# Conservation Law Models for Traffic Flow

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- Review of hyperbolic conservation laws
- Models of traffic flow, on a single road and on a network of roads
- Optimization problems
- Nash equilibria

## A PDE model for traffic flow



t = time, x = space variable along road,  $\rho = \rho(t, x) = density of cars$ 

#### Total number of cars is conserved:

 $\frac{d}{dt} \int_{a}^{b} \rho(t, x) dx = [\text{flux of cars entering at } a] - [\text{flux of cars exiting at } b]$ 

**flux:** = [number of cars crossing the point *x* per unit time]

$$=$$
 [density]  $imes$  [velocity]

Assume: velocity of cars depends only on their density:  $v = v(\rho)$ 

$$\frac{d}{dt} \int_{a}^{b} \rho(t, x) dx = [\text{flux of cars entering at } a] - [\text{flux of cars exiting at } b]$$
$$= \rho(t, a) v(\rho(t, a)) - \rho(t, b) v(\rho(t, b))$$
$$\int_{a}^{b} \frac{\partial}{\partial t} \rho \, dx = -\int_{a}^{b} \frac{\partial}{\partial x} [\rho v(\rho)] \, dx$$
$$\frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x} [\rho v(\rho)] = 0$$

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$$u_t + f(u)_x = 0 \qquad \qquad u = \text{conserved quantity}, \qquad f(u) = \text{flux}$$

$$\frac{d}{dt} \int_a^b u(t,x) \, dx = \int_a^b u_t(t,x) \, dx = -\int_a^b f(u(t,x))_x \, dx$$
$$= f(u(t,a)) - f(u(t,b)) = [\text{inflow at } a] - [\text{outflow at } b]$$



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conservation equation:  $u_t + f(u)_x = 0$ 

quasilinear form:  $u_t + f'(u)u_x = 0$ 

Conservation equation remains meaningful for u = u(t, x) discontinuous, in distributional sense:

$$\iint \left\{ u\phi_t + f(u)\phi_x \right\} \, dxdt = 0 \qquad \text{for all} \quad \phi \in \mathcal{C}^1_c$$

Need only : u, f(u) locally integrable

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$$u_t + f(u)_x = 0$$

Assume:  $u_n$  is a solution, for every  $n \ge 1$ ,

$$u_n \to u, \qquad f(u_n) \to f(u) \qquad \text{in} \quad \mathbf{L}^1_{loc}$$

then

$$\iint \left\{ u\phi_t + f(u)\phi_x \right\} \, dxdt = \lim_{n \to \infty} \iint \left\{ u_n\phi_t + f(u_n)\phi_x \right\} \, dxdt = 0$$
  
for all  $\phi \in \mathcal{C}^1_c$ 

(no need to check convergence of derivatives)

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Scalar Conservation Laws

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## Scalar Equation with Linear Flux

$$u_t + f(u)_x = 0$$
  $f(u) = \lambda u$ 

$$u_t + \lambda u_x = 0$$
  $u(0, x) = \phi(x)$ 

Explicit solution:  $u(t,x) = \phi(x - \lambda t)$ 



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#### The method of characteristics

$$u_t + f'(u)u_x = 0$$
  $u(0, x) = \phi(x)$ 

For each  $x_0$ , consider the straight line

$$t \mapsto x(t,x_0) = x_0 + tf'(\phi(x_0))$$

Set  $u = \phi(x_0)$  along this line, so that  $\dot{x}(t) = f'(u(t, x(t)))$ . As long as characteristics do not cross, this yields a solution:



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$$u_t + f'(u)u_x = 0$$

Assume: characteristic speed f'(u) is not constant



Global solutions only in a space of discontinuous functions

$$u(t,\cdot)\in BV$$



$$u(t,x) = \begin{cases} u^{-} & \text{if } x < \lambda t \\ u^{+} & \text{if } x > \lambda t \end{cases}$$

is a weak solution if and only if

 $\lambda \cdot [u^+ - u^-] = f(u^+) - f(u^-)$  Rankine - Hugoniot equations

[speed of the shock]  $\times$  [jump in the state] = [jump in the flux]

### Derivation of the Rankine - Hugoniot equation

$$\iint \left\{ u\phi_t + f(u)\phi_x \right\} dxdt = 0 \quad \text{for all} \quad \phi \in \mathcal{C}_c^1$$

$$\mathbf{v} \doteq \left( u\phi, f(u)\phi \right)$$

$$\mathbf{0} = \iint_{\Omega^+ \cup \Omega^-} \operatorname{div} \mathbf{v} dxdt = \int_{\partial \Omega^+} \mathbf{n}^+ \cdot \mathbf{v} ds + \int_{\partial \Omega^-} \mathbf{n}^- \cdot \mathbf{v} ds$$

$$= \int \left[ \lambda u^+ - f(u^+) \right] \phi(t, \lambda t) dt + \int \left[ -\lambda u^- + f(u^-) \right] \phi(t, \lambda t) dt$$

$$= \int \left[ \lambda (u^+ - u^-) - \left( f(u^+) - f(u^-) \right) \right] \phi(t, \lambda t) dt$$

$$\lambda (u^{+} - u^{-}) = f(u^{+}) - f(u^{-}) = \int_{0}^{1} f' (\theta u^{+} + (1 - \theta) u^{-}) \cdot (u^{+} - u^{-}) d\theta$$

The Rankine-Hugoniot conditions hold if and only if the speed of the shock is

$$\lambda = rac{f(u^+) - f(u^-)}{u^+ - u^-} = \int_0^1 f' ig( heta u^+ + (1 - heta) u^- ig) \, d heta$$

= [average characteristic speed]

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[speed of the shock] = [slope of secant line through  $u^-$ ,  $u^+$  on the graph of f]

= [average of the characteristic speeds between  $u^-$  and  $u^+$ ]

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## Points of approximate jump

The function u = u(t, x) has an **approximate jump** at a point  $(\tau, \xi)$  if there exists states  $u^- \neq u^+$  and a speed  $\lambda$  such that, calling

$$U(t,x) \doteq \begin{cases} u^{-} & \text{if } x < \lambda t, \\ u^{+} & \text{if } x > \lambda t, \end{cases}$$

there holds



**Theorem.** If u is a weak solution to a conservation law then the Rankine-Hugoniot equations hold at each point of approximate jump.

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### Weak solutions can be non-unique

Example: a Cauchy problem for Burgers' equation

$$u_t + (u^2/2)_x = 0$$
  $u(0,x) = \begin{cases} 1 & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}$ 

Each  $\alpha \in [0,1]$  yields a weak solution

$$u_{\alpha}(t,x) = \begin{cases} 0 & \text{if} \quad x < \alpha t/2 \\ \alpha & \text{if} \quad \alpha t/2 \le x < (1+\alpha)t/2 \\ 1 & \text{if} \quad x \ge (1+\alpha)t/2 \end{cases}$$



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## Stability conditions for shocks

Perturb the shock with left and right states  $u^-$ ,  $u^+$  by inserting an intermediate state  $u^* \in [u^-, u^+]$ 

Initial shock is stable  $\iff$ 

 $[{\rm speed \ of \ jump \ behind}] \ \geq \ [{\rm speed \ of \ jump \ ahead}]$ 



speed of a shock = slope of a secant line to the graph of f



Stability conditions:

- when  $u^- < u^+$  the graph of f should remain above the secant line
- when  $u^- > u^+$ , the graph of f should remain below the secant line

## The Lax admissibility condition



A shock connecting the states  $u^-, u^+$ , travelling with speed  $\lambda = \frac{f(u^+) - f(u^-)}{u^+ - u^-}$  is admissible if

$$f'(u^-) \geq \lambda \geq f'(u^+)$$

i.e. characteristics do not move out from the shock from either side

Cauchy problem:  $u_t + f(u)_x = 0$ ,  $u(0,x) = \bar{u}(x)$ Polygonal approximations of the flux function (Dafermos, 1972) Choose a piecewise affine function  $f_n$  such that

$$f_n(u) = f(u)$$
  $u = j \cdot 2^{-n}$ ,  $j \in \mathbb{Z}$ 

Approximate the initial data with a function  $\bar{u}_n : \mathbb{R} \mapsto 2^{-n} \cdot \mathbb{Z}$ 



## Front tracking approximations

piecewise constant approximate solutions:  $u_n(t, x)$ 



 $Tot.Var.(u_n(t, \cdot)) \leq Tot.Var.(\bar{u}_n) \leq Tot.Var.(\bar{u})$ 

⇒ as  $n \to \infty$ , a subsequence converges in  $L^1_{loc}([0, T] \times \mathbb{R})$ to a weak solution u = u(t, x)

$$u_t + f(u)_x = 0$$

Two initial data in  $L^1(\mathbb{R})$ :  $u_1(0,x) = \overline{u}_1(x), \quad u_2(0,x) = \overline{u}_2(x)$ 

 ${\bf L}^1$  - distance between solutions does not increase in time:

$$\|u_1(t,\cdot) - u_2(t,\cdot)\|_{\mathsf{L}^1(\mathbb{R})} \leq \|\bar{u}_1 - \bar{u}_2\|_{\mathsf{L}^1(\mathbb{R})}$$

(not true for the  $L^p$  distance, p > 1)

The  $\mathbf{L}^1$  distance between continuous solutions remains constant





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The  ${\rm L}^1$  distance decreases when a shock in one solution crosses the graph of the other solution



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## A related Hamilton-Jacobi equation

$$u_t + f(u)_x = 0$$
  $u(0,x) = \bar{u}(x)$ 

$$U(t,x) = \int_{-\infty}^{x} u(t,y) \, dy$$

$$U_t + f(U_x) = 0 \qquad \qquad U(0,x) = \overline{U}(x) = \int_{-\infty}^x \overline{u}(y) \, dy$$

f convex ===

U = U(t, x) is the value function for an optimization problem

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$$u \mapsto f(u) \in \mathbb{R} \cup \{+\infty\}$$
 convex

$$f^*(p) \doteq \max_u \{pu - f(u)\}$$



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# A representation formula

$$U_t + f(U_x) = 0$$
  $U(0,x) = \overline{U}(x)$ 

$$U(t,x) = \inf_{z(\cdot)} \left\{ \int_0^t f^*(\dot{z}(s)) \, ds + \overline{U}(z(0)); \quad z(t) = x \right\}$$
$$= \min_{y \in \mathbb{R}} \left\{ t \, f^*\left(\frac{x-y}{t}\right) + \overline{U}(y) \right\}$$

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# A geometric construction

$$U_{t} + f(U_{x}) = 0 \qquad U(0, x) = \overline{U}(x)$$
  
define  $h(s) \doteq -T f^{*} \left(\frac{-s}{T}\right)$   
 $u(T, x) = \inf_{y} \left\{ \overline{U}(y) - h(y - x) \right\}$ 

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Cauchy problem: 
$$\begin{cases} u_t + f(u)_x = 0, \\ u(0, x) = \overline{u}(x) \end{cases}$$

For each t > 0, and all but at most countably many values of  $x \in \mathbb{R}$ , there exists a unique y(t, x) s.t.

$$y(t, x) = \arg\min_{y \in \mathbb{R}} \left\{ t f^*\left(\frac{x-y}{t}\right) + \int_{-\infty}^{y} \overline{u}(s) ds \right\}$$

the solution to the Cauchy problem is

$$u(t, x) = (f')^{-1} \left( \frac{x - y(t, x)}{t} \right)$$
(1)



$$y(t, x) = \arg\min_{y \in \mathbb{R}} \left\{ t f^*\left(\frac{x-y}{t}\right) + \int_{-\infty}^{y} \overline{u}(s) ds \right\}$$

define the characteristic speed  $\xi \doteq \frac{x - y(t, x)}{t}$ 

if  $f'(\omega) = \xi$  then  $u(t, x) = \omega$ 

## Initial-Boundary value problem



P. Le Floch, Explicit formula for scalar non-linear conservation laws with boundary condition, *Math. Models Appl. Sci.* (1988)

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#### Systems of Conservation Laws

$$\begin{cases} \frac{\partial}{\partial t}u_1 + \frac{\partial}{\partial x}f_1(u_1, \dots, u_n) = 0, \\ & \ddots & \ddots \\ \\ \frac{\partial}{\partial t}u_n + \frac{\partial}{\partial x}f_n(u_1, \dots, u_n) = 0 \end{cases}$$

$$u_t + f(u)_x = 0$$

 $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$  conserved quantities

 $f = (f_1, \ldots, f_n) : \mathbb{R}^n \mapsto \mathbb{R}^n$  fluxes

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## Hyperbolic Systems

$$u_t + f(u)_x = 0$$
  $u = u(t, x) \in \mathbb{R}^n$ 

$$u_t + A(u)u_x = 0 \qquad A(u) = Df(u)$$

The system is **strictly hyperbolic** if each  $n \times n$  matrix A(u) has real distinct eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u)$$

right eigenvectors  $r_1(u), \ldots, r_n(u)$  (column vectors) left eigenvectors  $l_1(u), \ldots, l_n(u)$  (row vectors)

$$Ar_{i} = \lambda_{i}r_{i} \qquad I_{i}A = \lambda_{i}I_{i}$$
  
Choose bases so that  $I_{i} \cdot r_{j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ 

## A linear hyperbolic system

$$u_t + Au_x = 0$$
  $u(0, x) = \phi(x)$   
 $\lambda_1 < \cdots < \lambda_n$  eigenvalues  $r_1, \ldots, r_n$  eigenvectors

Explicit solution: linear superposition of travelling waves



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$$u_t + A(u)u_x = 0$$

#### eigenvalues depend on $u \implies$ waves change shape



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#### eigenvectors depend on $u \implies$ nontrivial wave interactions


## Global solutions to the Cauchy problem

$$u_t + f(u)_x = 0$$
  $u(0,x) = \bar{u}(x)$ 

- Construct a sequence of approximate solutions  $u_m$
- Show that (a subsequence) converges:  $u_m 
  ightarrow u$  in  $\mathsf{L}^1_{loc}$

 $\implies$  *u* is a weak solution



Need: a-priori bound on the total variation (J. Glimm, 1965)

### Building block: the Riemann Problem

$$u_t + f(u)_x = 0$$
  $u(0, x) = \begin{cases} u^- & \text{if } x < 0 \\ u^+ & \text{if } x > 0 \end{cases}$ 

**B. Riemann 1860:** 2 × 2 system of isentropic gas dynamics

**P. Lax 1957:**  $n \times n$  systems (+ special assumptions)

**T. P. Liu 1975**  $n \times n$  systems (generic case)

**S. Bianchini 2003** (vanishing viscosity limit for general hyperbolic systems, possibly non-conservative)

invariant w.r.t. symmetry: 
$$u^{ heta}(t,x) \doteq u( heta t, heta x) \qquad heta > 0$$

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### Riemann Problem for Linear Systems

$$u_{t} + Au_{x} = 0 \qquad u(0, x) = \begin{cases} u^{-} & \text{if } x < 0\\ u^{+} & \text{if } x > 0 \end{cases}$$

$$u_{t}^{1/1 = \lambda_{2}}$$

$$u^{-} - u^{-} = \sum_{j=1}^{n} c_{j}r_{j} \qquad (\text{sum of eigenvectors of } A)$$

$$u^{+} - u^{-} = \sum_{j=1}^{n} c_{j}r_{j} \qquad (\text{sum of eigenvectors of } A)$$

$$intermediate \text{ states }: \quad \omega_{i} \doteq u^{-} + \sum_{j \leq i} c_{j}r_{j}$$

$$i\text{-th jump:} \quad \omega_{i} - \omega_{i-1} = c_{i}r_{i} \quad \text{travels with speed } \lambda_{i}$$

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General solution of the Riemann problem: concatenation of elementary waves



**Glimm scheme:** piecing together solutions of Riemann problems on a fixed grid in the t-x plane



**Front tracking scheme:** piecing together piecewise constant solutions of Riemann problems at points where fronts interact



$$u_t + f(u)_x = 0, \qquad u(0, x) = \overline{u}(x)$$

#### Theorem (Glimm 1965).

Assume:

• system is strictly hyperbolic (+ some technical assumptions)

Then there exists  $\delta > 0$  such that, for every initial condition  $\bar{u} \in L^1(\mathbb{R}; \mathbb{R}^n)$  with

Tot. Var.  $(\bar{u}) \leq \delta$ ,

the Cauchy problem has an entropy admissible weak solution u = u(t, x) defined for all  $t \ge 0$ .

## Uniqueness and continuous dependence on the initial data

$$u_t + f(u)_x = 0$$
  $u(0, x) = \bar{u}(x)$ 

**Theorem (A.B.-** R.Colombo, B.Piccoli, T.P.Liu, T.Yang, 1994-1998). For every initial data  $\bar{u}$  with small total variation, the front tracking approximations converge to a unique limit solution  $u : [0, \infty[ \mapsto L^1(\mathbb{R}).$ 

The flow map  $(\bar{u}, t) \mapsto u(t, \cdot) \doteq S_t \bar{u}$  is a uniformly Lipschitz semigroup:

 $S_0 \bar{u} = \bar{u}, \qquad S_s(S_t \bar{u}) = S_{s+t} \bar{u}$ 

$$\left\|S_t\bar{u} - S_s\bar{v}\right\|_{\mathsf{L}^1} \leq L\cdot \left(\|\bar{u} - \bar{v}\|_{\mathsf{L}^1} + |t - s|\right) \qquad \text{ for all } \bar{u}, \bar{v}, \ s, t \geq 0$$

#### Theorem (A.B.- P. LeFloch, M.Lewicka, P.Goatin, 1996-1998).

Any entropy weak solution to the Cauchy problem coincides with the limit of front tracking approximations, hence it is unique

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# Vanishing viscosity approximations

Claim: weak solutions of the hyperbolic system

$$u_t + f(u)_x = 0$$

can be obtained as limits of solutions to the parabolic system

$$u_t^{\varepsilon} + f(u^{\varepsilon})_x = \varepsilon u_{xx}^{\varepsilon}$$

letting the viscosity  $\varepsilon \rightarrow 0+$ 



#### Theorem (S. Bianchini, A. Bressan, Annals of Math. 2005)

Consider a strictly hyperbolic system with viscosity

$$u_t + A(u)u_x = \varepsilon u_{xx} \qquad u(0, x) = \overline{u}(x). \qquad (CP)$$

If Tot.Var. $\{\bar{u}\}$  is sufficiently small, then (CP) admits a unique solution  $u^{\varepsilon}(t, \cdot) = S_t^{\varepsilon} \bar{u}$ , defined for all  $t \ge 0$ . Moreover

$$\mathsf{Tot.Var.}\left\{S_t^{\varepsilon}\bar{u}\right\} \leq C \mathsf{Tot.Var.}\left\{\bar{u}\right\}, \qquad \qquad (\mathsf{BV \ bounds})$$

$$\left\|S_t^{\varepsilon}\bar{u} - S_t^{\varepsilon}\bar{v}\right\|_{\mathsf{L}^1} \leq L \left\|\bar{u} - \bar{v}\right\|_{\mathsf{L}^1} \qquad (\mathsf{L}^1 \text{ stability})$$

**(Convergence)** If A(u) = Df(u), then as  $\varepsilon \to 0$ , the viscous solutions  $u^{\varepsilon}$  converge to the unique entropy weak solution of the system of conservation laws

$$u_t + f(u)_x = 0$$

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- Global existence of solutions to hyperbolic systems for initial data  $\bar{u}$  with **large total variation**
- Existence of entropy weak solutions for systems in **several space dimensions**



## Part 2 - Modeling traffic flow

- engineering models
- microscopic models
- kinetic models
- macroscopic models

D. Helbing, A. Hennecke, and V. Shvetsov, Micro- and macro-simulation of freeway traffic. *Math. Computer Modelling* **35** (2002).

N. Bellomo, M. Delitala, V. Coscia, On the mathematical theory of vehicular traffic flow I. Fluid dynamic and kinetic modelling. *Math. Models Appl. Sci.* **12** (2002).

M. Garavello and B. Piccoli, *Traffic Flow on Networks. Conservation Laws Models*. AIMS Series on Applied Mathematics, Springfield, Mo., 2006.

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Scalar Conservation Laws

X(t) = number of cars on a road at time t

If a new car enters at time t, it will exit at time t + D(X(t))



D(X) = delay = total time needed to travel along the road depends only on the total number of cars at the time of entrance

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# An ODE model (D. Merchant and G. Nemhauser, 1978)

 $X(t) = ext{total number of cars on a road at time } t$  $u(t) = ext{incoming flux}$   $g(X(t)) = ext{outgoing flux}$ 

 $\dot{X}(t) = u(t) - g(X(t))$  conservation equation





Models favored by engineers:

- simple to use, do not require knowledge of PDEs (or even ODEs)
- easy to compute, also on a large network of roads
- become accurate when the road is partitioned into short subintervals

### Microscopic models



**Goal:** describe the position and velocity of each car, writing a large system of ODEs

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### Car following models



Acceleration of *i*-th car depends on:

- its speed: *v<sub>i</sub>*
- speed of car in front:  $v_{i-1}$
- distance from car in front:  $x_{i-1} x_i$

$$\begin{cases} \dot{x}_{i} = v_{i} \\ \dot{v}_{i} = a(v_{i}, v_{i-1}, x_{i-1} - x_{i}) \end{cases} \quad i = 1, \dots, N$$

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i-th driver  $\left\{ \begin{array}{c} \quad \text{accelerates, up to the maximum speed } \bar{\nu} \\ \quad \text{decelerates, to keep a safe distance from the car in front} \end{array} \right.$ 

- $ar{v} = ext{maximum speed allowed on the road}$   $v_i \in [0, ar{v}]$
- $a = \max \max$  acceleration

$$\dot{v}_i = a \cdot \left[1 - \left(\frac{v_i}{\bar{v}}\right)^{\delta}\right] - a \cdot \left(\frac{s^*(v_i, \Delta v_i)}{s_i}\right)^2$$

 $s_i = x_{i-1} - x_i =$  actual gap from vehicle in front

$$s_i^* =$$
 desired gap

## Desired gap from the vehicle in front



$$s_i^* = \sigma_0 + \sigma_1 \sqrt{\frac{v_i}{\bar{v}} + Tv_i + \frac{v_i \Delta v_i}{2\sqrt{ab}}} = \text{desired gap}$$

 $\Delta v_i = v_i - v_{i-1} =$  speed difference with car in front

 $\sigma_0 = jam distance (bumper to bumper)$ 

 $\sigma_1 =$  velocity adjustment of jam distance

T = safe time headway

b = comfortable deceleration

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## Equilibrium traffic

Assume: all cars have the same speed, constant in time. Choose  $\sigma_0=\sigma_1=0,\;\delta=1$ 

$$\dot{v}_i = a \cdot \left[1 - rac{v_i}{ar{v}}\right] - a \cdot \left(rac{s^*(v_i, \Delta v_i)}{s_i}
ight)^2 = 0$$

Equilibrium gap from vehicle in front

$$s_e(v) = s^*(v,0) \cdot \left[1 - \frac{v_i}{\overline{v}}\right]^{-1/2}$$

Equilibrium velocity: 
$$v_e(s) = \frac{s^2}{2\bar{v}T^2} \left(-1 + \sqrt{\frac{4T^2\bar{v}^2}{s^2}}\right)$$

 $\implies$   $v_e = V_e(
ho)$   $ho pprox s^{-1} =$  macroscopic density

f = f(t, x, V) statistical distribution of position and velocity of vehicles

$$f(t, x, V) dx dV =$$
 number of vehicles which at time t  
are in the phase domain  $[x, x + dx] \times [V, V + dV]$ 

local density: 
$$\rho(t,x) = \int_0^\infty f(t,x,V) \, dV$$

average velocity: 
$$v(t,x) = \frac{1}{\rho(t,x)} \int_0^\infty V \cdot f(t,x,V) \, dV$$

$$\frac{\partial f}{\partial t} + V \frac{\partial f}{\partial x} + a(t, x) \frac{\partial f}{\partial V} = Q[f, \rho]$$

a(t,x) = acceleration (may depend on the entire distribution f)  $Q(f,\rho)$  models a trend to equilibrium (as for BGK model in kinetic theory)

$$Q = c_r(\rho) \cdot \left( f_e(V,\rho) - f(t,x,V) \right)$$

 $c_r$  = relaxation rate

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#### A conservation law model



t= time, x= space variable along road, ho=
ho(t,x)= density of cars

**flux:** = [number of cars crossing the point *x* per unit time]

$$= [\text{density}] \times [\text{velocity}] = \rho \cdot v \qquad v = V(\rho)$$

$$\rho_t + \left[\rho V(\rho)\right]_x = 0$$

Assume:  $\rho \mapsto \rho V(\rho)$  is concave

$$V'(\rho) < 0, \qquad 2V'(\rho) + \rho V''(\rho) < 0$$



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#### Characteristics vs. car trajectories



$$\begin{split} & [\rho V(\rho)]' \ = \ V(\rho) + \rho V'(\rho) \ < \ V(\rho) \\ & \text{characteristic speed} \ < \ & \text{speed of cars} \end{split}$$



### Adding a viscosity ?

$$\rho_t + [\rho V(\rho)]_x = 0$$
( =  $\varepsilon \rho_{xx}$ )

$$\rho_t + \left[\rho\left(V(\rho) - \varepsilon \frac{\rho_x}{\rho}\right)\right]_x = 0$$

effective velocity of cars: 
$$v = V(\rho) - \varepsilon \frac{\rho_x}{\rho}$$

can be negative, at the beginning of a queue



 $v = V(
ho) \implies$  velocity is instantly adjusted to the density

#### Models with acceleration

$$\begin{cases} \rho_t + (\rho v)_x = 0 \\ v_t + v v_x = a(\rho, v, \rho_x) \end{cases} a = \text{acceleration}$$

$$\begin{cases} \rho_t + (\rho v)_x = 0 \\ v_t + v v_x = \frac{1}{\tau} (V(\rho) - v) - \frac{p'(\rho)}{\rho} \rho_x \\ \text{[relaxation]} + \text{[pressure term]} \qquad p = \rho^{\gamma}, \quad \gamma > 0 \end{cases}$$

C. Daganzo, Requiem for second-order fluid approximation to traffic flow, 1995

$$\begin{cases} \rho_t + (\rho v)_x = 0\\ v_t + v v_x + \frac{p'(\rho)}{\rho} \rho_x = \frac{1}{\tau} (V(\rho) - v)\\ \begin{pmatrix} \rho_t\\ v_t \end{pmatrix} + \begin{pmatrix} v & \rho\\ p'(\rho)/\rho & v \end{pmatrix} \begin{pmatrix} \rho_x\\ v_x \end{pmatrix} = \begin{pmatrix} 0\\ 0 \end{pmatrix} \end{cases}$$

eingenvalues = characteristic speeds:  $v \pm \sqrt{p'(\rho)}$ 



Wrong predictions: • negative speeds

• perturbations travel faster than the speed of cars

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**Idea:** replace the partial derivative of the pressure  $\partial_x p$  with the *convective derivative*  $(\partial_t + v \partial_x)p$ 

$$\begin{cases} \partial_t \rho + \partial_x (\rho v) &= 0 \\ \partial_t (v + \rho(\rho)) + v \partial_x (v + \rho(\rho)) &= 0 \end{cases}$$

$$\begin{pmatrix} \rho_t \\ v_t \end{pmatrix} + \begin{pmatrix} v & \rho \\ 0 & v - \rho p'(\rho) \end{pmatrix} \begin{pmatrix} \rho_x \\ v_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

strictly hyperbolic for  $\rho > 0$ , positive speed:  $\nu + p(\rho) \ge 0$ 

eigenvalues:  $\lambda_1 = v - \rho p'(\rho), \quad \lambda_2 = v$ 

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- system is strictly hyperbolic (away from vacuum)
- the density  $\rho$  and the velocity v remain bounded and non-negative
- characteristic speeds (= eigenvalues) are smaller than car speed
   drivers are not influenced by what happens behind them.
- maximum speed of cars on an empty road depends on initial data

#### An improved model (R. M. Colombo, 2002)

Aw - Rascle:  

$$\begin{cases}
\partial_t \rho + \partial_x (v\rho) &= 0 \\
\partial_t q + \partial_x (vq) &= 0
\end{cases}$$

$$q = v\rho + \rho p(\rho) = \text{``momentum''}$$

Colombo: 
$$\begin{cases} \partial_t \rho + \partial_x (v\rho) = 0\\ \partial_t q + \partial_x (v(q - q_{max})) = 0 \end{cases}$$
$$v = \left(\frac{1}{\rho} - \frac{1}{\rho_{max}}\right) q$$

 $\rho_{max}$  = maximum density  $q_{max}$  = "maximum momentum"

 $\implies$  velocity can vanish only when  $\rho=\rho_{\max},$  and remains uniformly bounded

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Number of vehicles on a road << number of molecules in a gas

- microscopic models (solving an ODE for each car) are within computational reach
- kinetic models and macroscopic models are realistic on longer stretches of road, for densities away from vacuum
- optimization problems, dependence of solution on parameters, are better understood by studying macroscopic models
- Simple ODE models, delay models are popular among engineers. Scalar conservation laws are OK. Kinetic models, second order models, are a hard sell.

## Part 3 - Optimization problems for traffic flow

- Car drivers starting from a location A (a residential neighborhood) need to reach a destination B (a working place) at a given time T.
- There is a cost φ(τ<sub>d</sub>) for departing early and a cost ψ(τ<sub>a</sub>) for arriving late.



#### Elementary solution

L = length of the road, v = speed of cars

$$\tau_a = \tau_d + \frac{L}{v}$$

Optimal departure time:

$$au_d^{ ext{opt}} = \operatorname{argmin}_t \left\{ \varphi(t) + \psi\left(t + \frac{L}{v}\right) \right\}.$$

If everyone departs exactly at the same optimal time, a traffic jam is created and this strategy is not optimal anymore.

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#### An optimization problem for traffic flow

Problem: choose the departure rate  $\bar{u}(t)$  in order to minimize the total cost to all drivers.

$$u(t,x) \doteq \rho(t,x) \cdot v(\rho(t,x)) =$$
flux of cars

minimize: 
$$\int \varphi(t) \cdot u(t,0) dt + \int \psi(t)u(t,L) dt$$

for a solution of

$$\begin{cases} \rho_t + [\rho v(\rho)]_x = 0 \qquad x \in [0, L] \\ \rho(t, 0)v(\rho(t, 0)) = \overline{u}(t) \end{cases}$$

Choose the optimal departure rate  $\bar{u}(t)$ , subject to the constraint

$$\int ar{u}(t) \, dt \; = \; \kappa \; = \; [ ext{total number of drivers}]$$

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#### Boundary value problem for the density $\rho$ :

conservation law:  $\rho_t + [\rho v(\rho)]_x = 0,$   $(t, x) \in \mathbb{R} \times [0, L]$ 

control (on the boundary data):  $\rho(t, 0)v(\rho(t, 0)) = \bar{u}(t)$ 

#### Cauchy problem for the flux *u*:

conservation law:  $u_x + f(u)_t = 0$ ,  $u = \rho v(\rho)$ ,  $f(u) = \rho$ control (on the initial data):  $u(t, 0) = \bar{u}(t)$ 

Cost: 
$$J(u) = \int_{-\infty}^{+\infty} \varphi(t)u(t, 0) dt + \int_{-\infty}^{+\infty} \psi(t)u(t, L) dt$$
  
Constraint:  $\int_{-\infty}^{+\infty} \overline{u}(t) dt = \kappa$ 

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### The flux function and its Legendre transform



$$u = \rho v(\rho), \qquad \rho = f(u)$$

Legendre transform:  $f^*(p) \doteq \max_{u} \left\{ pu - f(u) \right\}$ 

Solution to the conservation law is provided by the Lax formula

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Scalar Conservation Laws

# The globally optimal (Pareto) solution

minimize: 
$$J(u) = \int \varphi(x) \cdot u(0, x) \, dx + \int \psi(x) \, u(T, x) \, dx$$
  
subject to: 
$$\begin{cases} u_t + f(u)_x &= 0 \\ u(0, x) &= \bar{u}(x), & \int \bar{u}(x) \, dx = \kappa \end{cases}$$

(A1) The flux function  $f : [0, M] \mapsto \mathbb{R}$  is continuous, increasing, and strictly convex. It is twice continuously differentiable on the open interval ]0, M[ and satisfies

$$f(0) = 0$$
,  $\lim_{u \to M^-} f'(u) = +\infty$ ,  $f''(u) \ge b > 0$  for  $0 < u < M$ 

(A2) The cost functions  $arphi,\psi$  satisfy  $\ arphi'\ <\ 0$ ,  $\ \psi,\psi'\ \geq\ 0$ ,

$$\lim_{x \to -\infty} \varphi(x) = +\infty, \qquad \lim_{x \to +\infty} \left( \varphi(x) + \psi(x) \right) = +\infty$$

**Theorem (A.B. and K. Han, 2011).** Let **(A1)-(A2)** hold. Then, for any given T,  $\kappa$ , there exists a unique admissible initial data  $\bar{u}$  minimizing the cost  $J(\cdot)$ . In addition,

**(**) No shocks are present, hence u = u(t, x) is continuous for t > 0. Moreover

 $\sup_{t\in[0,T],\,x\in\mathbb{R}}u(t,\,x)\ <\ M$ 

**3** For some constant  $c = c(\kappa)$ , this optimal solution admits the following characterization: For every  $x \in \mathbb{R}$ , let  $y_c(x)$  be the unique point such that

$$\varphi(y_c(x)) + \psi(x) = c$$

Then, the solution u = u(t, x) is constant along the segment with endpoints  $(0, y_c(x)), (T, x)$ . Indeed, either  $f'(u) \equiv \frac{x - y_c(x)}{T}$ , or  $u \equiv 0$ 

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### Necessary conditions



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# An Example

Cost functions:  $\varphi(t) = -t$ ,  $\psi(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ t^2, & \text{if } t > 0 \end{cases}$ L = 1,  $u = \rho(2 - \rho)$ , M = 1,  $\kappa = 3.80758$ **Bang-bang solution** Pareto optimal solution X L=1 L=1  $0 | \tau_1$  $\tau_0$ 0 τı t. t  $\tau_0 = -2.78836, \quad \tau_1 = 1.01924$  $\tau_0 = -2.8023, \quad \tau_1 = 1.5976$ total cost = 5.86767total cost = 5.5714

#### Does everyone pay the same cost?



Departure time vs. cost in the Pareto optimal solution

A solution u = u(t, x) is a Nash equilibrium if no driver can reduce his/her own cost by choosing a different departure time. This implies that all drivers pay the same cost.

To find a Nash equilibrium, write the conservation law  $u_t + f(u)_x = 0$ in terms of a Hamilton-Jacobi equation

$$U_t + f(U_x) = 0$$
  $U(0,x) = Q(x)$ 

$$U(t, x) \doteq \int_{-\infty}^{x} u(t, y) \, dy$$

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# A representation formula

$$U_t + f(U_x) = 0$$
  $U(0,x) = Q(x)$ 

$$U(T, x) = \inf_{z(\cdot)} \left\{ \int_{0}^{T} f^{*}(\dot{z}(s)) ds + Q(z(0)); \quad z(T) = x \right\}$$
$$= \min_{y \in \mathbb{R}} \left\{ T f^{*}\left(\frac{x - y}{T}\right) + Q(y) \right\}$$
$$\int_{0}^{t} \int_{0}^{t} \int_{0}^$$

Scalar Conservation Laws

No constraint can be imposed on the departing rate, so a queue can form at the entrance of the highway.

 $x \mapsto Q(x) =$  number of drivers who have started their journey before time x (joining the queue, if there is any).

$$Q(-\infty) = 0,$$
  $Q(+\infty) = \kappa$ 

 $x \mapsto U(T,x) =$  number of drivers who have reached destination within time x

$$U(T, x) = \min_{y \in \mathbb{R}} \left\{ T f^* \left( \frac{x - y}{T} \right) + Q(y) \right\}$$

# Characterization of a Nash equilibrium



 $\beta \in [0, \kappa]$  = Lagrangian variable labeling one particular driver  $x^{q}(\beta)$  = time when driver  $\beta$  departs (possibly joining the queue)

 $x^{a}(\beta) =$ time when driver  $\beta$  arrives at destination

# Existence and Uniqueness of Nash equilibrium

Departure and arrival times are implicitly defined by

$$Q(x^q(\beta)-) \leq \beta \leq Q(x^q(\beta)+), \qquad U(T,x^a(\beta)) = \beta$$

Nash equilibrium  $\implies \varphi(x^q(\beta)) + \psi(x^a(\beta)) \equiv c$ 

Theorem (A.B. - K. Han, SIAM J. Math. Anal. 2012).

Let the flux f and cost functions  $\varphi, \psi$  satisfy the assumptions (A1)-(A2). Then, for every  $\kappa > 0$ , the Hamilton-Jacobi equation

$$U_t + f(U_x) = 0$$

admits a unique Nash equilibrium solution with total mass  $\kappa$ 

### Sketch of the proof

**1.** For a given cost c, let  $Q_c^-$  be the set of all initial data  $Q(\cdot)$  for which every driver has a cost  $\leq c$ :

$$\varphi(\tau^q(\beta)) + \psi(\tau^a(\beta)) \leq c$$
 for a.e.  $\beta \in [0, Q(+\infty)]$ .

**2.** Claim: 
$$Q^*(t) \doteq \sup \left\{ Q(t); \quad Q \in \mathcal{Q}_c^- \right\}$$
 is the initial data for a Nash equilibrium with common cost  $c$ .



**3.** For each *c*, the Nash equilibrium solution where each driver has a cost = *c* is unique. Define  $\kappa(c) \doteq$  total number of drivers in this solution.

**4.** There exists a minimum cost  $c_0$  such that  $\kappa(c) = 0$  for  $c \le c_0$ .

The map  $c \mapsto \kappa(c)$  is strictly increasing and continuous from  $[c_0, +\infty[$  to  $[0, +\infty[$ .







$\tau_0 = -2.7$	$\tau_2 = -0.9074$
$\tau_3=0.9698$	$ au_{4} = 1.52303$
$\tau_1=1.56525$	$t_S = 2.0550$
$\delta_0=1.79259$	

total cost = 10.286

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$$Q(t) = 1.7 + \sqrt{t + 2.7} + 1/(4(\sqrt{t + 2.7} + 2.7))$$
$$Q'(t) = \left(1 - 1/(4(\sqrt{t + 2.7} + 2.7)^2)\right)/(2\sqrt{t + 2.7})$$

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# Globally optimal solution vs. Nash equilibrium



Globally optimal solution:

starting cost + arrival cost = constant for all characteristics

#### Nash equilibrium solution:

starting cost + arrival cost = constant for all car trajectories

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Total cost of the Pareto optimal solution:  $J^{opt} = 5.5714$ 

Total cost of the Nash equilibrium solution:  $J^{Nash} = 10.286$ 

Price of anarchy:  $J^{Nash} - J^{opt} \approx 4.715$ 

Can one eliminate this inefficiency, yet allowing freedom of choice to each driver ?

(goal of non-cooperative game theory: devise incentives)

Scientific American, Dec. 2010: Ten World Changing Ideas

"Building more roads won't eliminate traffic. Smart pricing will."

Suppose a fee b(t) is collected at a toll booth at the entrance of the highway, depending on the departure time.

New departure cost:  $\tilde{\varphi}(t) = \varphi(t) + b(t)$ 

#### **Problem:** We wish to collect a total revenue *R*.

How do we choose  $t \mapsto b(t) \ge 0$  so that the Nash solution with departure and arrival costs  $\tilde{\varphi}, \psi$  yields the minimum total cost to each driver?



p(t) = cost to a driver starting at time t, in the globally optimal solution Optimal pricing:  $b(t) = p_{max} - p(t) + C$ 

choosing the constant C so that [total revenue] = R.



### Continuous dependence of the Nash solution

 $\varphi_1(x), \ \varphi_2(x)$  costs for departing at time x

 $\psi_1(x), \ \psi_2(x)$  costs for arriving at time x

 $v_1(
ho), \ v_2(
ho)$  speeds of cars, when the density is  $ho \geq 0$ 

 $Q_1(x)$ ,  $Q_2(x)$  = number of cars that have departed up to time x, in the corresponding Nash equilibrium solutions (with zero total cost to all drivers)

#### Theorem (A.B., C.J.Liu, and F.Yu, *Quarterly Appl. Math.* 2012)

Assume all cars depart and arrive within the interval [a, b], and the maximum density is  $\leq \rho^*$ . Then

$$\begin{aligned} \|Q_{1}(x) - Q_{2}(x)\|_{\mathsf{L}^{1}([a,b])} \\ &\leq C \cdot \left( \|\varphi_{1} - \varphi_{2}\|_{\mathsf{L}^{\infty}([a,b])} + \|\psi_{1} - \psi_{2}\|_{\mathsf{L}^{\infty}([a,b])} + \|v_{1} - v_{2}\|_{\mathsf{L}^{\infty}([0,\rho^{*}])}^{1/2} \right) \end{aligned}$$

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- Fix:  $\kappa = \text{total number of drivers}$
- For any departure distribution

 $t \mapsto Q(t) =$  number of drivers who have departed within time t (possibly joining the queue at the entrance of the highway)

Define:  $\Phi(Q) \doteq \max \max \operatorname{cost}$ , among all drivers

#### Theorem (A.B., C.J.Liu, and F.Yu, Quarterly Appl. Math. 2012)

The starting distribution  $Q^*(\cdot)$  for the Nash equilibrium solution yields a global minimum of  $\Phi$ .

# Traffic Flow on a Network

Nodes:  $A_1, \ldots, A_m$  arcs:  $\gamma_{ij}$ 

 $L_{ij} = ext{length}$  of the arc  $\gamma_{ij}$ 



A viable path  $\Gamma$  is a concatenation of viable arcs

# Network loading problem

Given the departure times of N drivers, and the paths  $\Gamma_1, \ldots, \Gamma_N$  along which they travel, describe the overall traffic pattern.



**Delay Model:** If a drivers enters the arc  $\gamma_{ij}$  at time t, he will exit form that arc at time  $t + D_{ij}(n)$ 

n = number of cars present along the arc  $\gamma_{ij}$  at time t

### Conservation law model



Along the arc  $\gamma_{ij}$ , the density of cars satisfies the conservation law

$$\rho_t + [\rho v_{ij}(\rho)]_x = 0$$

 $v_{ij}(\rho) =$  velocity of cars, depending on the density

### Boundary conditions at nodes



Need: junction conditions

given the flux from incoming arcs, determine the flux along outgoing arcs

Simplest model: a queue is formed at the entrance of each outgoing arc if the flux is too large



An upper bound on the flow is imposed (by a crosslight) at the end of each incoming arc.

A queue is formed, if the flux is too large (with possible spill-over)



Cars from the incoming road having priority pass instantly through the intersection  $% \left( {{{\left[ {{{\left[ {{{c}} \right]}} \right]}_{i}}}_{i}}} \right)$ 

Cars from the access ramp wait in a queue



# Traffic Flow on a Network

n groups of drivers with different origins and destinations, and different costs



### Traffic Flow on a Network



drivers can use different paths  $\Gamma_1, \Gamma_2, \ldots$  to reach destination

Does there exist a globally optimal solution, and a Nash equilibrium solution for traffic flow on a network ?

 $G_k$  = total number of drivers in the k-th group, k = 1, ..., n

 $\Gamma_p$  = viable path (concatenation of viable arcs  $\gamma_{ij}$ ),  $p = 1, \ldots, N$ 

 $t \mapsto \bar{u}_{k,p}(t) =$  departure rate of k-drivers traveling along the path  $\Gamma_p$ The set of departure rates  $\{\bar{u}_{k,p}\}$  is **admissible** if

$$\overline{u}_{k,p}(t) \geq 0$$
,  $\sum_{p} \int_{-\infty}^{\infty} \overline{u}_{k,p}(t) dt = G_k$   $k = 1, \ldots, n$ 

Let  $\tau_p(t)$  = arrival time for a driver starting at time t, traveling along  $\Gamma_p$ 

(A1) Along each arc  $\gamma_{ij}$  the flux function  $\rho \mapsto \rho v_{ij}(\rho)$  is twice continuously differentiable and concave down.

$$v_{ij}(0) > 0,$$
  $v_{ij}(\rho_{max}) = 0$ 

(A2) The cost functions  $\varphi,\psi$  satisfy  $\varphi'$  < 0,  $\psi,\psi'$   $\geq$  0,

$$\lim_{x \to -\infty} \varphi(x) = +\infty, \qquad \lim_{x \to +\infty} \left( \varphi(x) + \psi(x) \right) = +\infty$$

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# Global optima and Nash equilibria on networks

An admissible family  $\{\bar{u}_{k,p}\}$  of departure rates is globally optimal if it minimizes the sum of the total costs of all drivers

$$J(\bar{u}) \doteq \sum_{k,p} \int \left(\varphi_k(t) + \psi_k(\tau_p(t))\right) \bar{u}_{k,p}(t) dt$$

An admissible family  $\{\bar{u}_{k,p}\}\$  of departure rates is a **Nash equilibrium** solution if no driver of any group can lower his own total cost by changing departure time or switching to a different path to reach destination.

#### **Theorem.** (A.B. - Ke Han, Networks & Heterogeneous Media, 2012).

On a general network of roads, there exists at least one globally optimal solution, and at least one Nash equilibrium solution.

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Scalar Conservation Laws



Let  $B \subset \mathbb{R}^n$  be a closed ball.

Every continuous map  $f : B \mapsto B$  admits a fixed point.



### A variational inequality

 $K \subset \mathbb{R}^n$  closed, bounded convex set,  $f: K \mapsto \mathbb{R}^n$  continuous Then there exists  $x^* \in K$  such that

 $\langle x - x^*, f(x^*) \rangle \leq 0$  for all  $x \in K$ 

Either  $f(x^*) = 0$ , or  $f(x^*)$  is an outer normal vector to K at  $x^*$ 



If f(x) is tangent, or points inward at every boundary point of K, then  $f(x^*) = 0$ 

# A constrained evolution

Trajectories of  $\dot{x} = f(x)$  are constrained to remain in K by a frictionless barrier



There exists a point  $x^* \in K$  that does not move.

# Finite dimensional approximations

On a family  $\mathcal{K}$  of admissible piecewise constant departure rates  $u = (u_{k,p})$ , define an evolution equation

$$\frac{d}{d\theta}u = \Psi(u)$$



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## Existence of a Nash equilibrium on a network

The map  $\Psi : \mathcal{K} \mapsto \mathbb{R}^N$  is continuous and inward-pointing hence it admits a zero:  $\Psi(\bar{u}) = 0$ 

The departure rates  $\bar{u} = (\bar{u}_{k,p})$  represent a Galerkin approximation to a Nash equilibrium

Letting the discretization step  $\Delta t$  approach zero, taking subsequences:

departure rates:  $\bar{u}_{k,p}^{\nu}(\cdot) \rightarrow \bar{u}_{k,p}(\cdot)$  weakly arrival times:  $\tau_{p}^{\nu}(\cdot) \rightarrow \tau_{p}(\cdot)$  uniformly

The departure rates  $\bar{u}_{k,p}(\cdot)$  provide a Nash equilibrium

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- More general conditions at junctions (K. Han, B. Piccoli)
- Necessary conditions for globally optimal solutions on networks No queues ? No shocks ?



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To justify the practical relevance of a Nash equilibrium, we need to

- analyze a suitable dynamic model
- check whether the rate of departures asymptotically converges to the Nash equilibrium

Assume: drivers can change their departure time on a day-to-day basis, in order to decrease their own cost (one group of drivers, one single road)

Introduce an additional variable  $\theta$  counting the number of days on the calendar.

$$\bar{u}(t, heta) \doteq$$
 rate of departures at time  $t$ , on day  $heta$ 

 $\Phi(t,\theta) \doteq$  cost to a driver starting at time t, on day  $\theta$ 

**Model 1:** drivers gradually change their departure time, drifting toward times where the cost is smaller.

If the rate of change is proportional to the gradient of the cost, this leads to the conservation law

$$\bar{u}_{\theta} + [\Phi_t \, \bar{u}]_t = 0$$



**Model 2:** drivers jump to different departure times having a lower cost. If the rate of change is proportional to the difference between the costs, this yields



**Question:** as  $\theta \to \infty$ , does the departure rate  $\overline{u}(t, \theta)$  approach the unique Nash equilibrium?

Flux function:  $f(\rho) = \rho (2 - \rho)$ 

Departure and arrival costs:  $\varphi(t) = -t$ ,  $\psi(t) = e^t$ 

## Numerical simulation: Model 1



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## Numerical simulation: Model 2



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main difficulty: non-local dependence

linearized equation:

$$\frac{d}{d\theta}Y(x) = \left[\alpha(x)\left(\beta(x)Y(x) - Y(z(x))\right)\right]_{x}$$

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