ number of vehicles which at time t are in the phase domain $[x,\ x + dx] \times [V,\ V + dV]$

local density: $$\rho(t, x) = \int_0^\infty f(t, x, V)\, dV$$

average velocity: $$v(t, x) = \frac{1}{\rho(t, x)} \int_0^\infty V \cdot f(t, x, V)\, dV$$

Evolution of the distribution function

$$\frac{\partial f}{\partial t} + V\,\frac{\partial f}{\partial x} + a(t,x)\frac{\partial f}{\partial V} ~=~ Q[f,\rho]$$

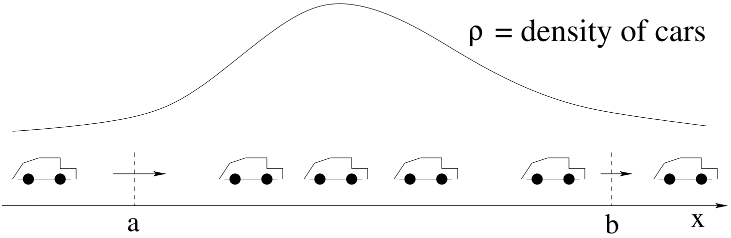
$a(t,x) =$ acceleration (may depend on the entire distribution f)

$Q(f,\rho)$ models a trend to equilibrium (as for BGK model in kinetic theory)

$$Q ~=~ c_r(\rho)\cdot\Big(f_e(V,\rho) - f(t,x,V)\Big)$$

$c_r =$ relaxation rate

A conservation law model (M. Lighthill and G. Witham, 1955)



$t =$ time, $x =$ space variable along road, $\rho = \rho(t, x) =$ density of cars

flux: $=$ [number of cars crossing the point x per unit time]

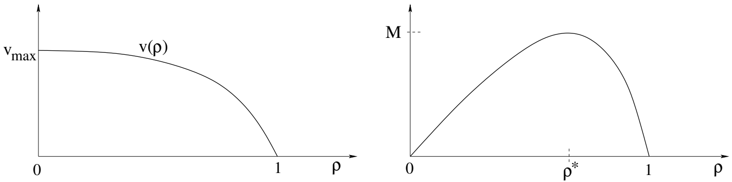
$=$ [density] $\times$ [velocity] $= \rho \cdot v$ $\quad v = V(\rho)$

$$\rho_t + \big[\rho\, V(\rho)\big]_x = 0$$

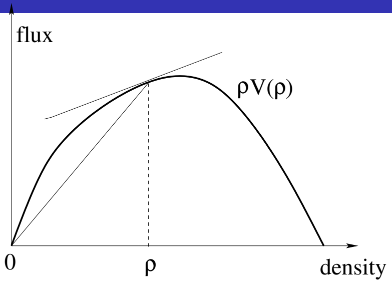
Flux function

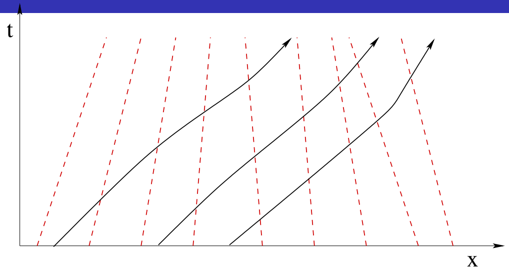
Assume: $\rho \mapsto \rho V(\rho)$ is concave

$$V'(\rho) \; < \; 0\,, \qquad\qquad 2V'(\rho) + \rho V''(\rho) \; < \; 0$$



Characteristics vs. car trajectories

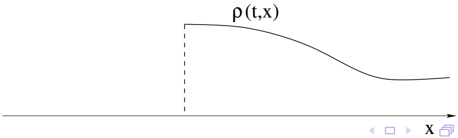




$$[\rho V(\rho)]' \quad = \quad V(\rho) + \rho V'(\rho) \quad < \quad V(\rho)$$

characteristic speed $\quad < \quad$ speed of cars

Weak solutions can have upward shocks



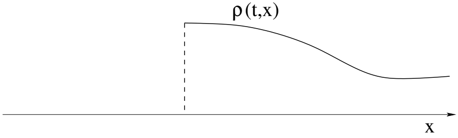
Adding a viscosity ?

$$\rho_t + \big[\rho\, V(\rho)\big]_x \;=\; 0 \qquad\qquad \big(\; = \; \varepsilon \rho_{xx} \;\big)$$

$$\rho_t + \left[\rho \left(V(\rho) - \varepsilon \frac{\rho_x}{\rho} \right)\right]_x \;=\; 0$$

effective velocity of cars: $v \;=\; V(\rho) - \varepsilon \dfrac{\rho_x}{\rho}$

can be negative, at the beginning of a queue



Second order models

$v = V(\rho) \quad \Longrightarrow$ velocity is instantly adjusted to the density

Models with acceleration

$$\begin{cases} \rho_t + (\rho v)_x = 0 \\ v_t + v\, v_x = a(\rho, v, \rho_x) \end{cases}$$

a = acceleration

$$\begin{cases} \rho_t + (\rho v)_x = 0 \\ v_t + v\, v_x = \frac{1}{\tau}(V(\rho) - v) - \frac{p'(\rho)}{\rho}\, \rho_x \end{cases}$$

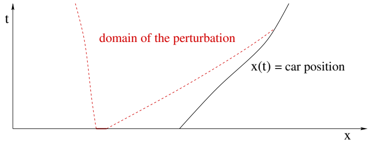
(Payne - Witham, 1971)

[relaxation] + [pressure term] $\qquad p = \rho^\gamma, \quad \gamma > 0$

$$\begin{cases} \rho_t + (\rho v)_x = 0 \\ v_t + v\, v_x + \frac{p'(\rho)}{\rho}\, \rho_x = \frac{1}{\tau}(V(\rho) - v) \end{cases}$$

$$\begin{pmatrix} \rho_t \\ v_t \end{pmatrix} + \begin{pmatrix} v & \rho \\ p'(\rho)/\rho & v \end{pmatrix} \begin{pmatrix} \rho_x \\ v_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

eingenvalues = characteristic speeds: $v \pm \sqrt{p'(\rho)}$



Wrong predictions: • negative speeds

• perturbations travel faster than the speed of cars

Idea: replace the partial derivative of the pressure $\partial_x p$ with the *convective derivative* $(\partial_t + v\partial_x)p$

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(v + p(\rho)) + v\partial_x(v + p(\rho)) = 0 \end{cases} \qquad \text{(Aw - Rascle)}$$

$$\begin{pmatrix} \rho_t \\ v_t \end{pmatrix} + \begin{pmatrix} v & \rho \\ 0 & v - \rho p'(\rho) \end{pmatrix} \begin{pmatrix} \rho_x \\ v_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

strictly hyperbolic for $\rho > 0$, positive speed: $v + p(\rho) \geq 0$

eigenvalues: $\lambda_1 = v - \rho p'(\rho), \quad \lambda_2 = v$

Properties of the Aw-Rascle model

- system is strictly hyperbolic (away from vacuum)
- the density ρ and the velocity v remain bounded and non-negative
- characteristic speeds (= eigenvalues) are smaller than car speed
$\Longrightarrow$ drivers are not influenced by what happens behind them.
- maximum speed of cars on an empty road depends on initial data

An improved model (R. M. Colombo, 2002)

Aw - Rascle: $$\begin{cases} \partial_t \rho + \partial_x (v\rho) & = & 0 \\ \partial_t q + \partial_x (vq) & = & 0 \end{cases}$$

$$q \;=\; v\rho + \rho p(\rho) \;=\; \text{“momentum”}$$

Colombo: $$\begin{cases} \partial_t \rho + \partial_x (v\rho) & = & 0 \\ \partial_t q + \partial_x (v(q - q_{max})) & = & 0 \end{cases}$$

$$v \;=\; \left(\frac{1}{\rho} - \frac{1}{\rho_{max}}\right) q$$

ρ_{max} = maximum density $\qquad$ q_{max} = “maximum momentum”

$\Longrightarrow$ velocity can vanish only when $\rho = \rho_{max}$,
and remains uniformly bounded

Concluding remarks

Number of vehicles on a road $<<$ number of molecules in a gas

- microscopic models (solving an ODE for each car) are within computational reach
- kinetic models and macroscopic models are realistic on longer stretches of road, for densities away from vacuum
- optimization problems, dependence of solution on parameters, are better understood by studying macroscopic models
- Simple ODE models, delay models are popular among engineers.
  Scalar conservation laws are OK.
  Kinetic models, second order models, are a hard sell.