6

The single conservation law

This chapter is concerned with the Cauchy problem for a scalar conservation law

$$u_t + f(u)_x = 0, \tag{6.1}$$

$$u(0, \cdot) = \bar{u}, \tag{6.2}$$

assuming that $f : \mathbb{R} \mapsto \mathbb{R}$ is locally Lipschitz continuous and that $\bar{u} \in \mathbf{L}^1_{loc}$. Recalling the analysis in the last section of Chapter 4, we define an *entropy solution* of (6.1)–(6.2) as a continuous map $u : [0, \infty) \mapsto \mathbf{L}^1_{loc}(\mathbb{R})$ which satisfies (6.2) together with

$$\iint \{|u - k|\phi_t + (f(u) - f(k))\operatorname{sgn}(u - k)\phi_x\}\, dx\, dt \geq 0, \tag{6.3}$$

for every $k \in \mathbb{R}$ and every non-negative function $\phi \in \mathcal{C}^1_c(\mathbb{R}^2)$, whose compact support is contained in the half plane where $t > 0$. In (6.3) we implicitly assume that both u and $f(u)$ are locally integrable on the half plane $[0, \infty[\, \times \mathbb{R}$. The inequality (6.3) means that

$$\eta(u)_t + q(u)_x \leq 0 \tag{6.4}$$

for every entropy of the form $\eta(u) = |u - k|$, with entropy flux $q(u) \doteq (f(u) - f(k))\operatorname{sgn}(u - k)$. If the function u is bounded, choosing $k < \inf u(t, x)$, it follows from (6.3) that

$$\iint \{u\phi_t + f(u)\phi_x\}\, dx\, dt \geq 0$$

for every $\phi \geq 0$ with support in the half plane where $t > 0$. Choosing $k > \sup u(t, x)$ we obtain the opposite inequality. Hence (6.3) implies that u is a distributional solution of (6.1).

The existence of solutions to (6.1)–(6.2) will be proved by the method of *wave-front tracking*. For given initial data $\bar{u} \in \mathbf{L}^1$, we will construct a sequence $(u_\nu)_{\nu \geq 1}$ of piecewise constant approximate solutions, with $u_\nu(0, \cdot) \to \bar{u}$. As $\nu \to \infty$, a compactness argument will yield a subsequence $(u_\mu)_{\mu \geq 1}$ converging in $\mathbf{L}^1_{loc}$ to an entropy solution.

The uniqueness and continuous dependence will then be proved by showing that, for any two bounded entropy solutions u, v of (6.1), one has

$$\int_{-\infty}^{\infty} |u(t, x) - v(t, x)|\, dx \leq \int_{-\infty}^{\infty} |u(0, x) - v(0, x)|\, dx$$

for every $t \geq 0$. In other words, the flow generated by a scalar conservation law is contractive w.r.t. the $\mathbf{L}^1$ distance. (Not true for systems, Temple, 198[illegible], No L^1-contractive metric for systems of conservation laws)

6.1 Piecewise constant approximations

Fix an integer $\nu \geq 1$ and let f_ν be the piecewise affine function which coincides with f at all nodes $2^{-\nu} j$ with j integer, i.e.

$$f_\nu(s) = \frac{s - 2^{-\nu} j}{2^{-\nu}} \cdot f(2^{-\nu}(j + 1)) + \frac{2^{-\nu}(j + 1) - s}{2^{-\nu}} \cdot f(2^{-\nu} j)$$
$$s \in [2^{-\nu} j, 2^{-\nu}(j + 1)]. \tag{6.5}$$

Let $\bar{u}$ be a piecewise constant function with compact support, taking values inside the discrete set $2^{-\nu}\mathbb{Z} \doteq \{2^{-\nu} j;\ j \text{ integer}\}$. We will show that the Cauchy problem

$$u_t + [f_\nu(u)]_x = 0 \tag{6.6}$$

with initial data $\bar{u}$ admits a piecewise constant entropy-admissible solution $u = u(t, x)$, still taking values within the discrete set $2^{-\nu}\mathbb{Z}$. As a preliminary, consider a Riemann problem for (6.6), with initial data

$$u(0, x) = \begin{cases} u^- & \text{if } x < 0, \\ u^+ & \text{if } x > 0, \end{cases} \qquad u^-, u^+ \in 2^{-\nu}\mathbb{Z}. \tag{6.7}$$

The functions f^*, f_* constructed below are illustrated in Fig. 6.1.

CASE 1: $u^- < u^+$. Let f_* be the largest convex function such that

$$f_*(s) \leq f_\nu(s) \quad \text{for all } s \in [u^-, u^+].$$

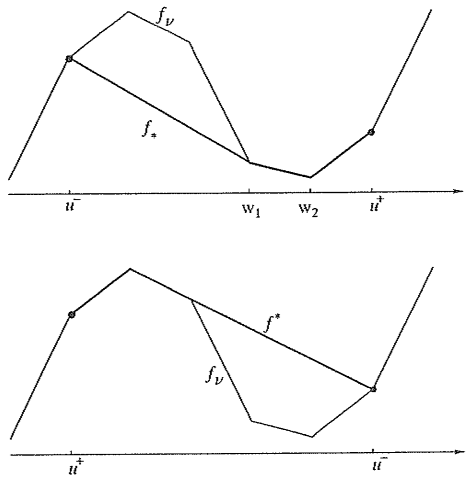

Figure 6.1

Observe that f_* is piecewise linear, being the convex hull of a piecewise linear function. By convexity, its derivative f_*' is a piecewise constant non-decreasing function, say with jumps at the points $w_0 \doteq u^- < w_1 < \cdots < w_q \doteq u^+$. We define the increasing sequence of shock speeds as

$$\lambda_\ell = \frac{f_\nu(w_\ell) - f_\nu(w_{\ell-1})}{w_\ell - w_{\ell-1}} \qquad \ell = 1, \ldots, q. \tag{6.8}$$

We claim that the function

$$\omega(t,x) \doteq \begin{cases} u^- & \text{if } x < t\lambda_1, \\ w_\ell & \text{if } t\lambda_\ell < x < t\lambda_{\ell+1} \ (1 \le \ell \le q-1), \\ u^+ & \text{if } t\lambda_q < x, \end{cases} \tag{6.9}$$

provides a weak, entropy-admissible solution of the Riemann problem (6.6)–(6.7). Indeed, fix any constant k and any $\mathcal{C}^1$ function $\phi \ge 0$ with compact support contained in the half plane where $t > 0$. We define the characteristic function

$$\chi_{[w_{\ell-1}, w_\ell]}(k) \doteq \begin{cases} 1 & \text{if } k \in [w_{\ell-1}, w_\ell], \\ 0 & \text{if } k \notin [w_{\ell-1}, w_\ell]. \end{cases}$$

From the above construction it follows that

$$\begin{aligned} &\iint \{|\omega - k|\phi_t + (f_\nu(\omega) - f_\nu(k))\operatorname{sgn}(\omega - k)\phi_x\}\, dx\, dt \\ &= \sum_{\ell=1}^{q} \int \{(|w_\ell - k| - |w_{\ell-1} - k|)\lambda_\ell - (f_\nu(w_\ell) - f_\nu(k))\operatorname{sgn}(w_\ell - k) \\ &\quad + (f_\nu(w_{\ell-1}) - f_\nu(k))\operatorname{sgn}(w_{\ell-1} - k)\}\phi(t, t\lambda_\ell)\, dt \\ &= \sum_{\ell=1}^{q} \int [(w_\ell + w_{\ell-1} - 2k)\lambda_\ell + 2 f_\nu(k) - f_\nu(w_\ell) - f_\nu(w_{\ell-1})] \\ &\quad \cdot \chi_{[w_{\ell-1}, w_\ell]}(k) \cdot \phi(t, t\lambda_\ell)\, dt \\ &\ge 0. \end{aligned}$$

Indeed, for $k \in [w_{\ell-1}, w_\ell]$ the definition of f_* implies

$$2 f_\nu(k) \ge 2 f_*(k) = [f_\nu(w_\ell) + (k - w_\ell)\lambda_\ell] + [f_\nu(w_{\ell-1}) + (k - w_{\ell-1})\lambda_\ell].$$

CASE 2: $u^- > u^+$. Let f^* be the smallest concave function such that

$$f^*(s) \ge f_\nu(s) \quad \text{for all } s \in [u^+, u^-].$$

The derivative of f^* is then a piecewise constant, non-increasing function, say with jumps at the points $w_0 \doteq u^+ < w_1 < \cdots < w_q \doteq u^-$. Letting the shock speeds λ_ℓ be as in (6.8), the function

$$\omega(t,x) \doteq \begin{cases} u^- & \text{if } x < t\lambda_q, \\ w_\ell & \text{if } t\lambda_{\ell+1} < x < t\lambda_\ell \ (1 \le \ell \le q-1), \\ u^+ & \text{if } t\lambda_1 < x \end{cases}$$

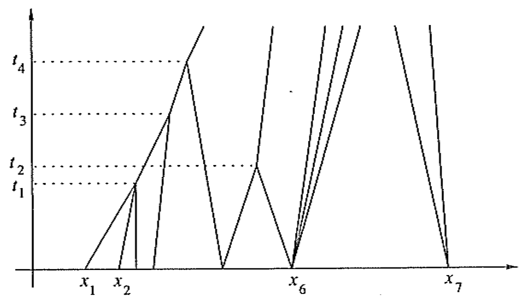

Figure 6.2

again provides an entropy solution to the Riemann problem (6.6)–(6.7). Observe that, in the above construction, all values w_ℓ lie within the set $2^{-\nu}\mathbb{Z}$.

Next, consider the more general Cauchy problem (6.6)–(6.2), still assuming that the initial condition $\bar{u}$ is piecewise constant, taking values within the set $2^{-\nu}\mathbb{Z}$. Let $x_1 < \cdots < x_N$ be the points where $\bar{u}$ has a jump. At each x_i, consider the left and right limits $\bar{u}(x_i-), \bar{u}(x_i+) \in 2^{-\nu}\mathbb{Z}$. Solving the corresponding Riemann problems, we thus obtain a local solution $u = u(t,x)$, defined for $t > 0$ sufficiently small. This solution can be prolonged up to a first time $t_1 > 0$ where two or more lines of discontinuity (emerging from different Riemann problems at $t = 0$) cross each other (Fig. 6.2). Since the values of $u(t,\cdot)$ always remain within the set $2^{-\nu}\mathbb{Z}$, we can again solve the new Riemann problems generated by the interactions, according to the above procedure. The solution is then prolonged up to a time $t_2 > t_1$ where a second set of wave-front interactions take place, and so on.

We claim that the total number of interactions is finite, and hence that the solution can be prolonged for all $t \ge 0$. Indeed, let

$$\xi_1(t) < \cdots < \xi_m(t) \qquad (t < \tau) \tag{6.10}$$

be the locations of m discontinuities, which interact all together at some time τ. For $t < \tau$, let $u_0, u_1, \ldots, u_m$ be the constant values taken by u and consider the jumps

$$u_i - u_{i-1} = u(t, \xi_i(t)+) - u(t, \xi_i(t)-) \qquad i = 1, \ldots, m. \tag{6.11}$$

Two cases can occur.

CASE 1: All jumps in (6.11) have the same sign. In this case (Fig. 6.3), we claim that the Riemann problem determined by the interaction is solved by a single jump, connecting u_0 with u_m. To fix the ideas, assume $u_0 < u_1 < \cdots < u_m$, the opposite case being entirely similar. By construction, all the incoming fronts are entropy admissible so that

$$\dot{\xi}_i = \frac{f_\nu(u_i) - f_\nu(u_{i-1})}{u_i - u_{i-1}},$$

$$f_\nu(s) \ge \frac{s - u_{i-1}}{u_i - u_{i-1}} \cdot f_\nu(u_i) + \frac{u_i - s}{u_i - u_{i-1}} \cdot f_\nu(u_{i-1}) \quad \text{for all } s \in [u_{i-1}, u_i].$$

Moreover, since all fronts ξ_i meet at the same point, (6.10) clearly implies $\dot\xi_1 > \cdots > \dot\xi_m$. From the above relations we deduce that

$$f_\nu(s) \geq \frac{s - u_0}{u_m - u_0} \cdot f_\nu(u_m) + \frac{u_m - s}{u_m - u_0} \cdot f_\nu(u_0) \quad \text{for all } s \in [u_0, u_m].$$

Therefore, the single jump (u_0, u_m), travelling with speed

$$\dot\xi = \frac{f_\nu(u_m) - f_\nu(u_0)}{u_m - u_0},$$

is entropy admissible, proving our claim.

We conclude that, in this case, the total variation of $u(t, \cdot)$ does not change as a consequence of the interaction. Moreover, the number of lines where u is discontinuous decreases at least by 1.

CASE 2: At least two of the jumps in (6.11) have opposite signs. In this case, the total number of wave-fronts may increase through the interaction. However, since the total strength of the outgoing fronts is given by $|u_m - u_0|$, the total variation of the solution must decrease by at least $2 \cdot 2^{-\nu}$, owing to a cancellation effect.

Figure 6.4 shows an example with two incoming fronts (i.e. $m = 2$) having opposite signs. The Riemann problem determined by the interaction is solved by three outgoing

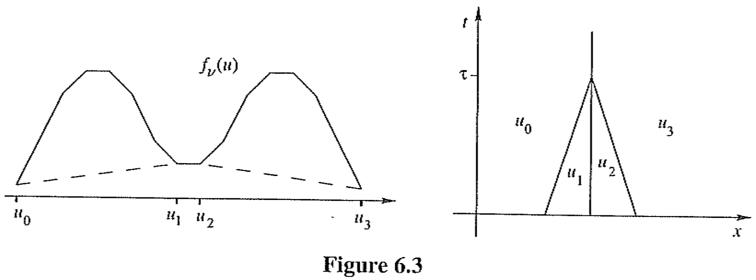

Figure 6.3

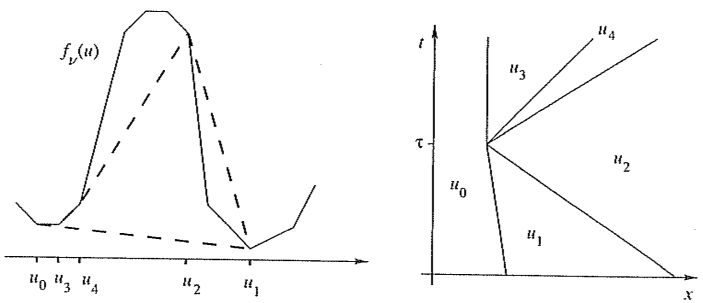

Figure 6.4

fronts, connecting the states $u_0 < u_3 < u_4 < u_2$. Observe that this configuration is possible owing to the particular shape of the function f, with two inflection points.

Since the total variation of $u(t, \cdot)$ is bounded when $t = 0$ and never increases, CASE 2 can occur only finitely many times, and hence also CASE 1. This proves that the total number of interactions is finite. The above method of wave-front tracking thus defines a piecewise constant solution to (6.6)–(6.2), with jumps occurring along a finite number of straight lines in the t–x plane.

6.2 Global existence of *BV* solutions

Relying on a compactness argument, we prove here an intermediate result concerning the global existence of entropy weak solutions, within a class of functions with bounded variation.

Theorem 6.1. *Let f be locally Lipschitz continuous and let $\bar u \in \mathbf{L}^1$ have bounded variation. Then the Cauchy problem (6.1)–(6.2) admits an entropy weak solution $u = u(t, x)$, defined for all $t \geq 0$, with*

$$\text{Tot. Var.}\{u(t, \cdot)\} \leq \text{Tot. Var.}\{\bar u\}, \quad \|u(t, \cdot)\|_{\mathbf{L}^\infty} \leq \|\bar u\|_{\mathbf{L}^\infty} \quad \textit{for all } t \geq 0. \tag{6.12}$$

Proof. Call $M \doteq \|\bar u\|_{\mathbf{L}^\infty}$. Recalling Lemma 2.2, we construct a sequence $(\bar u_\nu)_{\nu \geq 1}$ of piecewise constant functions such that

(i) $\bar u_\nu(x) \in 2^{-\nu}\mathbb{Z}$, for all x,
(ii) $\|\bar u_\nu - \bar u\|_{\mathbf{L}^1} \to 0$,
(iii) Tot. Var. $\{\bar u_\nu\} \leq$ Tot. Var. $\{\bar u\}$,
(iv) $\|\bar u_\nu\|_{\mathbf{L}^\infty} \leq M$.

For each ν, let $u_\nu = u_\nu(t, x)$ be the piecewise constant entropy solution of the conservation law (6.6) with initial data $u_\nu(0, \cdot) = \bar u_\nu$, constructed by the front tracking algorithm in the previous section. Observe that from (iii) and (iv) it follows that

$$\text{Tot. Var.}\{u_\nu(t, \cdot)\} \leq \text{Tot. Var.}\{\bar u\}, \qquad |u_\nu(t, x)| \leq M, \tag{6.13}$$

for all ν, t, x. Let L be a Lipschitz constant such that

$$|f(w) - f(w')| \leq L|w - w'| \quad \text{for all } w, w' \in [-M, M].$$

Clearly, L provides a Lipschitz constant also for all functions f_ν, on the interval $[-M, M]$. By (6.8), this implies that the speed of all discontinuities in $u_\nu(t, \cdot)$ is bounded by L. Using the bound (6.13) on the total variation, for every $t, t' \geq 0$ one obtains

$$\|u_\nu(t, \cdot) - u_\nu(t', \cdot)\|_{\mathbf{L}^1} \leq L|t - t'| \cdot \text{Tot. Var.}\{\bar u\}. \tag{6.14}$$

We can thus apply Theorem 2.4 and deduce the existence of a subsequence $(u_\mu)_{\mu \geq 1}$ which converges to some function u in $\mathbf{L}^1_{\text{loc}}([0, \infty[\, \times \mathbb{R})$. Clearly (6.13) implies (6.12). Observing that the convergence $f_\mu \to f$ is uniform on the interval $[-M, M]$

and recalling that each u_μ is an entropy solution of (6.6) (with ν replaced by μ), we obtain

$$\begin{aligned}&\iint \{|u-k|\phi_t + (f(u)-f(k))\operatorname{sgn}(u-k)\phi_x\}\,dx\,dt\\ &= \lim_{\mu\to\infty}\iint \{|u_\mu-k|\phi_t + (f_\mu(u_\mu)-f_\mu(k))\operatorname{sgn}(u_\mu-k)\phi_x\}\,dx\,dt\\ &\geq 0,\end{aligned}$$

for every C^1 function $\phi \geq 0$ with compact support contained in the half plane where $t > 0$. This proves that u is an entropy weak solution of (6.1). Finally, (6.14) and property (ii) of the approximating sequence imply that the initial condition (6.2) is attained. □

6.3 Uniqueness

The goal of this section is to prove the classical theorem of Kruzhkov, providing an estimate of the $\mathbf{L}^1$ distance between any two bounded entropy-admissible solutions of (6.1). In particular, we will show that the entropy solution of the Cauchy problem is unique, within a class of $\mathbf{L}^\infty$ functions.

Theorem 6.2 (Kruzhkov). *Let $f:\mathbb{R}\mapsto\mathbb{R}$ be locally Lipschitz continuous. Let u, v be entropy-admissible solutions of (6.1) defined for $t\geq 0$, and let M, L be constants such that*

$$|u(t,x)|\leq M,\qquad |v(t,x)|\leq M\qquad \text{for all } t,x, \tag{6.15}$$

$$|f(w)-f(w')|\leq L|w-w'|\quad \text{for all } w,w'\in[-M,M]. \tag{6.16}$$

Then, for every $R>0$ and $\tau\geq\tau_0\geq 0$, one has

$$\int_{|x|\leq R}|u(\tau,x)-v(\tau,x)|\,dx \leq \int_{|x|\leq R+L(\tau-\tau_0)}|u(\tau_0,x)-v(\tau_0,x)|\,dx. \tag{6.17}$$

Proof. To help the reader, we first give an intuitive sketch of the main arguments. Consider the trapezoid (Fig. 6.5)

$$\Omega \doteq \{(t,x);\ \tau_0\leq t\leq\tau,\ |x|\leq R+L(\tau-t)\}. \tag{6.18}$$

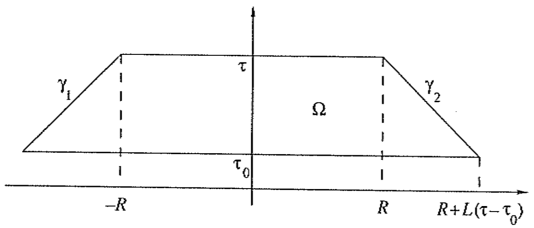

Figure 6.5

If u is an entropy solution, we can apply the divergence theorem to the vector field $\Phi \doteq (\eta(u), q(u))$ on the domain Ω. Using the inequality (6.4), we formally obtain

$$\begin{aligned}0 &\geq \iint_\Omega \{\eta(u)_t + q(u)_x\}\,dx\,dt\\ &= \int_{-R}^{R}\eta(u(\tau,x))\,dx - \int_{-R-L(\tau-\tau_0)}^{R+L(\tau-\tau_0)}\eta(u(\tau_0,x))\,dx\\ &\quad + \int_{\tau_0}^{\tau}\{L\,\eta(u(t,\gamma_1(t))) - q(u(t,\gamma_1(t)))\}\,dt + \int_{\tau_0}^{\tau}\{L\,\eta(u(t,\gamma_2(t))) + q(u(t,\gamma_2(t)))\}\,dt.\end{aligned} \tag{6.19}$$

As in Fig. 6.5, the lines

$$\gamma_1(t) \doteq -R-L(\tau-t),\qquad \gamma_2(t)\doteq R+L(\tau-t)$$

represent the two sides of Ω. Observe that (6.19) is valid for every entropy $\eta(u)=|u-k|$, with entropy flux $q(u) \doteq (f(u)-f(k))\operatorname{sgn}(u-k)$. By the assumption (6.16), f is Lipschitz continuous with constant L. As a consequence, the last two integrals on the right hand side of (6.19) are both ≥ 0. From (6.19) we thus obtain the inequality

$$\int_{|x|\leq R}|u(\tau,x)-k|\,dx \leq \int_{|x|\leq R+L(\tau-\tau_0)}|u(\tau_0,x)-k|\,dx, \tag{6.20}$$

valid for every $k\in\mathbb{R}$. Observe that (6.20) gives precisely the estimate (6.17) that we are looking for, in the special case where $v(t,x)\equiv k$ is a constant function.

Motivated by the previous analysis, the proof of the theorem will thus consist of two parts:

(i) Show that the inequality in (6.3) remains valid if the constant k is replaced by any entropy solution $v=v(t,x)$.

(ii) Make rigorous the formal derivation at (6.19).

To achieve (i), we first consider two entropy solutions $u=u(s,x)$ and $v=v(t,y)$ acting on distinct independent variables. The corresponding entropy inequalities can then be written on the product space $\mathbb{R}^2\times\mathbb{R}^2$, with variables (s,x,t,y). At this stage, the trick is to use test functions $\phi=\phi(s,x,t,y)$ which concentrate most of the mass along the diagonal where $s=t$ and $x=y$. By taking the limit over a suitable sequence of these test functions we shall obtain (6.27), providing the desired extension of (6.3). To achieve (ii), we shall use (6.27) in connection with test functions which approximate the characteristic function of the domain Ω. We can now begin with the actual details of the proof.

1. Let u, v be entropy solutions of (6.1). Given any constants $k, k'\in\mathbb{R}$ and any smooth function $\phi=\phi(s,x,t,y)\geq 0$ with compact support contained in the set where $s,t>0$,

by assumption one has

$$\iint \{|u(s,x)-k|\phi_s(s,x,t,y)+\operatorname{sgn}(u(s,x)-k)(f(u(s,x)) - f(k))\phi_x(s,x,t,y)\}\,dx\,ds \geq 0, \tag{6.21}$$

$$\iint \{|v(t,y)-k'|\phi_t(s,x,t,y)+\operatorname{sgn}(v(t,y)-k')(f(v(t,y)) - f(k'))\phi_y(s,x,t,y)\}\,dy\,dt \geq 0. \tag{6.22}$$

Set $k = v(t,y)$ in (6.21) and integrate w.r.t. t, y. Then set $k' = u(s,x)$ in (6.22) and integrate w.r.t. s, x. Adding the two results, one obtains

$$\iiiint \{|u(s,x)-v(t,y)|(\phi_s+\phi_t)(s,x,t,y)+[f(u(s,x))-f(v(t,y))] \cdot [(\phi_x+\phi_y)(s,x,t,y)]\operatorname{sgn}(u(s,x)-v(t,y))\}\,dx\,dy\,ds\,dt \geq 0. \tag{6.23}$$

2. Now choose a sequence of functions $(\delta_h)_{h\geq 1}$, approximating the Dirac mass at the origin. More precisely, let $\delta : \mathbb{R} \mapsto [0,1]$ be a $\mathcal{C}^\infty$ function such that

$$\int_{-\infty}^{\infty} \delta(z)\,dz = 1, \qquad \delta(z) = 0 \qquad \text{for all } z \notin [-1,1],$$

and define

$$\delta_h(z) = h\delta(hz), \qquad \alpha_h(z) = \int_{-\infty}^{z} \delta_h(s)\,ds. \tag{6.24}$$

Consider any non-negative smooth function $\psi = \psi(T,X)$ whose support is a compact subset of the open half plane where $T > 0$, and define

$$\phi(s,x,t,y) = \psi\left(\frac{s+t}{2}, \frac{x+y}{2}\right)\delta_h\left(\frac{s-t}{2}\right)\delta_h\left(\frac{x-y}{2}\right).$$

A direct computation yields

$$(\phi_s+\phi_t)(s,x,t,y) = \psi_T\left(\frac{s+t}{2}, \frac{x+y}{2}\right)\delta_h\left(\frac{s-t}{2}\right)\delta_h\left(\frac{x-y}{2}\right),$$

$$(\phi_x+\phi_y)(s,x,t,y) = \psi_X\left(\frac{s+t}{2}, \frac{x+y}{2}\right)\delta_h\left(\frac{s-t}{2}\right)\delta_h\left(\frac{x-y}{2}\right).$$

For h sufficiently large, the support of ϕ is contained in the set where $s > 0, t > 0$. From (6.23) it thus follows that

$$\iiiint \delta_h\left(\frac{s-t}{2}\right)\delta_h\left(\frac{x-y}{2}\right)\left\{|u(s,x)-v(t,y)|\psi_T\left(\frac{s+t}{2}, \frac{x+y}{2}\right) + [f(u(s,x))-f(v(t,y))]\operatorname{sgn}\big(u(s,x)-v(t,y)\big)\psi_X\left(\frac{s+t}{2}, \frac{x+y}{2}\right)\right\}dx\,dy\,ds\,dt \geq 0. \tag{6.25}$$

3. We now compute the limit of the left hand side of (6.25) as $h \to \infty$. Using the variables

$$T = \frac{s+t}{2}, \qquad S = \frac{s-t}{2}, \qquad X = \frac{x+y}{2}, \qquad Y = \frac{x-y}{2},$$

the inequality (6.25) becomes

$$\iiiint \{|u(T+S,X+Y)-v(T-S,X-Y)|\psi_T(T,X) + [f(u(T+S,X+Y))-f(v(T-S,X-Y))] \cdot \operatorname{sgn}(u(T+S,X+Y)-v(T-S,X-Y))\cdot\psi_X(T,X)\} \cdot \delta_h(S)\delta_h(Y)\,dX\,dY\,dS\,dT \geq 0. \tag{6.26}$$

Letting $h \to \infty$ in (6.26) and renaming the variables T, X, we thus obtain

$$\iint \{|u(t,x)-v(t,x)|\psi_t(t,x)+[f(u(t,x))-f(v(t,x))] \cdot \operatorname{sgn}(u(t,x)-v(t,x))\psi_x(t,x)\}\,dx\,dt \geq 0 \tag{6.27}$$

for every $\mathcal{C}^1$ function ψ with compact support contained in the half plane where $t > 0$.

4. Now let $0 < \tau_0 < \tau$ and $R > 0$ be given. We construct a smooth approximation ψ to the characteristic function of the trapezoid Ω in (6.18), by setting

$$\psi(t,x) \doteq [\alpha_h(t-\tau_0)-\alpha_h(t-\tau)]\cdot[1-\alpha_h(|x|-R+\bar{L}(\tau-t))].$$

Recall that α_h was defined at (6.24), so that $\alpha_h' = \delta_h \geq 0$. Using (6.27) with this particular test function ψ, one obtains

$$\iint |u(t,x)-v(t,x)|\,[\delta_h(t-\tau_0)-\delta_h(t-\tau)]\cdot[1-\alpha_h(|x|-R-L(\tau-t))]\,dx\,dt$$
$$\geq \iint \left\{\frac{x}{|x|}[f(u(t,x))-f(v(t,x))]\operatorname{sgn}(u(t,x)-v(t,x))+L|u(t,x)-v(t,x)|\right\} \cdot [\alpha_h(t-\tau_0)-\alpha_h(t-\tau)]\,\delta_h(|x|-R-L(\tau-t))\,dx\,dt. \tag{6.28}$$

By (6.15) and (6.16) we have $|f(u)-f(v)| \leq L|u-v|$. Moreover, (6.24) yields $\alpha_h(t-\tau_0)-\alpha_h(t-\tau) \geq 0$, $\alpha_h' = \delta_h \geq 0$. Hence

$$\iint |u(t,x)-v(t,x)|\,[\delta_h(t-\tau_0)-\delta_h(t-\tau)]\cdot[1-\alpha_h(|x|-R-L(\tau-t))]\,dx\,dt \geq 0. \tag{6.29}$$

Recalling that the maps $t \mapsto u(t,\cdot)$, $t \mapsto v(t,\cdot)$ are both continuous from $[0,\infty[$ into $\mathbf{L}^1_{\text{loc}}$, we now let $h \to \infty$ in (6.29) and obtain (6.17), in the case where $0 < \tau_0 < \tau$. By continuity, (6.17) still holds if $\tau_0 = \tau$ or if $\tau_0 = 0$. □

Corollary 6.1 (Uniqueness in $\mathbf{L}^\infty$). *Let $f : \mathbb{R} \mapsto \mathbb{R}$ be locally Lipschitz continuous. If u, v are bounded entropy solutions of (6.1) such that $\|u(0,\cdot) - v(0,\cdot)\|_{\mathbf{L}^1} < \infty$, then for every $t > 0$ we have*

$$\int_{-\infty}^{\infty} |u(t,x) - v(t,x)|\, dx \leq \int_{-\infty}^{\infty} |u(0,x) - v(0,x)|\, dx. \tag{6.30}$$

For all initial data $\bar{u} \in \mathbf{L}^\infty$, the Cauchy problem (6.1)–(6.2) has at most one bounded entropy solution.

Proof. By assumption, there exist constants M, L for which (6.15) and (6.16) hold. For every $R, t \geq 0$, by (6.17) one has

$$\int_{|x|\leq R} |u(t,x) - v(t,x)|\, dx \leq \int_{|x|\leq R+Lt} |u(0,x) - v(0,x)|\, dx. \tag{6.31}$$

Letting $R \to \infty$ in (6.31) we obtain (6.30), and hence the uniqueness of the solution. □

6.4 A contractive semigroup

By Theorem 6.1, a weak solution of (6.1) exists for every initial condition $\bar{u} \in \mathbf{L}^1$ with bounded variation. Since the $\mathbf{L}^1$ distance between any two such solutions does not increase in time, the solution operator can be extended by continuity to a much larger family of initial conditions. In particular, this yields the existence of a unique entropy solution to the Cauchy problem (6.1)–(6.2) for all initial data $\bar{u} \in \mathbf{L}^1 \cap \mathbf{L}^\infty$. We recall that weak solutions are defined up to equivalence in $\mathbf{L}^1_{\text{loc}}$. The results stated below are thus understood to be valid after possibly changing the values of the solutions on a set of measure zero.

Theorem 6.3. *Let $f : \mathbb{R} \mapsto \mathbb{R}$ be locally Lipschitz continuous. Then there exists a continuous semigroup $S : [0,\infty) \times \mathbf{L}^1 \mapsto \mathbf{L}^1$ with the following properties*

(i) $S_0\bar{u} = \bar{u}$, $S_s(S_t\bar{u}) = S_{s+t}\bar{u}$.

(ii) $\|S_t\bar{u} - S_t\bar{v}\|_{\mathbf{L}^1} \leq \|\bar{u} - \bar{v}\|_{\mathbf{L}^1}$.

(iii) *For each $\bar{u} \in \mathbf{L}^1 \cap \mathbf{L}^\infty$, the trajectory $t \mapsto S_t\bar{u}$ yields the unique bounded, entropy-admissible, weak solution of the corresponding Cauchy problem (6.1)–(6.2).*

(iv) *If $\bar{u}(x) \leq \bar{v}(x)$ for all $x \in \mathbb{R}$, then $S_t\bar{u}(x) \leq S_t\bar{v}(x)$ for every $x \in \mathbb{R}$, $t \geq 0$.*

Proof. For all initial data $\bar{w} \in \mathbf{L}^1 \cap BV$, let $S_t\bar{w}$ be the value at time t of the entropy solution to (6.1) with initial condition $\bar{w}$. The existence and uniqueness of such a solution are guaranteed by Theorems 6.1 and 6.2. By continuity, we can now extend the domain of the semigroup S to the entire space $\mathbf{L}^1$ by setting

$$S_t\bar{u} = \lim_{\substack{\bar{w}\to\bar{u}\\ \bar{w}\in BV}} S_t\bar{w}, \tag{6.32}$$

the convergence taking place in the $\mathbf{L}^1$ norm. Because of (6.30), the limit in (6.32) is well defined and satisfies (i) and (ii).

To prove (iii), let $\bar{u} \in \mathbf{L}^1 \cap \mathbf{L}^\infty$ be given, say with $|\bar{u}(x)| \leq M$ for all x. Consider a sequence of functions $\bar{u}_\nu \in BV$ with $\|\bar{u}_\nu\|_{\mathbf{L}^\infty} \leq M$, $\bar{u}_\nu \to \bar{u}$ in $\mathbf{L}^1$. Observe that the corresponding solutions $u_\nu(t,\cdot) = S_t\bar{u}_\nu$ all take values inside the interval $[-M, M]$. By the contractivity of the semigroup, for each ν and every $t \geq 0$ we now have

$$\|S_t\bar{u}_\nu - S_t\bar{u}\|_{\mathbf{L}^1} \leq \|\bar{u}_\nu - \bar{u}\|_{\mathbf{L}^1}, \qquad \|f(S_t\bar{u}_\nu) - f(S_t\bar{u})\|_{\mathbf{L}^1} \leq L \cdot \|\bar{u}_\nu - \bar{u}\|_{\mathbf{L}^1}, \tag{6.33}$$

where L is a Lipschitz constant for f on $[-M, M]$. Fix any $k \in \mathbb{R}$ and any $\mathcal{C}^1$ function $\phi \geq 0$ with compact support contained in the half plane where $t > 0$. Using (6.33) we obtain

$$\begin{aligned}
&\iint \{|S_t\bar{u} - k|\phi_t + (f(S_t\bar{u}) - f(k))\,\mathrm{sgn}(S_t\bar{u} - k)\phi_x\}\, dx\, dt \\
&= \lim_{\nu\to\infty} \iint \{|S_t\bar{u}_\nu - k|\phi_t + (f(S_t\bar{u}_\nu) - f(k))\,\mathrm{sgn}(S_t\bar{u}_\nu - k)\phi_x\}\, dx\, dt \\
&\geq 0,
\end{aligned}$$

showing that each trajectory of the semigroup is an entropy-admissible solution of (6.1).

Concerning (iv), by continuity it suffices to consider the case where both $\bar{u}$ and $\bar{v}$ have bounded variation. In this case, the corresponding solutions of (6.1)–(6.2) can be obtained as limits of the piecewise constant approximations constructed in Section 6.1. The proof is thus reduced to showing that for any given $\nu \geq 1$, if u, v are piecewise constant solutions of (6.6) and $u(0,x) \leq v(0,x)$ for all x, then

$$u(t,x) \leq v(t,x) \quad \text{for all } t \geq 0, \quad x \in \mathbb{R}. \tag{6.34}$$

If (6.34) fails, by continuity there exists a largest time τ such that $u(t,x) \leq v(t,x)$ for all $x \in \mathbb{R}$ and $t \leq \tau$. Since u are piecewise constant, on a small time interval $[\tau, \tau+\delta]$ it is obtained by piecing together the corresponding solutions of the Riemann problems at every point of jump of $u(\tau,\cdot)$. The same of course holds for v. To derive a contradiction, it thus suffices to prove a comparison result for solutions of two Riemann problems.

We thus consider two Riemann data (u^-, u^+), (v^-, v^+) for the conservation law (6.6). If $\max\{u^-, u^+\} \leq \min\{v^-, v^+\}$, it is obvious that the corresponding solutions satisfy $u(t,x) \leq v(t,x)$ for every $x \in \mathbb{R}$, $t > 0$. We thus need to consider two non-trivial cases:

CASE 1: $u^- \leq v^- < u^+ \leq v^+$.

CASE 2: $u^+ \leq v^+ < u^- \leq v^-$.

In the first case, we observe that the piecewise constant solution $u = u(t,x)$ constructed at (6.9) can be characterized as follows:

$$u(t,x) = w \quad \text{iff } f_\nu(w) - \frac{x}{t} \cdot w = \min_{s\in[u^-,u^+]} \left\{ f_\nu(s) - \frac{x}{t} \cdot s \right\}. \tag{6.35}$$

In other words, $u(t,x) = w$ iff the line with slope $\lambda = x/t$ supports the graph of f_ν (restricted to the interval $[u^-, u^+]$) at the point $(w, f(w))$. From (6.35) and the analogous property of v it follows that

$$u(t,x) = \arg\min_{s\in[u^-,u^+]} \left\{ f_\nu(s) - \frac{x}{t}\cdot s \right\} \le \arg\min_{s\in[v^-,v^+]} \left\{ f_\nu(s) - \frac{x}{t}\cdot s \right\} = v(t,x).$$

Similarly, in the second case we have

$$u(t,x) = \arg\max_{s\in[u^+,u^-]} \left\{ f_\nu(s) - \frac{x}{t}\cdot s \right\} \le \arg\max_{s\in[v^+,v^-]} \left\{ f_\nu(s) - \frac{x}{t}\cdot s \right\} = v(t,x).$$

In both cases we have obtained a contradiction with the maximality of τ. This establishes (6.34), completing the proof of the theorem. □

Problems

(1) Compute the unique entropy solution of the Riemann problem

$$u_t + (u^3 - 3u)_x = 0, \qquad u(0,x) = \begin{cases} -2 & \text{if } x < 0, \\ 2 & \text{if } x > 0. \end{cases}$$

(2) Let $f : \mathbb{R} \mapsto \mathbb{R}$ be globally Lipschitz continuous with constant L, and let S be the semigroup generated by (6.1). If $\bar{u} \in \mathbf{L}^1$ has support contained inside $[a,b]$, prove that the support of $S_t\bar{u}$ is contained in the interval $[a - Lt, b + Lt]$.

(3) Let f be locally Lipschitz continuous and consider the Riemann problem

$$u_t + f(u)_x = 0, \qquad u(0,x) = \begin{cases} u^- & \text{if } x < 0, \\ u^+ & \text{if } x > 0, \end{cases} \tag{6.36}$$

with $u^- < u^+$. Prove that the unique entropy solution of (6.36) has the form $u(t,x) = \psi(x/t)$, where

$$\psi(\lambda) \doteq \arg\min_{s\in[u^-,u^+]} f(s) - \lambda s.$$

Otherwise stated, $u(t,x) = w \in [u^-, u^+]$ iff the line with slope x/t supports the epigraph of f, restricted to the interval $[u^-, u^+]$, at the point $(w, f(w))$.
Hint: consider first the case where f is piecewise affine.

(4) Consider initial data $\bar{u} \in \mathbf{L}^\infty$ which is periodic with period p. Let $u = u(t,x)$ be an entropy solution of the Cauchy problem (6.1)–(6.2). For every $t \ge 0$, prove that $u(t,\cdot)$ is periodic with period p. Moreover,

$$\int_0^p u(t,x)\,dx = \int_0^p \bar{u}(x)\,dx.$$

Find an example where, for $t > 0$,

$$\int_0^p |u(t,x)|\,dx < \int_0^p |\bar{u}(x)|\,dx.$$

(5) Let $f \in C^2$ satisfy

$$f(0) = f'(0) = 0, \qquad f''(u) \ge c > 0 \quad \text{for all } u.$$

Consider the initial condition

$$\bar{u}(x) = \begin{cases} k(x-a) & \text{if } x \in [a,b], \\ 0 & \text{if } x \notin [a,b], \end{cases}$$

for some constant $k > 0$. Prove that the entropy solution of (6.1)–(6.2) is given by

$$u(t,x) = \begin{cases} \bar{u}(y(t,x)) & \text{if } x \in [a, b(t)], \\ 0 & \text{if } x \notin [a, b(t)], \end{cases}$$

where $y = y(t,x)$ and $b = b(t)$ are implicitly defined by

$$y + tf'(\bar{u}(y)) = x, \quad y \in [a,b],$$

$$\int_a^{b(t)} u(t,x)\,dx = \int_a^{b(t)} \bar{u}(y(t,x))\,dx = \int_a^b \bar{u}(x)\,dx = \frac{k(b-a)}{2}.$$

In addition, show that

$$\frac{u(t,x') - u(t,x)}{x' - x} \le \frac{1}{ct} \qquad x' > x, \quad t > 0. \tag{6.37}$$

Finally, prove the decay estimate

$$\|u(t,\cdot)\|_{\mathbf{L}^\infty} \le \sqrt{\frac{2\|\bar{u}\|_{\mathbf{L}^1}}{ct}} \qquad t > 0. \tag{6.38}$$

Hint: outside the shock at $x = b(t)$, the function u satisfies

$$(u_x)_t + f'(u)(u_x)_x = -f''(u)u_x^2 \le -cu_x^2. \tag{6.39}$$

Integrate (6.39) along the characteristics $x(t) = x_0 + tf'(u(x_0))$ to obtain $u_x(t,x) \le (ct)^{-1}$. To prove (6.38) observe that, if $u(t,x') = h > 0$ at some point x', then (6.37) implies

$$\frac{cth^2}{2} \le \|u(t,\cdot)\|_{\mathbf{L}^1} \le \|\bar{u}\|_{\mathbf{L}^1}. \tag{6.40}$$

(6) Let $f \in C^2$ satisfy $f''(u) \ge c > 0$ for all $u \in \mathbb{R}$. Fix an integer $\nu \ge 1$ and define the piecewise constant approximation f_ν as in (6.5). Let $u_\nu = u_\nu(t,x)$ be a piecewise constant solution of (6.6), taking values in the discrete set $2^{-\nu}\mathbb{Z}$. Lines $x = x_\alpha(t)$ where u_ν has an upward jump, i.e. $u_\nu(t, x_\alpha-) < u_\nu(t, x_\alpha+)$, are called *rarefaction fronts*. Lines where u_ν has a downward jump are called *shock fronts*.

(i) Show that all rarefaction fronts have strength $u_\nu(t, x_\alpha+) - u_\nu(t, x_\alpha-) = 2^{-\nu}$.

(ii) Show that, if two or more fronts collide, at least one of them is a shock. From the interaction, a single shock emerges (unless all fronts completely cancel each other).

(iii) If two adjacent fronts $x_\alpha, x_{\alpha+1}$ are both rarefactions, prove that their speeds satisfy $\dot{x}_{\alpha+1} - \dot{x}_\alpha \geq c \cdot 2^{-\nu}$.

(iv) From (iii) deduce the inequality

$$u_\nu(t, y) - u_\nu(t, x) \leq 2^{-\nu} + \frac{y-x}{ct} \qquad x < y, \quad t > 0.$$

(v) Letting $\nu \to \infty$ in (6.41), prove that every entropy solution u of (6.1) satisfies

$$u(t, y) - u(t, x) \leq \frac{y-x}{ct} \qquad x < y, \quad t > 0.$$

(vi) Using (6.42), show that the decay estimate (6.38) holds for every solution u of (6.1), provided that $f'' \geq c > 0$.

7

The Cauchy problem for systems

This chapter is concerned with the global existence of solutions to the Cauchy problem

$$u_t + f(u)_x = 0, \tag{7.1}$$

$$u(0, x) = \bar{u}(x), \tag{7.2}$$

under the assumptions

(♣) The $n \times n$ system of conservation laws (7.1) is strictly hyperbolic, with smooth coefficients, defined for u in an open set $\Omega \subseteq \mathbb{R}^n$. Each characteristic field is either genuinely non-linear or linearly degenerate.

By possibly performing a translation in the u-coordinates, it is not restrictive to assume that Ω contains the origin. Given an initial condition $\bar{u}$ with sufficiently small total variation, we will construct a weak, entropy-admissible solution u, defined for all $t \geq 0$. We recall that a function $u : [0, T] \times \mathbb{R} \mapsto \mathbb{R}^n$ is a weak solution to the Cauchy problem (7.1)–(7.2) if the map $t \mapsto u(t, \cdot)$ is continuous with values in $\mathbf{L}^1_{\mathrm{loc}}$, the initial condition (7.2) is satisfied and, for every $\mathcal{C}^1$ function ϕ with compact support contained in the open strip $]0, T[\, \times \mathbb{R}$, one has

$$\int_0^T \int_{-\infty}^{\infty} \{\phi_t(t, x)u(t, x) + \phi_x(t, x) f(u(t, x))\} \, dx \, dt = 0. \tag{7.3}$$

Given a convex entropy η for the system (7.1), with entropy flux q, we say that the solution u is η-admissible if it satisfies the entropy inequality

$$\eta(u)_t + q(u)_x \leq 0$$

in the distributional sense. For every non-negative $\mathcal{C}^1$ function ϕ with compact support contained in the strip $]0, T[\, \times \mathbb{R}$, we thus require

$$\int_0^T \int_{-\infty}^{\infty} \{\phi_t(t, x)\eta(u(t, x)) + \phi_x(t, x)q(u(t, x))\} \, dx \, dt \geq 0. \tag{7.4}$$

Most of this chapter is devoted to the proof of the following basic existence theorem.

The *characteristic function* of a set K is defined as

$$\chi_K(x) = \begin{cases} 1 & \text{if } x \in K, \\ 0 & \text{if } x \notin K. \end{cases}$$

We say that a function $f : \Omega \mapsto \mathbb{R}^n$ is *locally integrable*, and write $f \in \mathbf{L}^1_{\text{loc}}$, if, for every compact set $K \subset \Omega$, the product $f \cdot \chi_K$ of f with the characteristic function of K is integrable. A sequence of functions $(f_\nu)_{\nu \geq 1}$ converges to f in $\mathbf{L}^1_{\text{loc}}$ if the sequence $f_\nu \cdot \chi_K$ converges to $f \cdot \chi_K$ in $\mathbf{L}^1$ for every compact set K.

Every $f \in \mathbf{L}^1_{\text{loc}}(\Omega; \mathbb{R}^n)$ determines a distribution of order 0, defined as

$$\Lambda_f(\phi) = \int_\Omega f(x)\phi(x)\,dx \quad \text{for all } \phi \in \mathcal{D}(\Omega). \tag{2.10}$$

If α is a multi-index and $\Lambda \in \mathcal{D}'(\Omega)$, then the derivative $D^\alpha \Lambda$ is the distribution defined as

$$(D^\alpha \Lambda)(\phi) = (-1)^{|\alpha|} \Lambda(D^\alpha \phi) \quad \text{for all } \phi \in \mathcal{D}(\Omega). \tag{2.11}$$

If a function f is N times continuously differentiable and $|\alpha| \leq N$, then with the above notation one has $D^\alpha \Lambda_f = \Lambda_{D^\alpha f}$.

Example 2.1. A case that will be frequently encountered in applications is the following (Fig. 2.1). The open set Ω is contained in the plane $\mathbb{R}^2$ with coordinates t, x. Inside Ω we are given a C^1 curve

$$\{(t, x);\ x = \gamma(t)\ \ a < t < b\}$$

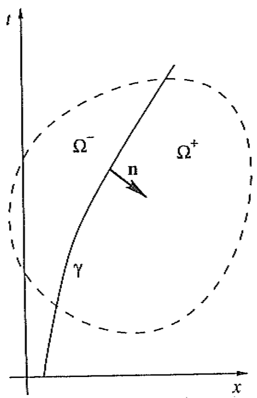

Figure 2.1

and two functions $g = g(t, x)$, $h = h(t, x)$ which are continuously differentiable for $x \neq \gamma(t)$ but possibly discontinuous on γ. We wish to compute the distribution Λ associated with the sum of distributional derivatives

$$D_t g + D_x h.$$

At each point $(t, \gamma(t))$ consider the jumps

$$\Delta g(t) = \lim_{x \to \gamma(t)+} g(t, x) - \lim_{x \to \gamma(t)-} g(t, x),$$
$$\Delta h(t) = \lim_{x \to \gamma(t)+} h(t, x) - \lim_{x \to \gamma(t)-} h(t, x),$$

assuming that the above limits exist for all $a < t < b$. Given any C^1 function $\phi = \phi(t, x)$ with compact support contained inside Ω, we apply the divergence theorem to the vector field $\mathbf{v} \doteq (\phi g,\ \phi h)$ on the domains

$$\Omega^+ \doteq \{(t, x) \in \Omega, x > \gamma(t)\}, \qquad \Omega^- \doteq \{(t, x) \in \Omega, x < \gamma(t)\}.$$

Let $d\sigma$ be the differential of the arc-length along the curve γ and denote the derivative w.r.t. time by an upper dot. Moreover, let $\mathbf{n}$ be the outer unit normal to the boundaries of Ω^-, Ω^+. By assumption, $\phi = 0$ on $\partial\Omega$; hence the only portion of these boundaries where ϕ does not vanish is the line $x = \gamma(t)$. On this line, an elementary calculation yields $\mathbf{n} \cdot d\sigma = \pm(-\dot{\gamma}, 1)\,dt$. Therefore

$$\Lambda(\phi) \doteq -\iint_\Omega \{g\phi_t + h\phi_x\}\,dx\,dt$$
$$= \iint_\Omega \{g_t + h_x\}\phi\,dx\,dt + \int_a^b \{\Delta h\,(t, \gamma(t)) - \dot{\gamma}(t)\Delta g\,(t, \gamma(t))\}\phi\,(t, \gamma(t))\,dt. \tag{2.12}$$

2.4 Functions with bounded variation

Consider a (possibly unbounded) interval $J \subseteq \mathbb{R}$ and a map $u : J \mapsto \mathbb{R}^n$. The *total variation* of u is then defined as

$$\text{Tot. Var.}\{u\} \doteq \sup \left\{ \sum_{j=1}^N |u(x_j) - u(x_{j-1})| \right\}, \tag{2.13}$$

where the supremum is taken over all $N \geq 1$ and all $(N+1)$-tuples of points $x_j \in J$ such that $x_0 < x_1 < \cdots < x_N$. If the right hand side of (2.13) is bounded, we say that u has bounded variation, and write $u \in BV$. Some elementary properties of BV functions are collected in the following lemmas.

Lemma 2.1. *Let $u :]a, b[\mapsto \mathbb{R}^n$ have bounded variation. Then, for every $x \in]a, b[$, the left and right limits*

$$u(x-) \doteq \lim_{y \to x-} u(y), \qquad u(x+) \doteq \lim_{y \to x+} u(y)$$

are well defined. Moreover, u has at most countably many points of discontinuity.

Proof. Let $x \in]a, b[$ be given and consider any strictly increasing sequence of points x_ν tending to x. Since

$$\sum_{\nu \geq 1} |u(x_\nu) - u(x_{\nu-1})| \leq \text{Tot. Var.}\{u\} < \infty,$$

the sequence $u(x_\nu)$ is Cauchy and converges to some limit u^-. Observing that any two such sequences $x_\nu \to x$, $x'_\nu \to x$ can be combined into a unique non-decreasing sequence, it is clear that the above limit is independent of the choice of the x_ν. This proves the existence of the left limit $u(x-)$. The case of $u(x+)$ is entirely similar.

To prove the last statement, for each $\nu \geq 1$ we observe that the number of points contained in the set

$$A_\nu \doteq \{x \in]a, b[\,;\, |u(x-) - u(x)| + |u(x+) - u(x)| > 1/\nu\}$$

cannot be bigger than $\nu \cdot \text{Tot. Var.}\{u\}$. Hence the set of points where u is discontinuous, being contained in the union of all A_ν, $\nu \geq 1$, is at most countable. □

Remark 2.1. By the above lemma, if u has bounded variation, we can redefine the value of u at each jump point by setting $u(x) \doteq u(x+)$. In particular, if we are only interested in the $\mathbf{L}^1$-equivalence class of a BV function u, by possibly changing the values of u at countably many points we can assume that u is right continuous.

Remark 2.2. If $u : \mathbb{R} \mapsto \mathbb{R}^n$ has bounded variation, the same arguments used in the proof of Lemma 2.1 show that the limits $u(-\infty)$, $u(\infty)$ are well defined.

Lemma 2.2. *Let $u : \mathbb{R} \mapsto \mathbb{R}^n$ be right continuous with bounded variation. Then, for every $\varepsilon > 0$, there exists a piecewise constant function v such that*

$$\text{Tot. Var.}\{v\} \leq \text{Tot.Var.}\{u\}, \qquad \|v - u\|_{\mathbf{L}^\infty} \leq \varepsilon. \tag{2.14}$$

If, in addition,

$$\int_{-\infty}^{0} |u(x) - u(-\infty)|\, dx + \int_{0}^{\infty} |u(x) - u(\infty)|\, dx < \infty,$$

then one can find v with the additional property

$$\|u - v\|_{\mathbf{L}^1} < \varepsilon. \tag{2.15}$$

Proof. Define the scalar function

$$U(x) \doteq \sup\left\{\sum_{j=1}^{N} |u(x_j) - u(x_{j-1})|;\ N \geq 1,\ x_0 < x_1 < \cdots < x_n = x\right\},$$

measuring the total variation of u on the interval $]-\infty, x]$. Observe that U is a right continuous, non-decreasing function which satisfies

$$\begin{aligned} &U(-\infty) = 0, \qquad U(\infty) = \text{Tot. Var.}\{u\}, \\ &|u(y) - u(x)| \leq U(y) - U(x) \quad \text{for all } x < y. \end{aligned} \tag{2.16}$$

Given $\varepsilon > 0$, let N be the largest integer $<$ Tot. Var. $\{u\}$ and consider the points

$$x_0 \doteq -\infty, \quad x_N \doteq \infty, \quad x_j \doteq \min\{x\,;\, U(x) \geq j\varepsilon\}, \quad j = 1, \ldots, N-1.$$

Defining

$$v(x) \doteq u(x_j) \quad \text{if } x \in [x_j, x_{j+1}[,$$

by (2.16) the two estimates in (2.14) are both satisfied.

To prove (2.15) we observe that, under the additional assumption of the lemma, one can find ρ large enough so that

$$\int_{-\infty}^{-\rho} |u(x) - u(-\infty)|\, dx + \int_{\rho}^{\infty} |u(x) - u(\infty)|\, dx < \frac{\varepsilon}{2}.$$

We can now construct a piecewise constant function $\tilde{v}$ such that

$$\|\tilde{v} - u\|_{\mathbf{L}^\infty} < \frac{\varepsilon}{2\rho}.$$

Defining

$$v(x) = \begin{cases} \tilde{v}(x) & \text{if } x \in [-\rho, \rho[, \\ u(-\infty) & \text{if } x < -\rho, \\ u(\infty) & \text{if } x \geq \rho, \end{cases}$$

one achieves the additional estimate (2.15). □

Lemma 2.3. *If $u : \mathbb{R} \mapsto \mathbb{R}^n$ has bounded variation, for every $\varepsilon > 0$ one has*

$$\frac{1}{\varepsilon} \cdot \int_{-\infty}^{\infty} |u(x + \varepsilon) - u(x)|\, dx \leq \text{Tot. Var.}\{u\}. \tag{2.17}$$

Proof. It is not restrictive to assume that u is right continuous. As in the proof of Lemma 2.2, define the non-decreasing scalar function $U(x)$ as the total variation of u on $]-\infty, x]$. By (2.16) we then have

$$\begin{aligned}\int_{-\infty}^{\infty} |u(x+\varepsilon) - u(x)|\, dx &\leq \int_{-\infty}^{\infty} [U(x+\varepsilon) - U(x)]\, dx \\ &= \text{meas}\{(x,y) \in \mathbb{R}^2\,;\, U(x) < y < U(x+\varepsilon)\} \\ &= \int_{U(-\infty)}^{U(\infty)} \text{meas}\{x\,;\, U(x) < y < U(x+\varepsilon)\}\, dy \\ &= \int_0^{\text{Tot.Var.}\{u\}} \varepsilon\, dy \\ &= \varepsilon \cdot \text{Tot. Var.}\{u\}.\end{aligned}$$

This establishes (2.17). □

Bounded sets of *BV* functions have a compactness property, stated in the following theorem, which will provide the key ingredient in the existence proof for weak solutions to systems of conservation laws.

Theorem 2.3 (Helly). *Consider a sequence of functions* $u_\nu : \mathbb{R} \mapsto \mathbb{R}^n$ *such that*

$$\text{Tot. Var.}\{u_\nu\} \leq C, \qquad |u_\nu(x)| \leq M \qquad \textit{for all } \nu, x, \tag{2.18}$$

for some constants C, M. *Then there exists a function* u *and a subsequence* u_μ *such that*

$$\lim_{\mu\to\infty} u_\mu(x) = u(x) \quad \textit{for every } x \in \mathbb{R}, \tag{2.19}$$

$$\text{Tot. Var.}\{u\} \leq C, \qquad |u(x)| \leq M \quad \textit{for all } x. \tag{2.20}$$

Proof. 1. For every $\nu \geq 1$, let

$$U_\nu(x) \doteq \sup\left\{\sum_{j=1}^N |u_\nu(x_j) - u_\nu(x_{j-1})|;\ N \geq 1, x_0 < x_1 < \cdots < x_n = x\right\}$$

be the total variation of u_ν on $]-\infty, x]$. Observe that each U_ν is non-decreasing and satisfies

$$0 \leq U_\nu(x) \leq C, \qquad |u_\nu(y) - u_\nu(x)| \leq U_\nu(p_2) - U_\nu(p_1) \qquad \text{for all } p_1 \leq x \leq y \leq p_2. \tag{2.21}$$

2. By a diagonal procedure, we construct a subsequence $U_{\nu'}$ whose limit exists at every rational point:

$$\lim_{\nu'\to\infty} U_{\nu'}(x) = U(x) \qquad x \in \mathbb{Q}.$$

Because of (2.21), the function U maps $\mathbb{Q}$ into $[0, C]$ and is non-decreasing. For each $n \geq 1$, consider the set of jump points

$$J_n \doteq \left\{x \in \mathbb{R};\ \lim_{y\to x+} U(y) - \lim_{y\to x-} U(y) \geq \frac{1}{n}\right\}$$

where, of course, the variable y ranges over $\mathbb{Q}$. By the properties of U, the set J_n can contain at most Cn points. Therefore, the set J of points $x \in \mathbb{R}$ where the right and left limits of U are distinct is at most countable, and indeed

$$J = \bigcup_{n\geq 1} J_n.$$

3. We now choose a further subsequence, say u_μ, such that the limit

$$u(x) \doteq \lim_{\mu\to\infty} u_\mu(x) \tag{2.22}$$

exists for each x in the countable set $J \cup \mathbb{Q}$. We claim that, for this subsequence, the limit (2.22) exists for every $x \in \mathbb{R}$ as well. Indeed, assume $x \notin J$. Then for each $n \geq 1$, since $x \notin J_n$, there exist rational points $p_1 < x < p_2$ such that $U(p_2) - U(p_1) < 2/n$. Using (2.21) and the fact that $u_\mu(p_1) \to u(p_1)$, we obtain

$$\begin{aligned}\limsup_{h,k\to\infty} |u_h(x) - u_k(x)| &\leq \limsup_{h\to\infty} |u_h(x) - u(p_1)| + \limsup_{k\to\infty} |u_k(x) - u(p_1)| \\ &= 2 \cdot \limsup_{\mu\to\infty} |u_\mu(x) - u_\mu(p_1)| \leq 2 \cdot \limsup_{\mu\to\infty} (U_\mu(p_2) - U_\mu(p_1)) \\ &= 2\,(U(p_2) - U(p_1)) < \frac{4}{n}.\end{aligned}$$

Since n was arbitrary, our claim is proved. This establishes the first part of the theorem.

4. For any given points $x_0 < x_1 < \cdots < x_N$, we now have

$$\begin{aligned}\sum_{j=1}^N |u(x_j) - u(x_{j-1})| &= \lim_{\mu\to\infty}\left(\sum_{j=1}^N |u_\mu(x_j) - u_\mu(x_{j-1})|\right) \\ &\leq \limsup_{\mu\to\infty} (\text{Tot. Var.}\{u_\mu\}) \leq C.\end{aligned}$$

This proves the first inequality in (2.20). The second is obvious. □

Theorem 2.4. *Consider a sequence of functions* $u_\nu : [0, \infty[\times \mathbb{R} \mapsto \mathbb{R}^n$ *with the following properties:*

$$\text{Tot. Var.}\{u_\nu(t,\cdot)\} \leq C, \qquad |u_\nu(t,x)| \leq M \qquad \textit{for all } t, x, \tag{2.23}$$

$$\int_{-\infty}^{\infty} |u_\nu(t,x) - u_\nu(s,x)|\, dx \leq L|t-s| \quad \textit{for all } t, s \geq 0, \tag{2.24}$$

for some constants C, M, L. Then there exists a subsequence u_μ which converges to some function u in $\mathbf{L}^1_{loc}([0,\infty)\times\mathbb{R};\ \mathbb{R}^n)$. This limit function satisfies

$$\int_{-\infty}^{\infty} |u(t,x)-u(s,x)|\,dx \le L|t-s| \quad \text{for all } t,s\ge 0. \tag{2.25}$$

The point values of the limit function u can be uniquely determined by requiring that

$$u(t,x)=u(t,x+)\doteq \lim_{y\to x+} u(t,y) \quad \text{for all } t,x. \tag{2.26}$$

In this case, one has

$$\text{Tot. Var.}\{u(t,\cdot)\}\le C,\qquad |u(t,x)|\le M \qquad \text{for all } t,x. \tag{2.27}$$

Proof. Using Theorem 2.3 we construct a subsequence $\{u_\mu\}$ such that $u_\mu(t,\cdot)\to u(t,\cdot)$ pointwise and hence also in $\mathbf{L}^1_{loc}(\mathbb{R};\ \mathbb{R}^n)$, at each rational time $t\ge 0$. This limit function clearly satisfies (2.25) and (2.27), restricted to $t,s\in\mathbb{Q}$. By continuity, it can thus be uniquely extended to a map $t\mapsto u(t,\cdot)$ from $[0,\infty[$ into $\mathbf{L}^1_{loc}(\mathbb{R};\ \mathbb{R}^n)$, satisfying (2.25). More precisely, for each $t\ge 0$ we consider a sequence of rational times $t_m\to t$ and define

$$u(t,\cdot)\doteq \lim_{m\to\infty} u(t_m,\cdot).$$

Because of (2.25), this limit exists and does not depend on the choice of the sequence. Observing that

$$\text{Tot. Var.}\{u(t_m,\cdot)\}\le C,\qquad |u(t_m,x)|\le M \qquad \text{for all } m\ge 1,\ x\in\mathbb{R},$$

by possibly modifying the limit function $u(t,\cdot)$ on a set of measure zero we achieve the bounds (2.27).

Finally, we define

$$u^\varepsilon(t,x)\doteq \frac{1}{\varepsilon}\int_x^{x+\varepsilon} u(t,x)\,dx.$$

Observe that each u^ε is uniformly Lipschitz continuous w.r.t. both variables t,x. Indeed

$$|u^\varepsilon(t,x)-u^\varepsilon(s,x)|\le \frac{1}{\varepsilon}\int_x^{x+\varepsilon}|u(t,x)-u(s,x)|\,dx\le \frac{L}{\varepsilon}\cdot|t-s|,$$

$$|u^\varepsilon(t,x)-u^\varepsilon(t,x+h)|\le \frac{1}{\varepsilon}\left(\int_x^{x+h}+\int_{x+\varepsilon}^{x+\varepsilon+h}\right)|u(t,y)|\,dy\le \frac{2M}{\varepsilon}\cdot h.$$

Moreover, for every t,x we have

$$\tilde u(t,x)\doteq \lim_{\varepsilon\to 0+} u^\varepsilon(t,x)=u(t,x+). \tag{2.28}$$

The function $\tilde u$, being the pointwise limit of continuous functions, is Borel-measurable. For each $t\ge 0$ the identity $\tilde u(t,x)=u(t,x)$ holds at all but countably many points x. By replacing u with $\tilde u$, all requirements in (2.25)–(2.27) are clearly satisfied. □

2.5 *BV* functions of two variables

In the following, we denote $y=(y_1,\dots,y_m)$ as the variable in $\mathbb{R}^m$. We say that a (vector-valued) function $u=u(y)$ has *locally bounded variation* if its distributional derivatives $D_{y_i}u$, $i=1,\dots,m$, are measures. By definition this is the case if, for every compact set $K\subset\mathbb{R}^2$, there exists a constant C_K such that

$$\left|\int u\cdot\frac{\partial\phi}{\partial y_i}\,dy\right|\le C_K\|\phi\|_{\mathcal{C}^0} \tag{2.29}$$

for each i and every $\phi\in\mathcal{C}^1_c$ with support contained in K.

Functions of bounded variation possess much better regularity properties than arbitrary measurable functions. Some basic results in this direction will be presented below.

Definition 2.1. We say that a function u has an *approximate jump discontinuity* at the point $\bar y$ if there exists vectors $u^+\neq u^-$ and a unit normal vector $\mathbf{n}\in\mathbb{R}^m$ such that, setting

$$U(y)\doteq\begin{cases}u^- & \text{if } y\cdot\mathbf{n}<0,\\ u^+ & \text{if } y\cdot\mathbf{n}>0,\end{cases} \tag{2.30}$$

the following holds:

$$\lim_{r\to 0+}\frac{1}{r^m}\int_{|y|<r}|u(\bar y+y)-U(y)|\,dy=0. \tag{2.31}$$

Moreover, we say that u is *approximately continuous* at the point $\bar y$ if the above relations hold with $u^+=u^-$ (and $\mathbf{n}$ arbitrary).

Observe that the above definitions depend only on the $\mathbf{L}^1$ equivalence class of u. Indeed, the limit (2.31) is unaffected if the values of u are changed on a set $\mathcal{N}\subset\mathbb{R}^m$ of Lebesgue measure zero. The standard example of an approximate jump point is the following.

Example 2.2. Let $f_1,f_2:\mathbb{R}^m\mapsto\mathbb{R}^n$ be continuous. Let $g:\mathbb{R}^m\mapsto\mathbb{R}$ be continuously differentiable. Consider the function

$$u(y)\doteq\begin{cases}f_1(y) & \text{if } g(y)\le 0,\\ f_2(y) & \text{if } g(y)>0.\end{cases}$$

At a point $\bar y$ where $g(\bar y)=0$, call $u^-\doteq f_1(\bar y)$, $u^+\doteq f_2(\bar y)$. If $u^+=u^-$, then u is continuous at $\bar y$, and hence also approximately continuous. On the other hand, if $u^+\neq u^-$ and $\nabla g(\bar y)\neq 0$, then u has an approximate jump at $\bar y$. Indeed, the limit (2.31) holds by choosing $\mathbf{n}$ as the unit vector in the direction of $\nabla g(\bar y)$.

The following theorem provides a useful description of the structure of BV functions of two independent variables. A proof, valid also for arbitrary space dimensions, can be found in Evans and Gariepy (1992) or Ziemer (1989).

Theorem 2.5. *Let Ω be an open subset of $\mathbb{R}^2$ and let $u : \Omega \mapsto \mathbb{R}^n$ be a BV function. Then there exists a set $\widetilde{\mathcal{N}} \subset \Omega$ whose one-dimensional Hausdorff measure is zero and such that, at each point $y \notin \widetilde{\mathcal{N}}$, the function u either is approximately continuous or has an approximate jump discontinuity.*

A class of BV functions of two variables, particularly important for applications to conservation laws, is now considered.

Theorem 2.6. *Let $u :]a, b[\times\mathbb{R} \mapsto \mathbb{R}^n$ satisfy*

$$\text{Tot. Var.}\,\{u(t,\cdot)\} \le M \qquad t \in]a, b[, \tag{2.32}$$

$$\int_{-\infty}^{\infty} |u(t,x) - u(s,x)|\,dx \le L|t-s| \qquad s, t \in]a, b[, \tag{2.33}$$

for some constants L, M. Then u is a BV function of the two variables t, x. Moreover, there exists a set $\mathcal{N} \subset]a, b[$ of measure zero such that, for every $(\tau, \xi) \in]a, b[\times\mathbb{R}$ with $\tau \notin \mathcal{N}$, calling

$$u^+ \doteq \lim_{x\to\xi+} u(\tau, x), \qquad u^- \doteq \lim_{x\to\xi-} u(\tau, x), \tag{2.34}$$

the following holds. There exists a finite speed $\lambda \in \mathbb{R}$ such that the function

$$U(t,x) \doteq \begin{cases} u^- & \text{if } x < \lambda t, \\ u^+ & \text{if } x > \lambda t \end{cases} \tag{2.35}$$

satisfies

$$\lim_{r\to 0+} \frac{1}{r^2} \int_{-r}^{r} \int_{-\lambda^* r}^{\lambda^* r} |u(\tau + t,\ \xi + x) - U(t,x)|\,dx\,dt = 0, \tag{2.36}$$

$$\lim_{r\to 0+} \frac{1}{r} \int_{-\lambda^* r}^{\lambda^* r} |u(\tau + r,\ \xi + x) - U(r,x)|\,dx = 0, \tag{2.37}$$

for every $\lambda^* > 0$.

Proof. To show that the distributional derivatives $D_t u$, $D_x u$ are measures, let $\phi \in C_c^1$ be any function with compact support contained in the strip $]a, b[\times\mathbb{R}$. We then have

$$\left|\iint u\phi_t\,dx\,dt\right| = \left|\lim_{h\to 0} \iint u(t,x)\cdot \frac{\phi(t+h,x) - \phi(t,x)}{h}\,dx\,dt\right|$$
$$= \left|\lim_{h\to 0} \iint \frac{u(t,x) - u(t-h,x)}{h} \cdot \phi(t,x)\,dx\,dt\right|$$

$$\le \int_a^b \left\{ \limsup_{h\to 0} \frac{1}{h} \int_{-\infty}^{\infty} |u(t,x) - u(t-h,x)|\,dx \right\} \cdot \|\phi\|_{C^0}\,dt$$
$$\le (b-a)L \cdot \|\phi\|_{C^0}.$$

Similarly,

$$\left|\iint u\phi_x\,dx\,dt\right| = \left|\lim_{h\to 0} \iint u(t,x)\cdot \frac{\phi(t,x+h) - \phi(t,x)}{h}\,dx\,dt\right|$$
$$= \left|\lim_{h\to 0} \iint \frac{u(t,x) - u(t,x-h)}{h} \cdot \phi(t,x)\,dx\,dt\right|$$
$$\le \int_a^b \left\{ \limsup_{h\to 0} \frac{1}{h} \int |u(t,x) - u(t,x-h)|\,dx \right\} \cdot \|\phi\|_{C^0}\,dt$$
$$\le (b-a)M \cdot \|\phi\|_{C^0}.$$

By the two previous estimates, u is a BV function.

We can now apply Theorem 2.5 to the case where the variable $(y_1, y_2) \doteq (t, x)$ ranges in $\mathbb{R}^2$. This yields the existence of a set $\widetilde{\mathcal{N}} \subset]a, b[\times\mathbb{R}$ of one-dimensional Hausdorff measure zero, such that u either is approximately continuous or has a jump discontinuity at every point $(\tau, \xi) \notin \widetilde{\mathcal{N}}$. Calling

$$\mathcal{N} \doteq \{t;\ (t,x) \in \widetilde{\mathcal{N}} \text{ for some } x \in \mathbb{R}\}$$

the projection of $\widetilde{\mathcal{N}}$ on the t-axis, it is clear that $\mathcal{N}$ has measure zero. Calling $y = (t, x)$ the variable in $\mathbb{R}^2$, at every point $\bar{y} = (\tau, \xi)$ with $\tau \notin \mathcal{N}$ the relations (2.30)–(2.31) hold for some states u^-, u^+ and some unit normal $\mathbf{n}$. In particular, (2.31) implies that (2.36) must hold for every $\lambda^* > 0$. If $u^+ = u^-$, we can trivially define U as in (2.35), choosing $\lambda \doteq 0$. In the case $u^+ \ne u^-$, we claim that $\mathbf{n}$ in (2.30) is not parallel to the t-axis. Indeed, assume that (2.36) holds with

$$U(t,x) \doteq \begin{cases} u^- & \text{if } t < 0, \\ u^+ & \text{if } t > 0. \end{cases} \tag{2.38}$$

By (2.33) the map $t \mapsto u(t,\cdot)$ is Lipschitz continuous w.r.t. the $\mathbf{L}^1$ distance. We thus have the estimate

$$E^* \doteq \limsup_{r\to 0+} \frac{1}{r^2} \int_0^r \int_{-\lambda^* r}^{\lambda^* r} |u(\tau + h, \xi + x) - u(\tau - h, \xi + x)|\,dx\,dh$$
$$\le \limsup_{r\to 0+} \frac{1}{r^2} \int_0^r 2Lh\,dh$$
$$= L. \tag{2.39}$$

On the other hand, by (2.36) it follows that

$$E^* \geq \liminf_{r\to 0+} \frac{1}{r^2} \int_0^r \int_{-\lambda^* r}^{\lambda^* r} |U(h,x) - U(-h,x)|\, dx\, dh$$

$$- \limsup_{r\to 0} \frac{1}{r^2} \int_{-r}^r \int_{-\lambda^* r}^{\lambda^* r} |u(\tau+h, \xi+x) - U(h,x)|\, dx\, dh$$

$$= 2|u^+ - u^-| \cdot \lambda^*. \tag{2.40}$$

Since λ^* can be arbitrarily large, from (2.39) and (2.40) we obtain a contradiction. Hence the function U has the form (2.35) for some $\lambda \in \mathbb{R}$ and some states u^-, u^+.

To show that the right and left states u^+, u^- are precisely given by (2.34), we define

$$v(t,x) \doteq u(\tau+t, \xi+x) - U(t,x),$$

$$E \doteq |u^+ - u(\tau, \xi+)| + |u^- - u(\tau, \xi-)|.$$

Observe that

$$\lim_{r\to 0+} \frac{1}{r} \int_{\xi-r}^{\xi+r} |v(0,x)|\, dx = E. \tag{2.41}$$

Moreover, the number $L' \doteq L + |\lambda|\,|u^+ - u^-|$ provides a Lipschitz constant for v, namely

$$\int_{\mathbb{R}} |v(t,x) - v(s,x)|\, dx \leq L'|t-s| \quad \text{for all } s, t. \tag{2.42}$$

By (2.36) and (2.41)–(2.42) it now follows that

$$0 = \limsup_{r\to 0+} \frac{1}{r^2} \int_{-r}^r \int_{-r}^r |v(t,x)|\, dx\, dt$$

$$\geq \limsup_{r\to 0+} \frac{1}{r^2} \int_0^{(E/L')r} \left(\int_{-r}^r |v(0,x)|\, dx - L'(t-\tau) \right) dt$$

$$= \frac{E^2}{2L'}.$$

Hence $E = 0$, proving (2.34).

The proof of (2.37) relies again on the Lipschitz continuity of v. If (2.37) failed, we could choose a constant $\delta < L'$ such that

$$0 < \delta \leq \limsup_{r\to 0+} \frac{1}{r} \int_{-\lambda^* r}^{\lambda^* r} |u(\tau+r, \xi+x) - U(r,x)|\, dx.$$

From (2.36) and (2.42) it then follows that

$$0 = \limsup_{r\to 0+} \frac{1}{r^2} \int_{-r}^r \int_{-\lambda^* r}^{\lambda^* r} |v(t,x)|\, dx\, dt$$

$$\geq \limsup_{r\to 0+} \frac{1}{r} \int_{r-(\delta/L')r}^{r} \frac{1}{r} \left(\int_{-\lambda^* r}^{\lambda^* r} |v(r,x)|\, dx - L'(r-t) \right) dt$$

$$\geq \frac{\delta^2}{2L'},$$

giving a contradiction. Hence (2.37) must hold. □

Example 2.3. Consider the scalar function (Fig. 2.2)

$$u(t,x) \doteq \begin{cases} 1 & \text{if } 0 < x < \min\{t^2, 1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then u satisfies the assumptions of Theorem 2.6 and hence is a BV function of the two variables t, x. Observe that u is continuous (hence also approximately continuous) at all points outside the two curves

$$\gamma_1 \doteq \{(t,x);\ x = 0\} \qquad \gamma_2 \doteq \{(t,x);\ x = \min\{t^2, 1\}\}.$$

Moreover, u has an approximate jump discontinuity at all points of the curves γ_1, γ_2 except at the origin, where u is approximately continuous, and at the points $P = (1,1)$, $Q = (-1,1)$. In this case, the set of irregular points is $\widetilde{\mathcal{N}} = \{P, Q\}$, which has one-dimensional Hausdorff measure zero.

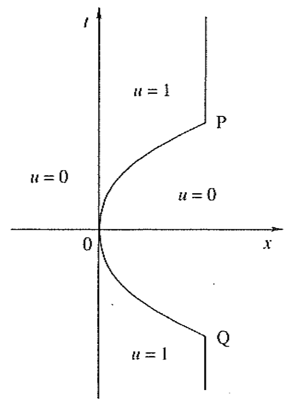

Figure 2.2

emma 2.10. *Let the function $g = g(t, x)$ be measurable in t and Lipschitz continuous x so that*

$$|g(t, x) - g(t, y)| \leq L|x - y| \quad \textit{for all } t, x, y.$$

t x_1, x_2 be two solutions of the differential equation (2.90), defined on a common terval $[t_0, T]$. Then

$$|x_1(t) - x_2(t)| \leq e^{L(t-t_0)} |x_1(t_0) - x_2(t_0)| \qquad t \in [t_0, T]. \tag{2.99}$$

particular, if $x_1(t_0) = x_2(t_0)$ then the two solutions coincide.

oof. Indeed, the absolutely continuous function $z(t) \doteq |x_1(t) - x_2(t)|$ satisfies

$$\dot{z}(t) \leq |\dot{x}_1(t) - \dot{x}_2(t)| \leq Lz(t).$$

plying Lemma 2.9 with $\alpha = L$, $\beta = 0$, $\gamma = |x_1(t_0) - x_2(t_0)|$ we obtain (2.99). □

oblems

) Let $\Phi : \Lambda \times X \mapsto X$ be as in Theorem 2.7. Let $x_0 \in X$ and let $(\lambda_\nu)_{\nu \geq 1}$ be a sequence in Λ, converging to λ_0. Prove that the sequence defined inductively by $x_{\nu+1} = \Phi(\lambda_\nu, x_\nu)$ converges to the unique point x_0, such that $\Phi(\lambda_0, x_0) = x_0$.

) Let $u : \mathbb{R} \mapsto \mathbb{R}^n$ have bounded variation. Consider a non-negative scalar function ϕ such that $\int \phi(y)\, dy = 1$ and define the convolution

$$(\phi * u)(x) = \int_{-\infty}^{\infty} u(x - y)\phi(y)\, dy.$$

Prove that Tot. Var. $(\phi * u) \leq$ Tot. Var. (u).

Let $u : \mathbb{R} \mapsto \mathbb{R}^n$ have bounded variation. Prove that, in addition to Lemma 2.3, the following holds:

$$\text{Tot. Var. } \{u\} = \sup_{\varepsilon > 0} \frac{1}{\varepsilon} \int_{-\infty}^{\infty} |u(x + \varepsilon) - u(x)|\, dx = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{-\infty}^{\infty} |u(x + \varepsilon) - u(x)|\, dx.$$

Let $u : \mathbb{R} \mapsto \mathbb{R}^n$ be right continuous. Show that $u \in BV$ iff

$$\sup \left\{ \int_{-\infty}^{\infty} u(x) \cdot \phi_x(x)\, dx;\ \phi \in \mathcal{C}^1,\ \|\phi(x)\|_{\mathbf{L}^\infty} \leq 1 \right\} < \infty. \tag{2.100}$$

Conversely, assume that a function $u \in \mathbf{L}^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^n)$ satisfies (2.100). Show that u coincides a.e. with a right continuous BV function.

(4) Let $u \in BV(\mathbb{R}; \mathbb{R}^n)$, $\lambda \in L(Du)$, ...

$$\lim_{\varepsilon \to 0} \int_a^b \frac{u(x + \varepsilon\lambda(x)) - u(x)}{\varepsilon}\, dx = \int_a^b \lambda(x)\, Du(x)$$

(it should be proved with similar arguments as for (3), considering the positive and negative variation of u and hence establishing above limit for increasing functions u cfr. p. 14)

(5) For each $\bar{x} \in \mathbb{R}$, denote by $t \mapsto S_t\bar{x}$ the unique strictly increasing solution of the Cauchy problem

$$\dot{x} = \sqrt{|x|}, \qquad x(0) = \bar{x}.$$

Show that S is a continuous semigroup on $\mathbb{R}$, i.e. the map $(t, \bar{x}) \mapsto S_t\bar{x}$ is well defined and continuous on $[0, \infty[\times \mathbb{R}$, and satisfies $S_0\bar{x} = \bar{x}$, $S_sS_t\bar{x} = S_{s+t}\bar{x}$.
In connection with the null solution $w(t) \equiv 0$, for any $\tau > 0$ compute the two quantities

$$|w(\tau) - S_\tau w(0)|, \qquad \int_0^\tau \left\{ \liminf_{h \to 0+} \frac{|w(t + h) - S_h w(t)|}{h} \right\} dt.$$

Compare this result with the statement of Theorem 2.9.

(6) Let $\Psi : \mathbb{R}^m \mapsto \mathbb{R}^n$ be a C^2 mapping with Lipschitz-continuous second derivatives. Assume that, for every $x_1, \ldots, x_m$, the following holds:

$$\Psi(x_1, 0, \ldots, 0) = \Psi(0, x_2, 0, \ldots, 0) = \cdots = \Psi(0, \ldots, 0, x_m) = 0.$$

Prove that, in a neighbourhood of the origin, one has the estimate

$$\Psi(x_1, \ldots, x_m) = \mathcal{O}(1) \cdot \sum_{i \neq j} |x_i x_j|.$$

Hint: observe that $f(x_1, \ldots, x_n) = [f(x_1, \ldots, x_n) - f(x_1, \ldots, x_{n-1}, 0)] + f(x_1, \ldots, x_{n-1}, 0)$. Use induction on n.

(7) Under the same assumptions as in Theorem 2.9, let $v : [0, T] \mapsto \mathcal{D}$ be a piecewise Lipschitz-continuous map, with jumps at the times $0 < t_1 < \cdots < t_m < T$. Prove the estimate

$$\|v(T) - S_T v(0)\| \leq L \cdot \int_0^T \left\{ \liminf_{h \to 0+} \frac{\|v(t + h) - S_h v(t)\|}{h} \right\} dt + L \cdot \sum_{i=1}^m \|v(t_i+) - v(t_i-)\|. \tag{2.101}$$

(3)' Let $u \in BV(\mathbb{R}; \mathbb{R}^n)$, $a, b, \lambda, \varepsilon \in \mathbb{R}$, $a < b$. Then one has

$$\frac{1}{|\lambda - \varepsilon|} \int_a^b |u(x - \lambda) - u(x - \varepsilon)|\, dx \leq \text{Tot. Var.} \{u; [a - \max\{|\lambda|, |\varepsilon|\}, b + \max\{|\lambda|, |\varepsilon|\}]\}$$

(cfr. Bressan & Lewicka, Shift differential of maps in BV space, Lemma 4, p. 12)