# Explicit Formula for Scalar Non-linear Conservation Laws with Boundary Condition

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We prove an uniqueness and existence theorem for the entropy weak solution of non-linear hyperbolic conservation laws of the form

$$\frac{\partial}{\partial t}u+\frac{\partial}{\partial x}f(u)=0,$$

with initial data and boundary condition. The scalar function $u=u(x,t)$, $x>0$, $t>0$, is the unknown; the function $f=f(u)$ is assumed to be strictly convex. We also study the weighted Burgers' equation: $\alpha\in\mathbb{R}$

$$\frac{\partial}{\partial t}(x^\alpha u)+\frac{\partial}{\partial x}\left(x^\alpha\frac{u^2}{2}\right)=0.$$

We give an explicit formula, which generalizes a result of Lax. In particular, a free boundary problem for the flux $f(u(.,.))$ at the boundary is solved by introducing a variational inequality. The uniqueness result is obtained by extending a semigroup property due to Keyfitz.

## 1. Introduction

We consider *scalar non-linear hyperbolic conservation laws* of the form

$$\frac{\partial}{\partial t}u+\frac{\partial}{\partial x}f(u)=0, \tag{1.1}$$

where the scalar function $u=u(x,t)$, $x>0$, $t>0$, is the unknown. The flux function $f=f(u)$ is assumed to be strictly convex (with $\lim_{|u|\to\infty} f(u)/|u|=+\infty$).

This paper is concerned with *the mixed problem* associated with equation (1.1): we are looking for a solution $u=u(x,t)$ of (1.1), satisfying initial data and a boundary condition. But, it is well known[3,5], that conservation laws of the type (1.1) do not possess classical solutions, even when the initial data are smooth: discontinuities appear in finite time. Hence, we consider only weak solutions of (1.1), that is solutions in the sense of distributions. And, for the sake of uniqueness, we have to add *an entropy condition* that selects the physical (or entropy) solution among all the

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solutions in the sense of distributions. For a flux function $f(.)$ convex, the entropy condition is written as[4,7]

$$u(x-0,t)\geqslant u(x+0,t), \quad x>0, t>0. \tag{1.2}$$

The main difficulty for an existence and uniqueness theory for equation (1.1) is to have *a good formulation of the boundary condition*. Namely, whereas we fix an initial condition as

$$u(x,0)=u_0(x), \quad x>0, \tag{1.3}$$

with a given function $u_0(.)$, we really cannot impose such a condition at the boundary. The boundary condition is necessarily linked to the entropy condition. We will establish that the functions $f(u(.,.))$ and $\operatorname{sgn} f'(u(.,.))$ possess traces at the boundary in a weak sense. And, we will follow ideas of Bardos, Leroux and Nedelec[1] (see equation 2.3).

To obtain an existence result concerning the entropy weak solution of the conservation laws (1.1) with initial data and boundary condition, *we extend the explicit representation* derived by Lax,[4] for conservation laws of the form (1.1) without boundary data, that is that the variable $x$ described $\mathbb{R}$.

Furthermore, for the sake of uniqueness, we establish an $L^1$-*semigroup property* in the class of piecewise regular solutions, which generalizes a previous result of Keyfitz.[2] If $u(.,.)$ and $v(.,.)$ are entropy weak solutions associated with (integrable) initial data $u_0(.)$ and $v_0(.)$ and (bounded) boundary data $\bar{u}_0(.)$ and $\bar{v}_0(.)$ (see equation (2.3)), we show that

$$\int_0^{\infty} |u(x,t)-v(x,t)|\,\mathrm{d}x \leqslant \int_0^{\infty} |u_0(x)-v_0(x)|\mathrm{d}x + \int_0^t |f(\bar{u}_0(s))-f(\bar{v}_0(s))|\,\mathrm{d}x, \quad t>0, \tag{1.4}$$

where we suppose—it is not a restriction—that $f'(\bar{u}_0(.))$ and $f'(\bar{v}_0(.))$ are positive functions.

Note that we have to solve *a free boundary problem*: an explicit formula is derived for the flux $f(u(.,.))$ at the boundary by using a *variational inequality*. This inequality determines at point $x=0$ and times $t>0$ if the value $u(0+,t)$ is, in particular, incoming or outgoing (see Theorem 2.3).

Then, in Section 3, we study an interesting model weighted equation, *the weighted Burgers' equation*, which appears for problems with spherical or cylindrical symmetry:[10]

$$\frac{\partial}{\partial t}(x^{\alpha}u(x,t))+\frac{\partial}{\partial x}\left(x^{\alpha}\frac{u(x,t)^2}{2}\right)=0, \quad x>0,\ t>0. \tag{1.5}$$

We show that a straightforward change of unknown function leads to an exact solution of this equation, by using the previous results of Section 2.

We refer to Reference 6 for an explicit formula for weighted conservation laws with singularity. A different approach can be found in the recent paper of Schonbek,[8] which uses the classical viscosity method and the theory of compensated compactness to obtain an existence result for conservation laws with singularity.

## 2. Scalar conservation laws in the half-space

### 2.1 *Formulation of the problem*

Let $f=f(u)$ be a continuously differentiable strictly convex function and take $a(.)=f'(.)$ and $b(.)=a^{-1}(.)$. Let $u_0=u_0(x)$ be the initial data and $\bar{u}_0=u_0(t)$ be the boundary condition (whose regularity will be specified).

Consider the following scalar mixed non-linear hyperbolic problem with the unknown $u=u(x,t)$:

$$\frac{\partial}{\partial t}u+\frac{\partial}{\partial x}f(u)=0, \quad u=u(x,t), \quad x>0,\ t>0; \tag{2.1}$$

$$u(x,0)=u_0(x), \text{ a.e. } x>0; \tag{2.2}$$

$$\left.\begin{array}{ll} & u(0,t)=\bar{u}_0(t) \quad \text{and} \quad a(\bar{u}_0(t))\geqslant 0, \\ \text{or} & a(u(0,t))\leqslant 0 \quad \text{and} \quad a(\bar{u}_0(t))\leqslant 0, \\ \text{or} & a(u(0,t))\leqslant 0, \quad a(\bar{u}_0(t))\geqslant 0 \quad \text{and} \quad f(u(0,t))\geqslant f(\bar{u}_0(t)), \quad \text{a.e. } t>0; \end{array}\right\} \tag{2.3}$$

$$u(x-0,t)\geqslant u(x+0,t), \quad x>0, \quad t>0. \tag{2.4}$$

In this section, we give a uniqueness and existence result for this problem, based on an explicit formula and an $L^1$-contraction semigroup property. In particular, we show that the boundary condition (2.3) is correct for the equation (2.1).

For motivations about the boundary condition (2.3), refer to the paper of Bardos, Leroux and Nedelec,[1] who have obtained the formulation (by studying the classical viscosity method for general quasi-linear first-order hyperbolic equations in several space variables) as follows:

$$\sup_{k\in I(u(0,t);\,\bar{u}_0(t))}\{\operatorname{sgn}(u(0,t)-k)(f(u(0,t))-f(k))\}=0, \quad \text{a.e. } t>0,$$

with $I(u(0,t);\bar{u}_0(t))=[\min\{u(0,t);\bar{u}_0(t)\};\max\{u(0,t);\bar{u}_0(t)\}]$. It is not difficult to see that this condition is equivalent to (2.3) when the flux function $f(.)$ is strictly convex. Moreover, it is not a restriction to suppose that the boundary data $\bar{u}_0(.)$ is always incoming, that is $a(\bar{u}_0(.))\geqslant 0$. Under this assumption, *the boundary condition* (2.3) can be rewritten as

$$\left.\begin{array}{ll} & u(0,t)=\bar{u}_0(t), \\ \text{or} & \\ & a(u(0,t))\leqslant 0 \quad \text{and} \quad f(u(0,t))\geqslant f(\bar{u}_0(t)), \quad \text{a.e. } t>0. \end{array}\right\} \tag{2.3'}$$

Henceforth, the boundary data will now be assumed to be always incoming and the formulation (2.3′), equivalent to (2.3), will be used.

### 2.2 *Existence, uniqueness and explicit representation*

**Theorem 2.1. (Existence).** *Assume that the function* $f(.)$ *satisfies* $\lim_{|u|\to+\infty} f(u)/|u|=+\infty$ *and that the initial data* $u_0(.)$ *and the boundary data* $\bar{u}_0(.)$ (*with* $\bar{u}_0(.)\geqslant b(0)$) *are* *measurable and bounded. Then, there exists a function $u(.,.)$, piecewise continuous having left and right limits at each point with respect to the variables x and t and possessing at the points $x=0$ and $t=0$ bounded measurable traces, which satisfies the properties* (2.1)–(2.4). *Moreover, the solution $u(.,.)$ is bounded as follows:*

$$\|u(.,.)\|_{\infty} \leqslant \sup\{\|u_0(.)\|_{\infty}, \|\bar{u}_0(.)\|_{\infty}\}. \tag{2.5}$$

Here, the traces of the function $u(.,.)$, at $t=0$ and at $x=0$, exist in the following weak sense: there exist two sets $\mathscr{E}$ and $\mathscr{F}$ with zero measure such that

$$\lim_{\substack{t\to 0_- \\ t\notin\mathscr{E}}} \int_0^x u(\xi,t)\,d\xi = \int_0^x u_0(\xi)\,d\xi, \quad x\geqslant 0; \tag{2.6}$$

$$\lim_{\substack{x\to 0_- \\ x\notin\mathscr{F}}} \int_0^t f(u(x,s))\,ds = \int_0^t \Upsilon(s)\,ds, \quad t\geqslant 0, \tag{2.7a}$$

where we will denote by $\Upsilon(.)$ the trace of the function $f(u(.,.))$ at $x=0$. Moreover, the function sgn $a(u(.,.))$ admits a measurable trace $\varepsilon(.)$ at $x=0$

$$\lim_{\substack{x\to 0^- \\ x\notin\mathscr{F}}} \operatorname{sgn} a(u(x,t)) = \varepsilon(t), \quad \text{a.e. } t>0. \tag{2.7b}$$

And, with this notation, the boundary condition (2.3) is

$$\left\{\begin{array}{lll} \Upsilon(t)=f(\bar{u}_0(t)) & \text{and} & \varepsilon(t)=+1, \\ \Upsilon(t)\geqslant f(\bar{u}_0(t)) & \text{and} & \varepsilon(t)=-1, \quad \text{a.e. } t>0. \end{array}\right\} \tag{2.3''}$$

**Remark 2.1.** With suitable regularity of the data, one can prove that the solution $u(.,.)$ given by Theorem 2.1 is more regular. And, with such assumptions, the previous properties (2.6) and (2.7) are equivalent to (2.2) and (2.3) respectively.

The uniqueness result concerning the problem (2.1)–(2.4) is specified by a *semigroup property* in $L^1$-space.

**Theorem 2.2. (Uniqueness).** *Let $u(.,.)$ and $v(.,.)$ be piecewise continuously differentiable solutions of the problem* (2.1)–(2.4), *associated with integrable initial data $u_0(.)$ and $v_0(.)$ and bounded boundary data $\bar{u}_0(.)$ and $\bar{v}_0(.)$, respectively. Then, for $0\leqslant t_1\leqslant t_2$, we have*

$$\int_0^{+\infty} |u(x,t_2)-v(x,t_2)|\,dx \leqslant \int_0^{+\infty} |u(x,t_1)-v(x,t_1)|\,dx + \int_{t_1}^{t_2} |f(\bar{u}_0(t)-f(\bar{v}_0(t))|\,dt. \tag{2.8}$$

Finally, let $g=g(v)$ be the Legendre transform of the function $f(.)$:

$$g(v)=\sup_{u\in\mathbb{R}}(uv-f(u)), \quad v\in\mathbb{R}. \tag{2.9}$$

One can show that it is convex and

$$g'(.)=b(.); \quad g(a(v))=a(v)v-f(v), \quad v\in\mathbb{R}. \tag{2.9'}$$

Then, for every bounded measurable function $\Upsilon = \Upsilon(t)$, the function $G = G(x, t; y)$, $x > 0$, $t > 0$, $y \in \mathbb{R}$, is defined as follows:

$$G(x,t;y) = \begin{cases} \displaystyle\int_0^y u_0(\xi)\,d\xi + tg\left(\frac{x-y}{t}\right), & y \geqslant 0, \\ \displaystyle -\int_0^\tau \Upsilon(s)\,ds + (t-\tau)g\left(\frac{x}{t-\tau}\right), & y \leqslant 0, \end{cases} \tag{2.10a}$$

where, for each non-positive number $y$, the number $\tau$ is given by

$$\frac{x-y}{t} = \frac{x}{t-\tau}. \tag{2.10b}$$

Note that $\tau$ belongs to the interval $[0; t[$. Note also that the formula (2.10a) has a sense with $x = 0$ and $y \geqslant 0$, which does not depend on the function $\Upsilon(.)$.

Then, the solution $u = u(u, t)$ given by Theorem 2.1 admits the following *explicit representation.*

**Theorem 2.3.** (**Explicit formula**).

A. Characterization of the boundary-flux $\Upsilon(.)$.

(A1) *For any positive number $t$, let $y = y(t)$ be a point which minimizes the function:* $[0; +\infty[ \ni y \mapsto G(0, t; y)$.

(A2) *Let $m = m(t)$, $t > 0$, be the unique continuous function, almost everywhere differentiable such that*

$$\left.\begin{aligned} & m(0+) = 0+, \\ & \left\{\frac{d}{dt} m(t) - f(\bar{u}_0(t)) + f\left(b\left(\frac{-y(t)}{t}\right)\right)\right\} \times \{\varphi - m(t)\} \geqslant 0, \quad \forall \varphi \geqslant 0, \quad \text{a.e. } t > 0 \end{aligned}\right\}, \tag{2.11}$$

(A3) *The function $\Upsilon = \Upsilon(t)$ is defined by*

$$\Upsilon(t) = \frac{d}{dt} m(t) + f\left(b\left(\frac{-y(t)}{t}\right)\right), \quad \text{a.e. } t > 0. \tag{2.12}$$

B. Explicit representation inside. *We have*

$$u(x,t) = b\left(\frac{x - y(x,t)}{t}\right), \quad x > 0, \quad t > 0, \tag{2.13}$$

*where the point $y(x, t)$ realizes the minimum value of the function $G(x, t; .)$ defined by* (2.10) *with $\Upsilon(.)$ given by* (2.12).

Moreover, by using this explicit formula, we can specify the entropy condition (2.4) for a solution $u = u(x, t)$ given by Theorem 2.1.

**Proposition 2.1. (Entropy condition).** *There exists a positive constant k, depending only on $\|u_0\|_\infty$ and $a(.)$ such that*

$$\frac{u(x_2,t)-u(x_1,t)}{x_2-x_1} \leqslant \frac{k}{t-\tau(x_1,t)}, \quad t>0, \quad 0<x_1<x_2, \tag{2.14a}$$

*with $\tau=\tau(x,t)$ given by*

$$\frac{x-y(x,t)}{t}=\frac{x}{t-\tau(x,t)}, \quad x>0, \quad t>0. \tag{2.14b}$$

At the boundary ($x=0$), the difficulty stems from the fact that the value $u=u(0,t)$ can be either incoming, that is $\operatorname{sgn}(a(u(0,t)))=+1$ or outgoing, that is $\operatorname{sgn}(a(u(0,t)))=-1$. According as the value is incoming or outgoing, the boundary condition (2.3) is different; so, there is *a free boundary problem* at $x=0$. Here, we resolve this problem by introducing the function $m(.)$ that satisfies *the variational inequality* (2.11). Note that the inequality (2.11) is equivalent to the assertion

$$\left.\begin{aligned} & m(t)>0 \text{ and } \frac{\mathrm{d}}{\mathrm{d}t}m(t)=f(\bar{u}_0(t))-f\left(b\left(\frac{-y(t)}{t}\right)\right), \\ \text{or} \\ & m(t)=0 \text{ and } \frac{\mathrm{d}}{\mathrm{d}t}m(t)\geqslant f(\bar{u}_0(t))-f\left(b\left(\frac{-y(t)}{t}\right)\right), \quad \text{a.e. } t>0. \end{aligned}\right\} \tag{2.11'}$$

The plan of the proofs of the previous results is the following one: first, the explicit formula with boundary condition (Theorem 2.3) is derived; then we prove that this formula leads to a solution of the problem (2.1)–(2.4) (Theorem 2.1), which satisfies the property (2.14) (Proposition 2.1); finally, we show the $L^1$-semigroup property (Theorem 2.2) (compare with References 2 and 5).

## 2.3. *First step: derivation of the explicit formula*

Here we will suppose that the hypotheses of Theorem 2.1 are satisfied. Let $u(.,.)$ be a (sufficiently regular) solution of (2.1)–(2.4) and consider the function $U=U(\xi,s)$ given by

$$U(\xi,s)=\int_0^\xi u(x,s)\,\mathrm{d}x, \quad \xi>0, \quad s>0.$$

By integration of the equation (2.1) on the interval $[0;\xi]$, we obtain

$$0=U_s(\xi,s)+f(U_\xi(\xi,s))-\Upsilon(s),$$

with $\Upsilon(s)=f(u(0,s))$. But, the flux function being convex, we have for $\bar{u}$ and $\bar{v}$ in $\mathbb{R}$

$$f(\bar{u})\geqslant f(\bar{v})+a(\bar{v})(\bar{u}-\bar{v}).$$

So, taking $\bar{u}=U_\xi(\xi,s)$, we deduce the following inequality

$$U_s(\xi,s)+a(\bar{v})U_\xi(\xi,s)\leqslant a(\bar{v})\bar{v}-f(\bar{v})+\Upsilon(s), \tag{2.15}$$

for all positive numbers $\xi$ and $s$ and by any real number $v$.

**2.3.1. Explicit formula inside the domain.** Let $x, t$ be positive. For $\bar{v}$ belonging to $\mathbb{R}$, we distinguish between two cases:

(a) If $-\infty < a(\bar{v}) \leqslant x/t$, then we integrate the inequality (2.15) along the line joining the point $(y, 0) = (x - ta(\bar{v}), 0)$ to the point $(x, t)$. Thus, we have

$$U(x,t) - U(y,0) \leqslant t(a(\bar{v})\bar{v} - f(\bar{v})) + \int_0^t \Upsilon(s)\,ds.$$

Using the property of (2.9′) of the Legendre transformation $g(.)$ defined in (2.9), and since

$$a(\bar{v}) = \frac{x-y}{t},$$

we deduce

$$a(\bar{v})\bar{v} - f(\bar{v}) = g\left(\frac{x-y}{t}\right).$$

But, when $a(\bar{v})$ describes the interval $]-\infty; x/t]$, the point $y$ describes $[0, \infty[$; hence, the following inequality holds:

$$U(x,t) - \int_0^t \Upsilon(s)\,ds \leqslant \int_0^y u_0(\xi)\,d\xi + tg\left(\frac{x-y}{t}\right), \tag{2.16}$$

for any non-negative number $y$.

(b) If $a(\bar{v}) \geqslant x/t > 0$, then we integrate the inequality (2.15) along the line joining the point $(0, \tau) = (0, t - (x/a(\bar{v}))$ to the point $(x, t)$. Thus, we have

$$U(x,t) - U(0,\tau) \leqslant (t-\tau)(a(\bar{v})\bar{v} - f(\bar{v})) + \int_\tau^t \Upsilon(s)\,ds.$$

When $a(\bar{v})$ describes the interval $[x/t; +\infty[$, the point $\tau$ describes $[0; t[$, so the following inequality holds:

$$U(x,t) - \int_0^t \Upsilon(s)\,ds \leqslant -\int_0^\tau \Upsilon(s)\,ds + (t-\tau)\,g\left(\frac{x}{t-\tau}\right), \tag{2.17}$$

for any $\tau$ belonging to $[0; t[$.

Now, we define the function $G(.,.,..)$ by (2.10). And, using (2.16) and (2.17), we conclude that the following inequality holds:

$$U(x,t) - \int_0^t \Upsilon(s)\,ds \leqslant G(x,t;y), \tag{2.18}$$

for all positive numbers $x$ and $t$ and any real $y$.

Finally, we see that, for the value of $y$ for which $\bar{v}$ equals $u(x,t)$, the inequality holds in (2.18) along the whole line over which we integrate to obtain (2.16) or (2.17), and thus the equality also holds in (2.18). Hence, from that fact, we deduce that the value $u(x,t)$ is obtained by minimizing the function $G(x,t;.)$. So, we have obtained the formula B of Theorem 2.3.

**2.3.2. Explicit formula at the boundary.** Let $t$ be positive; the previous inequalities (2.16) and (2.17) hold when $x$ equals zero:

$$0 \leqslant \int_0^y u_0(\xi)\,d\xi + tg\left(\frac{-y}{t}\right) + \int_0^t \Upsilon(s)\,ds, \quad \forall\, y \geqslant 0, \tag{2.16'}$$

and

$$0 \leqslant (t-\tau)g(0) + \int_\tau^t \Upsilon(s)\,ds, \quad \forall\, \tau \in [0; t[. \tag{2.17'}$$

By the same arguments as previously in subsection 2.3.1, we know that the values $u(0,t)$, $t>0$, are obtained by minimizing the right members of inequalities (2.16′) and (2.17′). Moreover, note that the equality holds in (2.17′) with $\tau=t$; and if the value is outgoing, that is $a(u(0,t)) \leqslant 0$, we necessarily have

$$u(0,t) = b\left(\frac{-y(t)}{t}\right),$$

with $y(t)$ minimizing: $y \geqslant 0 \to G(0,t;y)$.

Let us define the function $m = m(t)$ by

$$m(t) = \int_0^t \Upsilon(s)\,ds + \min_{y \geqslant 0} G(0,t;y), \quad t>0. \tag{2.19}$$

And, for each positive number $t$, take a point $y(t) \geqslant 0$ minimizing the function: $y \geqslant 0 \mapsto G(0,t;y)$. Then, the functions $y(.)$ and $m(.)$ satisfy the following properties:

**Proposition 2.2.** *The function $y=y(t)$ is non-decreasing (thus, it is 'well-defined' almost everywhere). The function $m(.)$ is non-negative, continuous, almost everywhere differentiable, equal to zero at $t=0$, and its derivative is*

$$\frac{d}{dt} m(t) = \Upsilon(t) - f\left(b\left(\frac{-y(t)}{t}\right)\right), \tag{2.20}$$

*for almost every number $t$.*

Before proving this result, we now deduce the formula A from the equalities (2.19) and (2.20) and the boundary condition (2.3′). For any $t>0$, we distinguish two cases:

1. If $m(t)>0$, then, because of the definition (2.19), the equality does not hold in (2.16′); thus, $a(u(0,t))$ is positive and by virtue of the condition (2.3′), we deduce $u(0,t)=\bar{u}_0(t)$, and so

$$\frac{d}{dt} m(t) = f(\bar{u}_0(t)) - f\left(b\left(\frac{-y(t)}{t}\right)\right). \tag{2.21}$$

2. If $m(t)=0$, then, the function $m(.)$ being non-negative, we have

$$\frac{d}{dt} m(t) \geqslant 0.$$

That is (using (2.20))

$$\Upsilon(t) \geqslant f\left(b\left(\frac{-y(t)}{t}\right)\right). \tag{2.22}$$

Again, distinguish between two cases:

(a) If $(\mathrm{d}/\mathrm{d}t)\, m(t)>0$, then the inequality is strict in (2.22), and we deduce from (2.3′) that

$$u(0,t)=\bar{u}_0(t);$$

and again we obtain (2.21).

(b) If $(\mathrm{d}/\mathrm{d}t)\, m(t)=0$, then with (2.22) we have

$$\Upsilon(t)=f\left(b\left(\frac{-y(t)}{t}\right)\right)$$

and, using the boundary condition (2.3′), we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\, m(t)-f(\bar{u}_0(t))+f\left(b\left(\frac{-y(t)}{t}\right)\right)\geqslant 0.$$

So the variational inequality (2.11) is proved; and (2.12) results from (2.20).

So, we have obtained the formula A of Theorem 2.3.

*Proof of Proposition* 2.2. We show that the function $y(.)$ is non-decreasing, so it is continuous everywhere except eventually at a few points belonging to a countable set. Thus, for almost every $t$ positive, the minimum in (2.19) holds for a unique point $y=y(t)$, and this function is 'well-defined' almost everywhere.

Take $0<t_1<t_2$, and denote by

$$y_1=y(t_1) \quad \text{and} \quad y_2=y(t_2).$$

To prove $y_1\leqslant y_2$, it is sufficient to show that

$$G(0,t_2;y_1)<G(0,t_2;y)$$

for any number $y$ in $[0;y_1[$. This inequality results from

$$G(0,t_2;y_1)+G(0,t_1;y)<G(0,t_2;y)+G(0,t_1;y_1),$$

which is equivalent to

$$\frac{g\left(\frac{-y}{t_1}\right)-g\left(\frac{-y_1}{t_1}\right)}{\left(\frac{-y}{t_1}\right)-\left(\frac{-y_1}{t_1}\right)}<\frac{g\left(\frac{-y}{t_2}\right)-g\left(\frac{-y_1}{t_2}\right)}{\left(\frac{-y}{t_2}\right)-\left(\frac{-y_1}{t_2}\right)}. \tag{2.23}$$

But the points $-y/t_1$, $-y_1/t_1$, $-y/t_2$ and $-y_1/t_2$ satisfy

$$\frac{-y}{t_1}\geqslant\frac{-y_1}{t_1},\quad \frac{-y}{t_2}\geqslant\frac{-y_1}{t_2},\quad \frac{-y}{t_2}>\frac{-y}{t_1},\quad \frac{-y_1}{t_1}<\frac{-y_1}{t_2}.$$

So that (2.23) is Jensen's inequality (it expresses the strict convexity of $g(.)$).

By virtue of the inequality (2.16′), the function $m(.)$ is non-negative. Moreover, the continuity of $m(.)$ is a classical fact, because the function $G(0,t;y)$ is regular.

In (2.19), the first term $\int_0^t \Upsilon(s)\,\mathrm{d}s$ tends to zero with $t$, and the second is

$$\min_{y\geq 0} G(0,t;y)\leq G(0,t;t)=\int_0^t u_0(\xi)\,\mathrm{d}\xi+tg(-1),$$

which also tends to zero. The function $m(.)$ being non-negative, we conclude

$$\lim_{t\to 0+} m(t)=0+.$$

Let $t$ be given positive and such that the function $y(.)$ is continuous at $t$. We now compute $(\mathrm{d}/\mathrm{d}t)\,m(t)$. (At a point of discontinuity, consider $y(t-0)$ or $y(t+0)$ and also $(\mathrm{d}/\mathrm{d}t)\,m(t-0)$ and $(\mathrm{d}/\mathrm{d}t)\,m(t+0)$, and the same results are valid). In (2.19), the first term is differentiable and its derivative is $\Upsilon(t)$ which is the first term of the right member of (2.20), so it is sufficient to show that

$$\Omega(h)=\frac{1}{h}\left[G(0;t+h,y(t+h))-G(0,t;y(t))\right]+f\left(b\left(\frac{-y(t)}{t}\right)\right) \tag{2.24}$$

tends to zero with $h$. Namely, we have first

$$\Omega(h)\leq\frac{1}{h}\left[G(0,t+h,y(t))-G(0,t;y(t))\right]+f\left(b\left(\frac{-y(t)}{t}\right)\right);$$

clearly, the right member of this inequality tends to

$$\frac{\partial G}{\partial t}(0,t;y(t))+f\left(b\left(\frac{-y(t)}{t}\right)\right)$$
$$=g\left(\frac{-y(t)}{t}\right)+b\left(\frac{-y(t)}{t}\right)\frac{y(t)}{t}+f\left(b\left(\frac{-y(t)}{t}\right)\right)=0.$$

Secondly, we have

$$\Omega(h)\geq\frac{1}{h}\left[G(0;t+h,y(t+h))-G(0,t;y(t+h))\right]+f\left(b\left(\frac{-y(t)}{t}\right)\right);$$

and the right member of this inequality tends to

$$\frac{\partial G}{\partial t}(0,t;y(t))+f\left(b\left(\frac{-y(t)}{t}\right)\right)=0,$$

by using the continuity of $y(.)$ at $t$.

## 2.4. *Second step: existence of a solution*

Under the assumptions of Theorem 2.1, we now show that the formulae A and B of Theorem 2.3 define a function $u(.,.)$ and we specify its regularity; then we prove that this function is a solution of the problem (2.1)–(2.4).

**2.4.1. Existence of a function given by the explicit formula.** In each formula A or B the existence of (at least) a point $y(t)$ (respectively $y(x,t)$) minimizing the function: $y\geq 0\mapsto G(0,t;y)$ (respectively $G(x,t;.)$) results from the facts that the function $g(.)$ is convex with

$$\lim_{|u|\to+\infty} \frac{g(u)}{|u|}=+\infty$$

and the initial data are bounded (so $(1/|y|)\int_0^y u_0(\xi)\,d\xi$ remains bounded for large $y$).

Thus, for defining a function $u(.,.)$ by A and B, it suffices to remark that

**Proposition 2.3.** *There exists one and only one positive function $m=m(t)$, continuous and having a bounded measurable derivative, which verifies the variational inequality* (2.11). *Moreover, we have*

$$f(b(0))\leqslant\frac{d}{dt}m(t)+f\left(b\left(\frac{-y(t)}{t}\right)\right)\leqslant\max\{\|f(u_0(.))\|_\infty,\|f(\bar{u}_0(.))\|_\infty\}.$$

The proof is classical; we omit it.

We recall that the function $y=y(t)$, given by (A 1), is non-decreasing (see Proposition 2.2), and thus piecewise continuous.

Finally, in order to prove that the point $y(x,t)$ minimizing the function $G(x,t;.)$ is unique almost everywhere in $x,t$, we verify that, for $t$ positive, the function $y(.,t)$ is piecewise continuous. Henceforth, the function $u(.,.)$ given by A and B, is defined almost everywhere. These facts result immediately from the following proposition.

**Proposition 2.4.** *Let $t$ be positive and take $x_0(t)\geqslant 0$ a solution of*

$$y(x_0(t),\ t)=0.$$

*Then,*

1. *The function:* $\begin{cases}[x_0(t);+\infty[\longrightarrow[0;+\infty[\\ x\longmapsto y(x,t),\end{cases}$

   *is non-decreasing;*

2. *The function:* $\begin{cases}[0;x_0(t)]\longrightarrow[0;t[\\ x\longmapsto \tau(x,t),\end{cases}$

   *is non-increasing.*

Here, the point $\tau(x,t)$ is defined by the relation (2.14b).

*Proof of Proposition* 2.4. First, we take $x_0\leqslant x_1<x_2$ and we show that for all $y$ belonging to $[0;\ y_1[$, with $y_1=y(x_1,t)$, we have

$$G(x_2,t;y_1)<G(x_2,t;y); \tag{2.25}$$

then, we can deduce that $y(x_2,t)\geqslant y(x_1,t)$. Namely, the inequality (2.25) results from

$$G(x_2,t;y_1)+G(x_1,t;y)<G(x_2,t;y)+G(x_1,t;y_1),$$

which is equivalent to the following one:

$$g\left(\frac{x_2-y_1}{t}\right)+g\left(\frac{x_1-y}{t}\right)<g\left(\frac{x_2-y}{t}\right)+g\left(\frac{x_1-y_1}{t}\right). \tag{2.26}$$

This is exactly Jensen's inequality:

$$\frac{g\left(\frac{x_1-y}{t}\right)-g\left(\frac{x_2-y_1}{t}\right)}{\frac{x_1-y}{t}-\frac{x_2-y}{t}}<\frac{g\left(\frac{x_1-y}{t}\right)-g\left(\frac{x_2-y_1}{t}\right)}{\frac{x_1-y}{t}-\frac{x_2-y}{t}}.$$

Secondly, we take $0\leqslant x_1<x_2\leqslant x_0$; and we show that, for all $\tau$ belonging to $]\tau_1;t]$ with $\tau_1=\tau(x_1,t)$, we have

$$G(x_1,t;\tau_1)<G(x_1,t;\tau). \tag{2.27}$$

Here, we write, for example, $G(x_1,\ t;\ \tau_1)$ instead of $G(x_1,t;y(x_1,t))$. Then, we can deduce that

$$\tau(x_1,t)\geqslant\tau(x_2,t).$$

Namely, the inequality (2.27) is a result of

$$G(x_1,t;\tau_1)+G(x_2,t;\tau)<G(x_1,t;\tau)+G(x_2,t;\tau_1),$$

which is equivalent to

$$(t-\tau_1)\,g\left(\frac{x_1}{t-\tau_1}\right)+(t-\tau)\,g\left(\frac{x_2}{t-\tau}\right)<(t-\tau)\,g\left(\frac{x_1}{t-\tau}\right)+(t-\tau_1)\,g\left(\frac{x_2}{t-\tau_1}\right),$$

that is

$$\frac{g\left(\frac{x_2}{t-\tau}\right)-g\left(\frac{x_1}{t-\tau}\right)}{\frac{x_2}{t-\tau}-\frac{x_1}{t-\tau}}<\frac{g\left(\frac{x_2}{t-\tau_1}\right)-g\left(\frac{x_1}{t-\tau_1}\right)}{\frac{x_2}{t-\tau_1}-\frac{x_1}{t-\tau_1}}, \tag{2.28}$$

which is exactly Jensen's inequality.

Moreover, we specify the regularity of the function $u(.,.)$ as follows:

**Proposition 2.5.**

1. *Let $x$ be positive and take $t_0\geqslant 0$ a solution of $y(x,t_0)=0$. Then, the function $y(x,.)$ is non-decreasing (respectively non-increasing) for $t\in[0;\ t_0]$ (respectively $t\in[t_0;\ +\infty]$).*
2. *The function $u(.,.)$, given by* A *and* B, *is piecewise continuous having left and right limits at any point with respect to the variables $x$ and $t$. Furthermore, it is a bounded function that satisfies (2.5).*

*Proof of Proposition* 2.5. The first part is analogous to the proof of Proposition 2.4, by using the following inequality:

$$\frac{g\left(\frac{x-y}{t_1}\right)-g\left(\frac{x-y_1}{t_1}\right)}{\frac{x-y}{t_1}-\frac{x-y_1}{t_1}}<\frac{g\left(\frac{x-y}{t_2}\right)-g\left(\frac{x-y_1}{t_2}\right)}{\frac{x-y}{t_2}-\frac{x-y_1}{t_2}}, \tag{2.29}$$

with $0<t_1<t_2$, $0<x$ and $0\leqslant y<y_1$.

The piecewise regularity of the function $u(.,.)$ immediately results from the monotonicity properties of the function $y(.,.)$ and from (2.13). To obtain the majorization (2.5), let $(x, t; y)$ be a point such that

$$\left| b\left(\frac{x-y}{t}\right)\right| \geqslant \sup(\|u_0\|_\infty, \|\bar{u}_0\|_\infty).$$

Then, for any number $y'$ such that

$$\left|\frac{x-y'}{t}\right| >_{\neq} \left|\frac{x-y}{t}\right|,$$

we show the inequality

$$G(x, t; y') >_{\neq} G(x, t; y). \tag{2.30}$$

From this fact, we immediately conclude that (2.5) holds because of the minimization property of the point $y(x, t)$.

Namely, we now prove (2.30), when $y$ and $y'$ are, for example, positive (when $y$ or $y'$ are negative, the computation is similar):

$$\begin{aligned} G(x, t; y') - G(x, t; y) &= \int_y^{y'} u_0(\xi)\, d\xi + t\left(g\left(\frac{x-y'}{t}\right) - g\left(\frac{x-y}{t}\right)\right) \\ &>_{\neq} -|y'-y| \times \|u_0\|_\infty + t\left| b\left(\frac{x-y}{t}\right)\right| \times \left|\frac{x-y'}{t} - \frac{x-y}{t}\right| \\ &= |y'-y|\left(-\|u_0\|_\infty + \left| b\left(\frac{x-y}{t}\right)\right|\right) \geqslant 0. \end{aligned}$$

**2.4.2. Existence of a solution of the problem.** *First $u(.,.)$ is a solution of the conservation law* (2.1) *in the sense of distributions.* We define the functions $\Omega_N(.,.)$, $u_N(.,.)$ and $f_N(.,.)$, for $N > 0$, by

$$\Omega_N(x, t) = \int_0^{+\infty} \exp\{-NG(x, t; y)\}\, dy - \int_0^t \exp\{-NG(x, t; \tau)\}\, d\tau,$$

$$\begin{aligned} u_N(x, t) = \frac{1}{\Omega(x, t)}\Bigg|\int_0^{+\infty} & b\left(\frac{x-y}{t}\right) \exp\{-NG(x, t; y)\}\, dy \\ & - \int_0^t b\left(\frac{x}{t-\tau}\right) \exp\{-NG(x, t; \tau)\}\, d\tau\Bigg|, \end{aligned}$$

and

$$\begin{aligned} f_N(x, t) = \frac{1}{\Omega(x, t)}\Bigg|\int_0^{+\infty} & f\left(b\left(\frac{x-y}{t}\right)\right) \exp\{-NG(x, t; y)\}\, dy \\ & - \int_0^t f\left(b\left(\frac{x}{t-\tau}\right)\right) \exp\{-NG(x, t; \tau)\}\, dt\Bigg|. \end{aligned}$$

(We write $G(x, t; \tau)$ instead of $G(x, t; y)$, with $y$ given by (2.10b)). For any positive numbers $x$ and $t$, the points $y(x, t)$ or $\tau(x, t)$ minimize the function $G(x, t; .)$, so that we have, in the sense of distributions,

$$u_N(.,.) \to u(.,.), \quad f_N(.,.) \to f(u(.,.)),$$

when $N \to +\infty$. Moreover, by differentiation, one can show (as in Reference 5) that the distributions $u_N(.,.)$ and $f_N(.,.)$ satisfy

$$\frac{\partial u_N}{\partial t} + \frac{\partial}{\partial x} f_N = 0.$$

And this equation leads to (2.1) when $N \to +\infty$.

*Secondly, the initial condition* (2.2) *is satisfied in the weak sense* (2.6). To begin with, we fix a point $x_0$; we take

$$\delta(t) = \sup_{0<x<x_0} |y(x,t) - x|, \quad t>0,$$

and we prove that

$$\lim_{t \to 0+} \delta(t) = 0. \tag{2.31}$$

Let $\delta$ be positive. For all positive numbers $x$ and $y$, such that $|x-y| \geqslant \delta$, $0<x<x_0$, $0<y<y(x_0, 1)$, for any $t$ in $]0; 1[$, we have

$$G(x,t;y) = \int_0^y u_0(\xi)\,\mathrm{d}\xi + |x-y| \frac{g\left(\dfrac{x-y}{t}\right)}{\left|\dfrac{x-y}{t}\right|}.$$

Thus

$$G(x,t;y) \geqslant \int_0^{y(x_0,1)} |u_0(\xi)|\,\mathrm{d}\xi + \delta \inf_{|\xi| \geqslant \frac{\delta}{t}} \frac{g(\xi)}{\xi}. \tag{2.32}$$

But the right member of (2.32) tends to infinity uniformly in $x, y$, when $t$ tends to zero. Furthermore, by the monotonicity of the function $y(.,.)$ (see Propositions 2.4 and 2.5), the inequality (2.32) holds with $y = y(x,t)$. Hence, from the minimization property of $y(x,t)$, we deduce (2.31).

We now estimate $G(x_0, t; y(x_0, t))$ when $t$ tends to zero (and the following inequalities hold only for sufficiently small $t$). On the other hand, we majorize:

$$G(x_0, t; y(x_0, t)) \leqslant G(x_0, t; x) = \int_0^{x_0} u_0(\xi)\,\mathrm{d}\xi + tg(0) \tag{2.33}$$

and the right member of (2.33) tends to $\int_0^{x_0} u_0(\xi)\,\mathrm{d}\xi$, when $t$ tends to zero. On the other hand, by taking

$$m = \inf g(.),$$

and by denoting by $\eta(\delta)$ the oscillation of the uniformly continuous function $x>0 \to \int_0^x u_0(\xi)\mathrm{d}\xi$, over an interval of length $\delta>0$ (remark that $\eta(\delta)$ tends to zero with the parameter $\delta>0$), we minorize:

$$G(x_0,\ t;\ y(x_0,\ t)) \geqslant \int_0^{x_0} u_0(\xi)\,\mathrm{d}\xi - \eta(\delta(t)) + mt \tag{2.34}$$

And by virtue of (2.31), the right member of (2.34) tends to $\int_0^{x_0} u_0(\xi)\,d\xi$, when $t$ tends to zero.

But $G(x_0, t; y(x_0, t))$ tends to $\int_0^{x_0} u(\xi, t)\,d\xi$ when $t$ tends to zero (see (2.18)), and since the functions and $u(.,.)$ $y(.,.)$ are defined only almost everywhere, we conclude with (2.33) and (2.34) that there exists a set $\mathscr{E}$ of zero measure such that the property (2.6) holds.

*Thirdly, the boundary condition* (2.3) *is satisfied.* By arguments similar to the previous ones used for proving (2.6), we deduce that there exists a set of null measure $\mathscr{F}$ such that the property (2.7) is satisfied. And, because of the monotonicity of the function $y(.,.)$, the function $x \mapsto \operatorname{sgn} a(u(x,t))$, admits a limit at the point $x=0$, $\varepsilon(t)$, for almost every $t>0$. Henceforth, the function $\Upsilon(.)$ being given by the variational inequality (2.11), we now can prove, for almost every $t>0$, the condition (2.3′) rewritten as follows (see (2.3″)):

$$\left.\begin{aligned} &\Upsilon(t)=f(\bar{u}_0(t)) \quad \text{and} \quad \varepsilon(t)=\lim_{x\to 0+} \operatorname{sgn} a(u(x,t))=1,\\ &\text{or}\\ &\Upsilon(t)\geqslant f(\bar{u}_0(t)) \quad \text{and} \quad \varepsilon(t)=\lim_{x\to 0+} \operatorname{sgn} a(u(x,t))=-1. \end{aligned}\right\} \tag{2.35}$$

To this purpose, we define the function $M=M(x,t)$ by

$$M(x,t)=\int_0^t \Upsilon(s)\,ds+\min_{y\geqslant 0} G(x,t;y), \quad x\geqslant 0, \quad t>0.$$

Because of the regularity of the function: $y\geqslant 0 \mapsto G(x,t;y)$, for $x\geqslant 0$, $t>0$, this function $M(.,.)$ is continuous ($x\geqslant 0$, $t>0$), and we have

$$M(0,t)=m(t), \quad t>0. \tag{2.36}$$

Namely, using (2.12), we obtain

$$M(0,t)=\int_0^t \frac{dm}{dt}(s)\,ds+\int_0^t f\left(b\left(\frac{-y(s)}{s}\right)\right)ds+\min_{y\geqslant 0} G(0,t;y).$$

Hence we have

$$M(0,t)=m(t)+\int_0^t f\left(b\left(\frac{-y(s)}{s}\right)\right)ds+\min_{y\geqslant 0} G(0,t;y).$$

But, by virtue of Proposition 2.2, the function

$$0<t\mapsto \min_{y\geqslant 0} G(0,t;y),$$

is differentiable and its derivative is

$$-f\left(b\left(\frac{-y(t)}{t}\right)\right).$$

So, we deduce (2.36).

Now, in order to prove (2.35), the following two cases need to be considered for almost every $t$ positive (since the function $m(.)$ is continuous):

1. $m(t)>0$: Using the variational inequality (2.11), we have

$$\frac{d}{dt}m(t)=f(\bar{u}_0(t))-f\left(b\left(\frac{-y(t)}{t}\right)\right).$$

Hence, by (2.12)

$$\Upsilon(t)=f(\bar{u}_0(t))$$

and, the boundary condition is, in that case, satisfied. Moreover, using (2.36), we can conclude that $M(0, t)$ is positive, and thus $\varepsilon(t)$ is positive.

2. $m(t')=0$, $\forall t'\in[t-\varepsilon;\ t+\varepsilon]$, with $\varepsilon>0$: From (2.11) and (2.12) we deduce that

$$f\left(b\left(\frac{-y(t)}{t}\right)\right)\geqslant f(\bar{u}_0(t)), \tag{2.37}$$

and (since $m(.)$ equals zero in a neighbourhood of $t$)

$$\Upsilon(t)=f\left(b\left(\frac{-y(t)}{t}\right)\right).$$

Moreover, using the continuity of $M(., t)$ and the monotonicity of $y(., t)$, we arrive at

$$\lim_{x\to 0+} a(u(x, t))\leqslant 0.$$

Namely, to prove this fact, we remark that $y(0, t)$ is obtained by minimizing the function:

$$\mathbb{R}\ni y\mapsto\int_0^t \Upsilon(s)\,ds+G(0, t; y).$$

But, for $y<0$, it is equal to

$$\int_0^t \Upsilon(s)ds+\int_0^\tau -\Upsilon(s)ds+(t-\tau)g(0)=\int_\tau^t\{\Upsilon(s)-f(b(0))\}\,ds$$

which is always positive by virtue of Proposition 2.3. For $y\geqslant 0$, its minimal value is $M(0, t)=m(t)$ (see (2.36)). Thus, $y(0, t)$ is positive, and so $\varepsilon(t)$ is negative. And the boundary condition (2.35) is satisfied.

*Fourthly, the entropy condition* (2.4) *is satisfied.* More precisely, we now prove the property (2.14). Define $k$ by

$$k=\sup\{b'(z)/|z|\leqslant a(\pm\|u_0\|_\infty),\ a(\pm\|\bar{u}_0\|_\infty)\};$$

and for any positive $t$, take $x_0(t)$ a solution of $y(x_0(t), t)=0$. Then, for $0\leqslant x_1<x_2$, we distinguish between the following cases:

(a) If $x_0\leqslant x_1<x_2$, the function $y(., t)$ is non-decreasing, by virtue of Proposition 2.4; thus, we have $y(x_0)\leqslant y(x_1)\leqslant y(x_2)$

and

$$u(x_1, t) = b\left(\frac{x_1 - y(x_1, t)}{t}\right) \geqslant b\left(\frac{x_1 - y(x_2, t)}{t}\right)$$

$$\geqslant b\left(\frac{x_2 - y(x_2, t)}{t}\right) - k\frac{(x_2 - x_1)}{t}$$

$$= u(x_2, t) - k\frac{(x_2 - x_1)}{t}.$$

So we obtain

$$\frac{u(x_2, t) - u(x_1, t)}{x_2 - x_1} \leqslant \frac{k}{t}. \tag{2.38}$$

(b) If $0 \leqslant x_1 < x_2 \leqslant x_0$, the function $\tau(., t)$ being non-increasing by virtue of Proposition 2.4; we deduce that

$$\tau(x_1, t) \geqslant \tau(x_2, t) \geqslant \tau(x_0, t),$$

and

$$u(x_1, t) = b\left(\frac{x_1}{t - \tau(x_1, t)}\right) \geqslant b\left(\frac{x_1}{t - \tau(x_2, t)}\right)$$

$$\geqslant b\left(\frac{x_2}{t - \tau(x_2, t)}\right) - k\frac{(x_2 - x_1)}{t - \tau(x_2, t)}.$$

So we obtain

$$\frac{u(x_2, t) - u(x_1, t)}{x_2 - x_1} \leqslant \frac{k}{t - \tau(x_2, t)}. \tag{2.39}$$

(c) If $0 \leqslant x_1 \leqslant x_0 \leqslant x_2$, by virtue of (2.38), we have on the one hand

$$u(x_2, t) - u(x_0, t) \leqslant \frac{k}{t}(x_2 - x_0),$$

and on the other hand by virtue of (2.39)

$$u(x_0, t) - u(x_1, t) \leqslant \frac{k}{t - \tau(x_0, t)}(x_0 - x_1) = \frac{k}{t}(x_0 - x_1).$$

So we deduce that

$$\frac{u(x_2, t) - u(x_1, t)}{x_2 - x_1} \leqslant \frac{k}{t}. \tag{2.40}$$

## 2.5. Third step: Uniqueness of the solution

We here prove Theorem 2.2. Let $u(.,.)$ and $v(.,.)$ be piecewise $C^1$ solutions of (2.1)–(2.4), associated with integrable initial data $u_0(.)$ and $v_0(.)$ and bounded boundary data $\bar{u}_0(.)$ and $\bar{v}_0(.)$, respectively. We will suppose that $\bar{u}_0(.) \geqslant b(0)$ and $\bar{v}_0(.) \geqslant b(0)$ (and so we will use (2.3′)).

We only indicate the proof by comparing with the one given by Keyfitz[2] (see also Reference 5). Because the single new fact happens along the boundary at $x=0$, the arguments in Reference 2 immediately lead to the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{+\infty} |u(x,t)-v(x,t)|\,\mathrm{d}x \leqslant \mu(t), \quad t>0, \tag{2.41}$$

with the function $\mu(.)$ given by

$$\mu(t)=\operatorname{sgn}(u(0,t)-v(0,t))\{f(u(0,t))-f(v(0,t))\}, \quad t>0. \tag{2.42}$$

We are now going to prove that $\mu(.)$ satisfies

$$\mu(t)\leqslant |f(\bar{u}_0(t))-f(\bar{v}_0(t))|, \quad t>0. \tag{2.43}$$

Hence, the semigroup property (2.8) will be derived from (2.41) and (2.43) by integration.

Using the boundary condition (2.3′), we distinguish between four cases: ($t>0$).

1. If $u(0,t)=\bar{u}_0(t)$ and $v(0,t)=\bar{v}_0(t)$, the equality holds in (2.43):

$$\mu(t)=|f(\bar{u}_0(t))-f(\bar{v}_0(t))|.$$

2. If $u(0,t)\neq\bar{u}_0(t)$ and $v(0,t)=\bar{v}_0(t)$, then we have with (2.3′)

$$a(u(0,t))\leqslant 0, \quad f(u(0,t))\geqslant f(\bar{u}_0(t)).$$

Hence, we deduce that

$$\begin{aligned}\mu(t)&=f(v(0,t))-f(u(0,t))=f(\bar{v}_0(t))-f(u(0,t))\\ &=\{f(\bar{v}_0(t))-f(\bar{u}_0(t))\}+\{f(\bar{u}_0(t))-f(u(0(t))\}\\ &\leqslant f(\bar{v}_0(t))-f(\bar{u}_0(t))\leqslant |f(\bar{u}_0(t))-f(\bar{v}_0(t))|.\end{aligned}$$

3. If $u(0,t)=\bar{u}_0(t)$ and $v(0,t)\neq\bar{v}_0(t)$, we prove (2.43) as in the previous case 2.

4. If $u(0,t)\neq\bar{u}_0(t)$ and $v(0,t)\neq\bar{v}_0(t)$, then we necessarily have with (2.3′)

$$a(u(0,t))\leqslant 0 \quad \text{and} \quad a(v(0,t))\leqslant 0.$$

Thus

$$\mu(t)\leqslant 0.$$

**Remark 2.2.**

1. *For the proof of (2.8) without the assumption of the piecewise regularity of the solutions, see Reference 1.*
2. *If the boundary data are not assumed to be more than $b(0)$, then the second term in (2.8) is replaced by*

$$k\int_{t_1}^{t_2} |\bar{u}_0(s)-\bar{v}_0(s)|\,\mathrm{d}s,$$

*with*

$$k=\sup\{|a(w)/|w|\leqslant \sup(\|\bar{u}_0(.)\|_\infty, \|\bar{v}_0(.)\|_\infty)\}.$$

## 3. Study of the weighted Burgers' equation

### 3.1. *Preliminary*

The mixed problem associated with the weighted Burgers' equation (1.5) is now solved by using a change of function that reduces this equation to the classical Burgers' equation.

First, we specify the result concerning the classical Burgers' equation. Let $u_0(.)$ and $\bar{u}_0(.)$ be bounded measurable functions with $\bar{u}_0(.) \geqslant 0$ (it is not a restriction). And consider the following problem with the unknown $u(.,.)$:

$$\frac{\partial}{\partial t} u + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0, \qquad x > 0, \qquad t > 0, \tag{3.1}$$

$$u(x, 0) = u_0(x), \text{ a.e. } x > 0, \tag{3.2}$$

$$u(0, t) = \bar{u}_0(t) \quad \text{or} \quad u(0, t) \leqslant -\bar{u}_0(t), \quad \text{a.e. } t > 0, \tag{3.3}$$

$$u(x-0, t) \geqslant u(x+0, t), \quad x > 0, \quad t > 0. \tag{3.4}$$

By virtue of Theorems 2.1–2.3, the unique weak solution of this problem (3.1)–(3.4) is given as follows:

**Proposition 3.1**

A. *Characterization of the boundary flux* $u(0, .)^2$.

(A1) $y = y(t)$ *minimizes the function*

$$0 \leqslant y \rightarrow \int_0^y u_0(\xi)\mathrm{d}\xi + \frac{y^2}{2t}, \quad t > 0.$$

(A2) *The function* $m = m(t) \geqslant 0$ *satisfies* $m(0) = 0$ *and*

$$\left\{\frac{\mathrm{d}}{\mathrm{d}t} m(t) - \frac{\bar{u}_0(t)^2}{2} + \frac{y(t)^2}{2t^2}\right\} \times \{\varphi - m(t)\} \geqslant 0, \quad \varphi \geqslant 0, \quad t > 0. \tag{3.5}$$

(A3) *The function* $u(0,.)^2$ *is given by*

$$\frac{1}{2} u(0, t)^2 = \frac{\mathrm{d}}{\mathrm{d}t} m(t) + \frac{y(t)^2}{2t^2}, \quad t > 0. \tag{3.5'}$$

B. *The explicit representation inside.*

$$u(x, t) = \frac{x - y(x, t)}{t}, \quad x > 0, \quad t > 0,$$

*where* $y = y(x, t)$ *minimizes the function*

$$y \in \mathbb{R} \mapsto \begin{cases} \displaystyle\int_0^y u_0(\xi)\mathrm{d}\xi + \frac{(x-y)^2}{2t}, & \text{if } y \geqslant 0, \\ \displaystyle -\int_0^{\frac{-yt}{x-y}} \frac{1}{2} u(0, s)^2 \mathrm{d}s + \frac{(x-y)^2}{2t}, & \text{if } y \leqslant 0. \end{cases} \tag{3.6}$$

3.2. *Existence and uniqueness*

Let $u_0 = u_0(x)$ and $w = w(t)$ be such that

$$x^\alpha u_0(.) \in L^1(0, \infty), \quad x^{\alpha/2} u_0(.) \in L^\infty(0, \infty), \quad w(.) \in L^\infty(0, \infty), \quad w(.) \geqslant 0,$$

where $\alpha$ is an arbitrary real, and consider the following problem (3.7)–(3.10) with the unknown $u = u(x, t) \in \mathbb{R}$:

$$\frac{\partial}{\partial t}(x^\alpha u(x, t)) + \frac{\partial}{\partial x}\left(x^\alpha \frac{u(x, t)^2}{2}\right) = 0, \quad x > 0, \quad t > 0. \tag{3.7}$$

$$u(x, 0) = u_0(x), \text{ a.e. } x > 0. \tag{3.8}$$

if $\alpha > -2$:

$$\lim_{x \to 0+} (x^{\alpha/2} u(x, t)) = w(t),$$

or

$$\lim_{x \to 0+} (x^{\alpha/2} u(x, t)) \leqslant -w(t), \quad \text{a.e. } t > 0. \tag{3.9}$$

if $\alpha < -2$:

$$\lim_{x \to +\infty} (x^{\alpha/2} u(x, t)) = -w(t),$$

or

$$\lim_{x \to +\infty} (x^{\alpha/2} u(x, t)) \geqslant w(t), \quad \text{a.e. } t > 0.$$

$$u(x - 0, t) \geqslant u(x + 0, t), \quad x > 0, \quad t > 0. \tag{3.10}$$

In this section, we briefly indicate the results for this problem.

**Theorem 3.1.** *There exists one function $u = u(x, t)$, piecewise continuous, such that the function $x^{\alpha/2} u(x, t)$ has a trace at $x = 0$, and which satisfies the properties* (3.7)–(3.10) *Moreover, it satisfies*

$$x^{\alpha/2} u(.,.) \in L^\infty((0, \infty) \times (0, \infty))$$

*and*

$$(t \to x^\alpha u(., t)) \in L^\infty((0, \infty); L^1(0, \infty)).$$

*We have the semigroup property: if $u(.,.)$ and $v(.,.)$ are solutions of* (3.7)–(3.10), *with initial data $u_0(.)$, $v_0(.)$ and boundary data $w(.)$, $z(.)$, then for $0 \leqslant t_1 \leqslant t_2$*

*if $\alpha \neq -2$;*

$$\int_0^\infty |u(x, t_2) - v(x, t_2)|x^\alpha \mathrm{d}x \leqslant \int_0^\infty |u(x, t_1) - v(x, t_1)|x^\alpha \mathrm{d}x + \int_{t_1}^{t_2} |w(s) - z(s)| \mathrm{d}s. \tag{3.11}$$

*if $\alpha = -2$:*

$$\int_0^\infty |u(x, t_2) - v(x, t_2)|x^\alpha \mathrm{d}x \leqslant \int_0^\infty |u(x, t_1) - v(x, t_1)|x^\alpha \mathrm{d}x. \tag{3.12}$$

*Proof of the Theorem* 3.1. We deduce these results from those of section 2 by making, in equation (3.7), a change of function that reduces it to equation (3.1). An explicit formula for the solution will be possible (see below). We distinguish two cases.

First: $\alpha \neq -2$. We consider the following change of function:

$$v(x_1, t) = a x^{\alpha/2} u(x, t), \quad x_1 = x^{\alpha}, \quad a = \frac{\alpha}{2} + 1 \neq 0. \tag{3.13}$$

It does not modify the notion of weak solution.

About the classical notion of entropic solution (see equation (3.10)), it is also obvious that it does not modify (3.10) when $a$ is assumed positive, that is $\alpha > -2$. And, $a$ is negative, note that

$$v(x_1 + 0, t) = a x^{\alpha/2} u(x - 0, t)$$

and $\quad v(x_1 - 0, t) = a x^{\alpha/2}(x + 0, t).$

Hence, from the inequality (3.10), we deduce an analogous inequality for the function $v(.,.)$.

Using (3.13) with a solution $u = u(x, t)$ of (3.7), we obtain

$$\frac{\partial}{\partial t} v(x_1, t) + \frac{1}{2} \frac{\partial V}{\partial x_1}(x_1, t)^2 = 0, \quad x_1 > 0, \quad t > 0. \tag{3.14}$$

But, this equation is exactly *the classicial Burgers' equation in a half-space* (see equation (3.1)).

Second: $\alpha = -2$. Take

$$v(x_1, t) = \frac{u(x, t)}{x}, \quad x_1 = \log x; \tag{3.15}$$

then the equation (3.7) can be rewritten as

$$\frac{\partial}{\partial t} v(x_1, t) + \frac{1}{2} \frac{\partial}{\partial x} v(x_1, t)^2 = 0, \quad x_1 \in \mathbb{R}, \quad t > 0. \tag{3.16}$$

Equation (3.16) is exactly Burgers' equation and, hence, we do not have to give any boundary condition in this case.

### 3.3. *Explicit representation*

Now, using the change of functions (3.13) or (3.15), we deduce from Proposition 3.1 the explicit representation of the solution $u(.,.)$ given by Theorem 3.1.

**First case:** $\alpha > -2$. (In particular, it includes the 'physical' cases: $\alpha = 0$, 1 or 2.)

A. *Characterization of the flux at* $x = 0$.

(A1) $y = y(t)$ minimizes the function

$$y \geq 0 \rightarrow \int_0^y u_0(\xi)\xi^{\alpha} d\xi + \frac{y^{2a}}{2a^2 t}, \quad \left(a = \frac{\alpha}{2} + 1\right), \text{ a.e. } t > 0.$$

(A2) $m=m(t)\geqslant 0$ satisfies $m(0)=0$ and

$$\forall \varphi \geqslant 0, \left\{\frac{\partial}{\partial t} m(t)-\frac{w(t)^2}{2}+\frac{y(t)^{2a}}{2t^2}\right\} \times \{\varphi-m(t)\} \geqslant 0 \quad \text{a.e. } t>0.$$

(A3) $$\lim_{x\to 0^+}\left(x^{\alpha}\frac{u(x,t)^2}{2}\right)=\frac{\partial}{\partial t} m(t)+\frac{y(t)^{2a}}{2t^2}, \quad \text{a.e. } t>0.$$

B. *The explicit representation inside.*

$$u(x,t)=\frac{x^a-y^a}{atx^{a-1}}, \quad x>0, \quad t>0,$$

where $y=y(x,t)$ minimizes the following function:

$$y\in\mathbb{R}\mapsto\begin{cases} \displaystyle\int_0^y u_0(\xi)\xi^{\alpha}\mathrm{d}\xi+\frac{1}{2t}\left(\frac{x^a-y^a}{a}\right)^2, & \text{if } y\geqslant 0,\\ \displaystyle -\int_0^{\frac{-y^a t}{x^a-y^a}}\lim_{x\to 0^+}\left(x^{a}\frac{u(x,s)^2}{2}\right)\mathrm{d}s+\frac{1}{2t}\left(\frac{x^a-y^a}{a}\right)^2, & \text{if } y\leqslant 0.\end{cases}$$

**Second case:** $\alpha=-2$. We have

$$u(x,t)=\frac{x}{t}\log\frac{x}{y}, \quad x>0, \quad t>0,$$

where $y=y(x,t)$ minimizes the function

$$0<y<\infty\to\int_0^y\frac{u_0(\xi)}{\xi}\mathrm{d}\xi+\frac{1}{2t}\left(\log\frac{x}{y}\right)^2.$$

**Third case:** $\alpha<-2$.

A. *Characterization of the flux at* $x=+\infty$.

(A1) $y=y(t)$ minimizes the function

$$0<y\leqslant+\infty\to\int_0^y u_0(\xi)\xi^{\alpha}\mathrm{d}\xi+\frac{y^{2a}}{2a^2t}, \quad \text{a.e. } t>0.$$

(A2) $m=m(t)\geqslant 0$, $m(0)=0$ and we have

$$\forall \varphi \geqslant 0, \left\{\frac{\partial}{\partial t} m(t)-\frac{w(t)^2}{2}+\frac{y(t)^{2a}}{2t^2}\right\} \times \{\varphi-m(t)\} \geqslant 0, \quad \text{a.e. } t>0.$$

(A3) $$\lim_{x\to+\infty}\left(x^{\alpha}\frac{u(x,t)^2}{2}\right)=\frac{\partial}{\partial t} m(t)+\frac{y(t)^{2a}}{2t^2}, \quad \text{a.e. } t>0.$$

B. *Explicit representation inside.*

$$u(x,t)=\frac{x^a-y^a}{atx^{a-1}},$$

where $y = y(x, t)$ minimizes the function

$$y \in [-\infty; \infty] \underset{y \neq 0}{\rightarrow} \begin{cases} -\displaystyle\int_y^{\infty} u_0(\xi)\xi^{\alpha} \mathrm{d}\xi + \frac{1}{2t}\left(\frac{x^a - y^a}{a}\right)^2, & \text{if } y \in ]0; +\infty], \\ -\displaystyle\int_0^{\frac{-y^{a}t}{x^a - y^a}} \lim_{x \to +\infty} \left(x^{\alpha}\frac{u(x, s)^2}{2}\right)\mathrm{d}s + \frac{1}{2t}\left(\frac{x^a - y^a}{a}\right)^2, & \text{if } y \in [-\infty, 0[. \end{cases}$$

Moreover, we can specify *the entropy condition* (3.10) as follows: there exists $k > 0$ such that, for $0 \leqslant x_1 < x_2$, we have

If $\alpha \neq -2$:

$$a\left(\frac{x_2^{2a}u(x_2, t) - x_1^{2a}u(x_1, t)}{x_2^{2a} - x_1^{2a}}\right) \leqslant \frac{k}{t - \tau(x_2, t)}$$

with $\tau(x_2, t) = \max(0, \tau)$ and $\tau \in \mathbb{R}$ given by

$$\frac{x}{t - \tau} = \frac{x - y(x, t)}{t}.$$

if $\alpha = -2$:

$$\frac{\dfrac{u(x_2, t)}{x_2} - \dfrac{u(x_1, t)}{x_1}}{\log x_2 - \log x_1} \leqslant \frac{k}{t}.$$

## References


1. Bardos, Cl., Leroux, A. Y. and Nedelec, J. C., 'First order quasi linear equations with boundary conditions', *Comm. P.D.E.* 4, (9), 1017–1034 (1979).
2. Keyfitz, B. L., 'Solutions with shocks', *Comm. P.A.M.*, **24**, 125–132 (1971).
3. Kruskov, S. N., 'First order quasilinear equations in several independent variables', *Math. USSR Sbornik*, **10**, 217–243 (1970).
4. Lax, P. D., 'Hyperbolic systems of conservation laws', *Comm. P.A.M.*, **10**, 537–556 (1957).
5. Lax, P. D., 'Hyperbolic systems of conservation laws', *S.I.A.M.*, **13**, (1973).
6. Le Floch, P. H. and Nedelec, J. C., 'Weighted conservation laws', in *Trans. A.M.S.* to appear.
7. Oleinik, O. A., 'Discontinuous solutions of conservation laws', *A.M.S. Transl.*, 2(26), 95–170 (1964).
8. Schonbek, M. E., 'Existence of solutions to singular conservation laws', *J. Math. Anal.*, **15**, (6), 1125–1139 (1984).
9. Smoller, J. A., *Shock Waves and Reaction–Diffusion Equations*, Springer-Verlag, 1983, p. 258.
10. Whitham, G. B. *Linear and Non-linear Waves*, New York, Wiley Interscience, 1974.