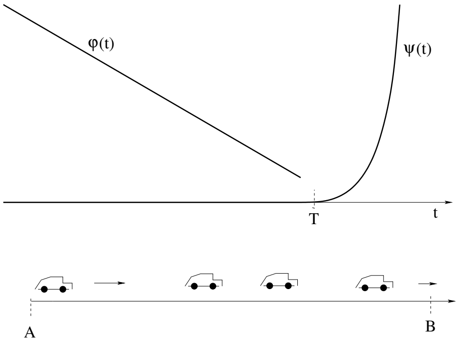
Part 3 - Optimization problems for traffic flow

- Car drivers starting from a location A (a residential neighborhood) need to reach a destination B (a working place) at a given time T.
- There is a cost $\varphi(\tau_d)$ for departing early and a cost $\psi(\tau_a)$ for arriving late.



Elementary solution

L = length of the road, $\qquad$ v = speed of cars

$$\tau_a \;=\; \tau_d + \frac{L}{v}$$

Optimal departure time:

$$\tau_d^{\rm opt} \;=\; \operatorname*{argmin}_t \left\{ \varphi(t) + \psi\left(t + \frac{L}{v}\right) \right\}.$$

If everyone departs exactly at the same optimal time,
a traffic jam is created and this strategy is not optimal anymore.

An optimization problem for traffic flow

Problem: choose the departure rate $\bar{u}(t)$ in order to minimize the total cost to all drivers.

$$u(t,x) \doteq \rho(t,x)\cdot v(\rho(t,x)) = \text{flux of cars}$$

$$\text{minimize:} \qquad \int \varphi(t)\cdot u(t,0)\,dt + \int \psi(t)u(t,L)\,dt$$

for a solution of

$$\begin{cases} \rho_t + [\rho\, v(\rho)]_x &= 0 \qquad\qquad x\in[0,L] \\ \rho(t,0)v(\rho(t,0)) &= \bar{u}(t) \end{cases}$$

Choose the optimal departure rate $\bar{u}(t)$, subject to the constraint

$$\int \bar{u}(t)\,dt = \kappa = [\text{total number of drivers}]$$

Equivalent formulations

Boundary value problem for the density ρ:

conservation law: $\rho_t + [\rho v(\rho)]_x \; = \; 0, \qquad (t, x) \in \mathbb{R} \times [0, L]$

control (on the boundary data): $\rho(t, 0) v(\rho(t, 0)) \; = \; \bar{u}(t)$

Cauchy problem for the flux u:

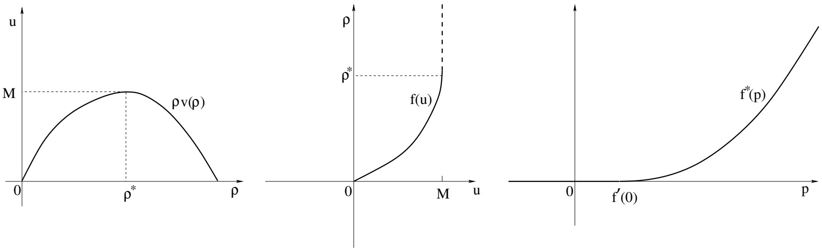
conservation law: $u_x + f(u)_t \; = \; 0, \qquad u \; = \; \rho\, v(\rho)\,, \qquad f(u) \; = \; \rho$

control (on the initial data): $u(t, 0) \; = \; \bar{u}(t)$

$$\text{Cost:} \quad J(u) \; = \; \int_{-\infty}^{+\infty} \varphi(t) u(t, 0)\, dt + \int_{-\infty}^{+\infty} \psi(t) u(t, L)\, dt$$

$$\text{Constraint:} \quad \int_{-\infty}^{+\infty} \bar{u}(t)\, dt \; = \; \kappa$$

The flux function and its Legendre transform



$$u \;=\; \rho\, v(\rho)\,, \qquad\qquad \rho \;=\; f(u)$$

Legendre transform: $\qquad f^*(p) \;\doteq\; \max_u \Big\{ pu - f(u) \Big\}$

Solution to the conservation law is provided by the Lax formula

The globally optimal (Pareto) solution

$$\text{minimize:} \qquad J(u) \;=\; \int \varphi(x)\cdot u(0,x)\,dx + \int \psi(x)\,u(T,x)\,dx$$

$$\text{subject to:} \begin{cases} u_t + f(u)_x \;=\; 0 \\ \qquad u(0,x) \;=\; \bar u(x)\,, \qquad\qquad \displaystyle\int \bar u(x)\,dx \;=\; \kappa \end{cases}$$

(A1) The flux function $f : [0, M] \mapsto \mathbb{R}$ is continuous, increasing, and strictly convex. It is twice continuously differentiable on the open interval $]0, M[$ and satisfies

$$f(0) = 0\,, \quad \lim_{u\to M-} f'(u) = +\infty, \qquad\qquad f''(u) \geq b > 0 \quad \text{for} \quad 0 < u < M$$

(A2) The cost functions φ, ψ satisfy $\varphi' \;<\; 0, \quad \psi, \psi' \;\geq\; 0,$

$$\lim_{x\to-\infty} \varphi(x) \;=\; +\infty\,, \qquad\qquad \lim_{x\to+\infty} \Big(\varphi(x) + \psi(x)\Big) \;=\; +\infty$$

Existence and characterization of the optimal solution

Theorem (A.B. and K. Han, 2011). Let **(A1)-(A2)** hold. Then, for any given T, κ, there exists a unique admissible initial data $\bar{u}$ minimizing the cost $J(\cdot)$. In addition,

1. No shocks are present, hence $u = u(t,x)$ is continuous for $t > 0$. Moreover

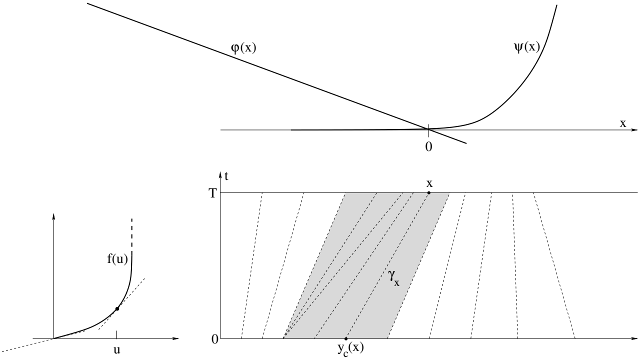
$$\sup_{t\in[0,T],\, x\in\mathbb{R}} u(t,\, x) \;<\; M$$

2. For some constant $c = c(\kappa)$, this optimal solution admits the following characterization: For every $x \in \mathbb{R}$, let $y_c(x)$ be the unique point such that

$$\varphi(y_c(x)) + \psi(x) \;=\; c$$

   Then, the solution $u = u(t,x)$ is constant along the segment with endpoints $(0, y_c(x))$, (T, x).
   Indeed, either $f'(u) \;\equiv\; \frac{x - y_c(x)}{T}$, or $u \equiv 0$

Necessary conditions



$$\varphi(y_c(x)) + \psi(x) \;=\; c$$

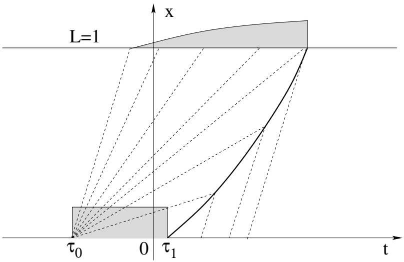
$$f'(u) \;=\; \frac{x - y_c(x)}{T} \qquad \text{on the characteristic segment} \;\; \gamma_x$$

An Example

Cost functions: $\varphi(t) = -t, \qquad \psi(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ t^2, & \text{if } t > 0 \end{cases}$

$$L = 1, \quad u = \rho(2-\rho), \quad M = 1, \quad \kappa = 3.80758$$

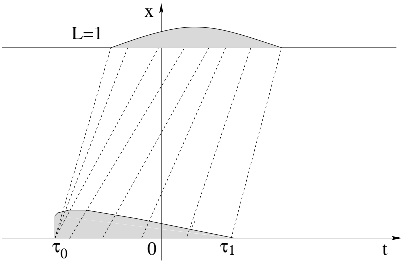
Bang-bang solution



$\tau_0 = -2.78836, \qquad \tau_1 = 1.01924$

total cost $= 5.86767$

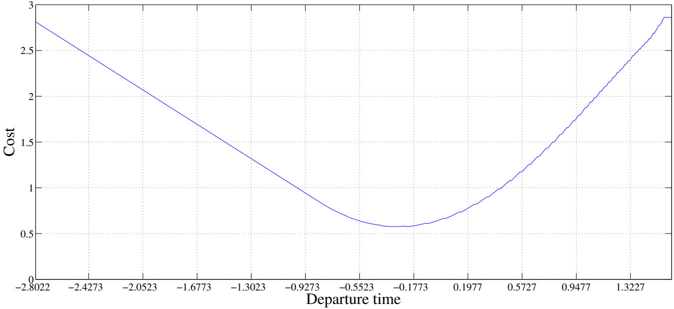
Pareto optimal solution



$\tau_0 = -2.8023, \qquad \tau_1 = 1.5976$

total cost $= 5.5714$

Does everyone pay the same cost?



Departure time vs. cost in the Pareto optimal solution

The Nash equilibrium solution

A solution $u = u(t,x)$ is a Nash equilibrium if no driver can reduce his/her own cost by choosing a different departure time.
This implies that all drivers pay the same cost.

To find a Nash equilibrium, write the conservation law $u_t + f(u)_x = 0$ in terms of a Hamilton-Jacobi equation

$$U_t + f(U_x) = 0 \qquad\qquad U(0,x) = Q(x)$$

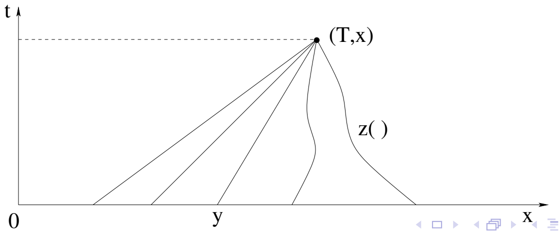
$$U(t,\, x) \doteq \int_{-\infty}^{x} u(t,\, y)\, dy$$

A representation formula

$$U_t + f(U_x) \;=\; 0 \qquad\qquad U(0,x) \;=\; Q(x)$$

$$U(T,x) \;=\; \inf_{z(\cdot)} \left\{ \int_0^T f^*(\dot z(s))\,ds + Q(z(0))\,; \qquad z(T)=x \right\}$$

$$= \; \min_{y\in\mathbb{R}} \left\{ T\, f^*\left(\frac{x-y}{T}\right) + Q(y) \right\}$$



No constraint can be imposed on the departing rate, so a queue can form at the entrance of the highway.

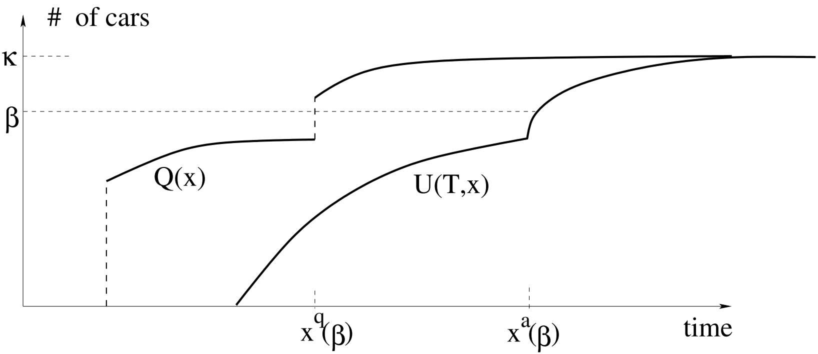
$x \mapsto Q(x) =$ number of drivers who have started their journey before time x (joining the queue, if there is any).

$$Q(-\infty) = 0, \qquad\qquad Q(+\infty) = \kappa$$

$x \mapsto U(T,x) =$ number of drivers who have reached destination within time x

$$U(T,\, x) = \min_{y\in\mathbb{R}} \left\{ T\, f^*\left(\frac{x-y}{T}\right) + Q(y)\right\}$$

Characterization of a Nash equilibrium



$\beta \in [0, \kappa]$ = Lagrangian variable labeling one particular driver

$x^q(\beta)$ = time when driver β departs (possibly joining the queue)

$x^a(\beta)$ = time when driver β arrives at destination

Existence and Uniqueness of Nash equilibrium

Departure and arrival times are implicitly defined by

$$Q(x^q(\beta)-) \ \leq \ \beta \ \leq \ Q(x^q(\beta)+), \qquad\qquad U(T, x^a(\beta)) \ = \ \beta$$

$$\text{Nash equilibrium} \qquad \Longrightarrow \qquad \varphi(x^q(\beta)) + \psi(x^a(\beta)) \ \equiv \ c$$

Theorem (A.B. - K. Han, *SIAM J. Math. Anal.* 2012).

Let the flux f and cost functions φ, ψ satisfy the assumptions (A1)-(A2). Then, for every $\kappa > 0$, the Hamilton-Jacobi equation

$$U_t + f(U_x) \ = \ 0$$

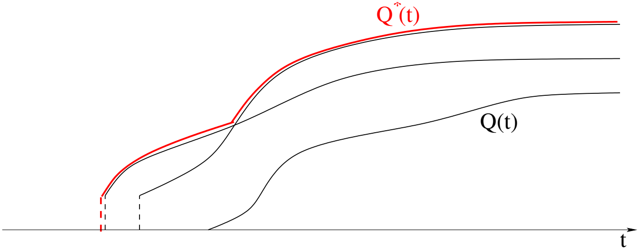
admits a unique Nash equilibrium solution with total mass κ

Sketch of the proof

1. For a given cost c, let $\mathcal{Q}_c^-$ be the set of all initial data $Q(\cdot)$ for which every driver has a cost $\leq c$:

$$\varphi(\tau^q(\beta)) + \psi(\tau^a(\beta)) \ \leq \ c \qquad\qquad \text{for a.e. } \beta \in [0,\ Q(+\infty)]\,.$$

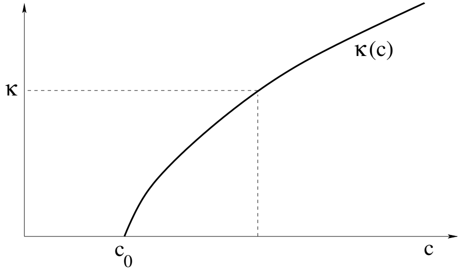
2. Claim: $\quad Q^*(t) \ \doteq \ \sup\Big\{Q(t)\,;\ \ Q \in \mathcal{Q}_c^-\Big\}$
is the initial data for a Nash equilibrium with common cost c.



3. For each c, the Nash equilibrium solution where each driver has a cost $= c$ is unique. Define $\kappa(c) \doteq$ total number of drivers in this solution.

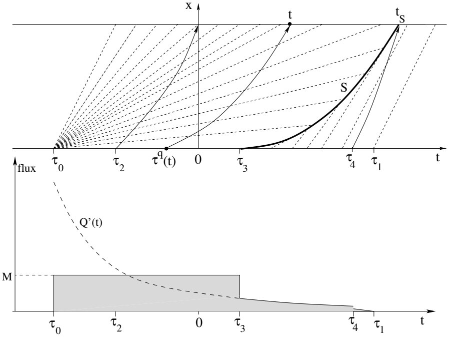
4. There exists a minimum cost c_0 such that $\kappa(c) = 0$ for $c \le c_0$.

The map $c \mapsto \kappa(c)$ is strictly increasing and continuous from $[c_0\,, +\infty[$ to $[0, +\infty[$.



Numerical results

$$L = 1, \quad u(\rho) = \rho(2-\rho), \quad M = 1, \quad \kappa = 3.80758, \quad c = 2.7$$



$\tau_0 = -2.7 \qquad \tau_2 = -0.9074$

$\tau_3 = 0.9698 \qquad \tau_4 = 1.52303$

$\tau_1 = 1.56525 \qquad t_S = 2.0550$

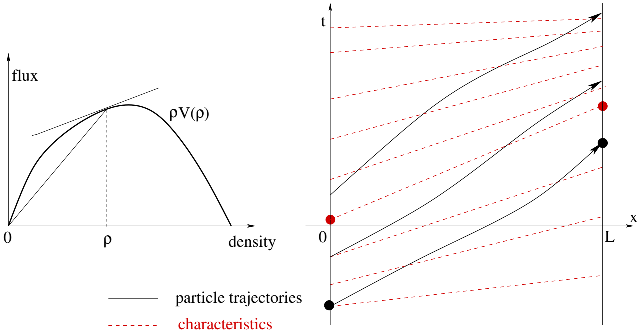
$\delta_0 = 1.79259$

total cost $=$ 10.286

$$Q(t) = 1.7 + \sqrt{t+2.7} + 1/(4(\sqrt{t+2.7} + 2.7))$$

$$Q'(t) = \Big(1 - 1/(4(\sqrt{t+2.7} + 2.7)^2)\Big)/(2\sqrt{t+2.7})$$

Globally optimal solution vs. Nash equilibrium



Globally optimal solution:

starting cost + arrival cost = constant for all characteristics

Nash equilibrium solution:

starting cost + arrival cost = constant for all car trajectories

A comparison

Total cost of the Pareto optimal solution: $J^{opt} \;=\; 5.5714$

Total cost of the Nash equilibrium solution: $J^{Nash} \;=\; 10.286$

Price of anarchy: $J^{Nash} - J^{opt} \;\approx\; 4.715$

Can one eliminate this inefficiency,
yet allowing freedom of choice to each driver ?

(goal of non-cooperative game theory: devise incentives)

Optimal pricing

Scientific American, Dec. 2010: Ten World Changing Ideas

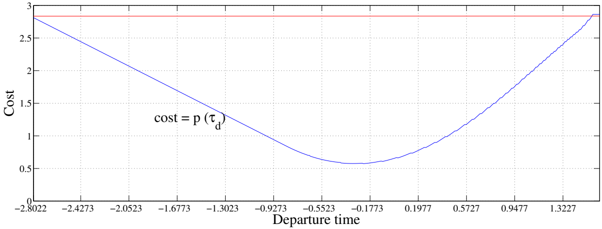
"Building more roads won't eliminate traffic. Smart pricing will."

Suppose a fee $b(t)$ is collected at a toll booth at the entrance of the highway, depending on the departure time.

$$\text{New departure cost:} \qquad \tilde{\varphi}(t) \;=\; \varphi(t) + b(t)$$

Problem: We wish to collect a total revenue R.

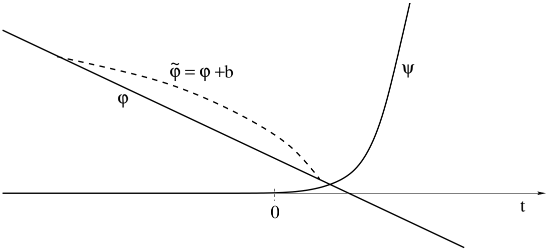
How do we choose $t \mapsto b(t) \geq 0$ so that the Nash solution with departure and arrival costs $\tilde{\varphi}, \psi$ yields the minimum total cost to each driver?



$p(t) \;=$ cost to a driver starting at time t, in the globally optimal solution

Optimal pricing: $\quad b(t) = p_{max} - p(t) + C$

choosing the constant C so that $\quad$ [total revenue] $= R$.



Continuous dependence of the Nash solution

$\varphi_1(x), \ \varphi_2(x)$ costs for departing at time x

$\psi_1(x), \ \psi_2(x)$ costs for arriving at time x

$v_1(\rho), \ \ v_2(\rho)$ speeds of cars, when the density is $\rho \geq 0$

$Q_1(x), \ Q_2(x) =$ number of cars that have departed up to time x, in the corresponding Nash equilibrium solutions (with zero total cost to all drivers)

Theorem (A.B., C.J.Liu, and F.Yu, *Quarterly Appl. Math.* 2012)

Assume all cars depart and arrive within the interval $[a, b]$, and the maximum density is $\leq \rho^*$. Then

$$\begin{aligned} & \|Q_1(x) - Q_2(x)\|_{\mathbf{L}^1([a,b])} \\ & \leq \ C \cdot \left(\|\varphi_1 - \varphi_2\|_{\mathbf{L}^\infty([a,b])} + \|\psi_1 - \psi_2\|_{\mathbf{L}^\infty([a,b])} + \|v_1 - v_2\|^{1/2}_{\mathbf{L}^\infty([0,\rho^*])} \right) \end{aligned}$$

A min-max property of Nash equilibrium solutions

Fix: $\kappa =$ total number of drivers

For any departure distribution

$t \;\mapsto\; Q(t) \;=\;$ number of drivers who have departed within time t

(possibly joining the queue at the entrance of the highway)

Define: $\Phi(Q) \;\doteq$ maximum cost, among all drivers

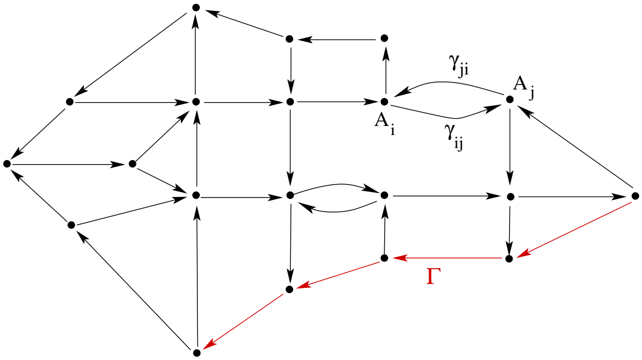
Theorem (A.B., C.J.Liu, and F.Yu, *Quarterly Appl. Math.* 2012)

The starting distribution $Q^*(\cdot)$ for the Nash equilibrium solution yields a global minimum of Φ.

Traffic Flow on a Network

Nodes: $A_1, \ldots, A_m$ arcs: γ_{ij}

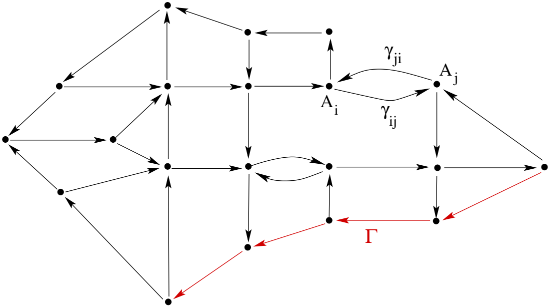
L_{ij} = length of the arc γ_{ij}



A viable path Γ is a concatenation of viable arcs

Network loading problem

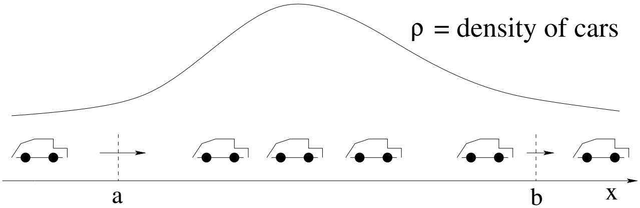
Given the departure times of N drivers, and the paths $\Gamma_1, \ldots, \Gamma_N$ along which they travel, describe the overall traffic pattern.



Delay Model: If a drivers enters the arc γ_{ij} at time t,
he will exit form that arc at time $\quad t + D_{ij}(n)$

$n \;=\;$ number of cars present along the arc γ_{ij} at time t

Conservation law model

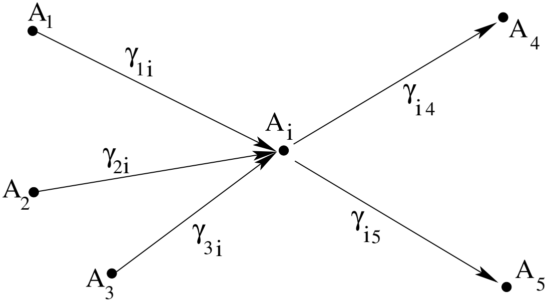


Along the arc γ_{ij}, the density of cars satisfies the conservation law

$$\rho_t + [\rho v_{ij}(\rho)]_x \;=\; 0$$

$v_{ij}(\rho) \;=\;$ velocity of cars, depending on the density

Boundary conditions at nodes

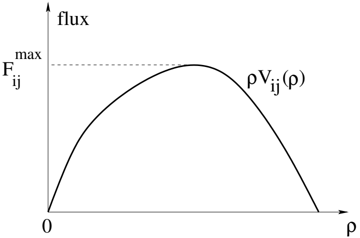


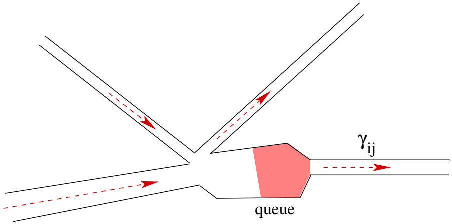
Need: junction conditions

given the flux from incoming arcs, determine the flux along outgoing arcs

A queue at the entrance of each arc

Simplest model: a queue is formed at the entrance of each outgoing arc if the flux is too large

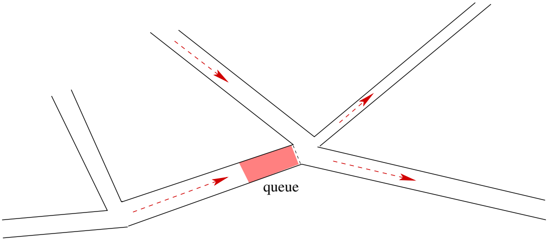




A queue at the exit of each arc

An upper bound on the flow is imposed (by a crosslight) at the end of each incoming arc.

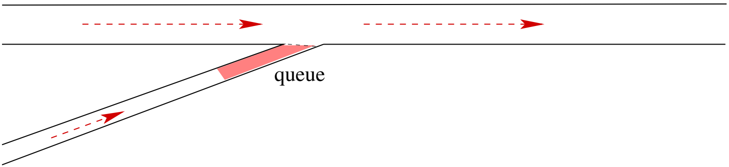
A queue is formed, if the flux is too large (with possible spill-over)



Priority among different incoming roads

Cars from the incoming road having priority pass instantly through the intersection

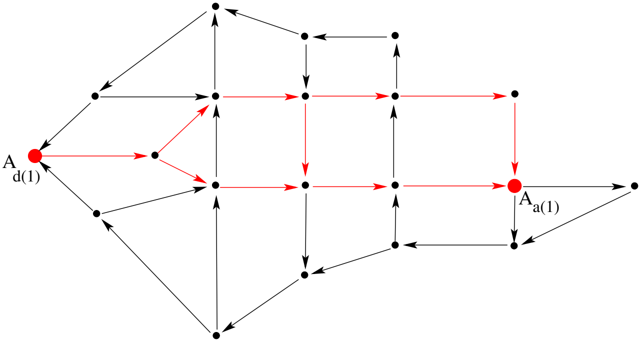
Cars from the access ramp wait in a queue



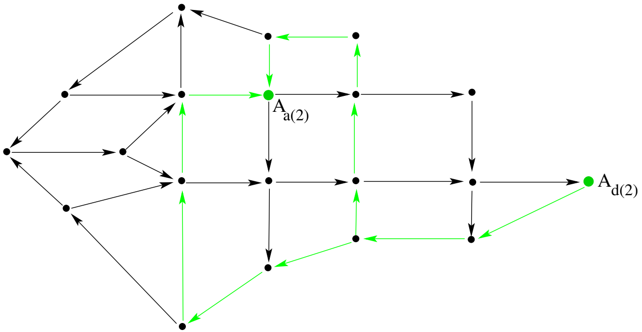
Traffic Flow on a Network

n groups of drivers with different origins and destinations, and different costs

$$k\text{-drivers: } \begin{cases} \text{depart from } A_{d(k)} \text{ and arrive to } A_{a(k)} \\ \text{departure cost: } \varphi_k(t), \qquad \text{arrival cost: } \psi_k(t). \end{cases}$$



Traffic Flow on a Network



drivers can use different paths $\Gamma_1, \Gamma_2, \ldots$ to reach destination

Does there exist a globally optimal solution, and a Nash equilibrium solution for traffic flow on a network ?

Admissible departure rates

G_k = total number of drivers in the k-th group, $\quad k = 1, \ldots, n$

Γ_p = viable path (concatenation of viable arcs γ_{ij}), $\quad p = 1, \ldots, N$

$t \mapsto \bar{u}_{k,p}(t)$ = departure rate of k-drivers traveling along the path Γ_p

The set of departure rates $\{\bar{u}_{k,p}\}$ is **admissible** if

$$\bar{u}_{k,p}(t) \geq 0\,, \qquad \sum_p \int_{-\infty}^{\infty} \bar{u}_{k,p}(t)\, dt \;=\; G_k \qquad k = 1, \ldots, n$$

Let $\tau_p(t)$ = arrival time for a driver starting at time t, traveling along Γ_p

Main assumptions

(A1) Along each arc γ_{ij} the flux function $\rho \mapsto \rho\, v_{ij}(\rho)$ is twice continuously differentiable and concave down.

$$v_{ij}(0) \ > \ 0, \qquad\qquad v_{ij}(\rho_{max}) \ = \ 0$$

(A2) The cost functions φ, ψ satisfy $\quad \varphi' \ < \ 0, \quad \psi, \psi' \ \geq \ 0,$

$$\lim_{x\to-\infty} \varphi(x) \ = \ +\infty\,, \qquad\qquad \lim_{x\to+\infty} \Big(\varphi(x) + \psi(x)\Big) \ = \ +\infty$$

Global optima and Nash equilibria on networks

An admissible family $\{\bar{u}_{k,p}\}$ *of departure rates is* **globally optimal** *if it minimizes the sum of the total costs of all drivers*

$$J(\bar{u}) \doteq \sum_{k,p} \int \Big(\varphi_k(t) + \psi_k(\tau_p(t))\Big) \bar{u}_{k,p}(t)\, dt$$

An admissible family $\{\bar{u}_{k,p}\}$ *of departure rates is a* **Nash equilibrium solution** *if no driver of any group can lower his own total cost by changing departure time or switching to a different path to reach destination.*

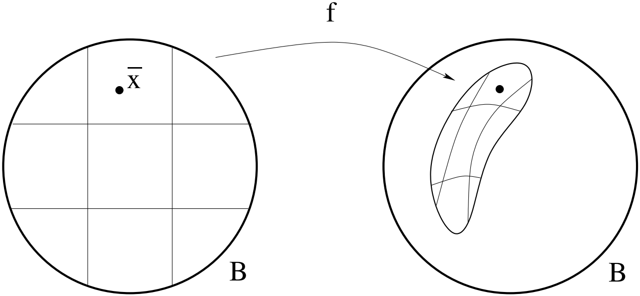
Theorem. *(A.B. - Ke Han, Networks & Heterogeneous Media, 2012).*

On a general network of roads, there exists at least one globally optimal solution, and at least one Nash equilibrium solution.

Two classical theorems in topology

Theorem (Luitzen Egbertus Jan Brouwer, 1912)

Let $B \subset \mathbb{R}^n$ be a closed ball.
Every continuous map $f : B \mapsto B$ admits a fixed point.



$f(\bar{x}) = \bar{x}$

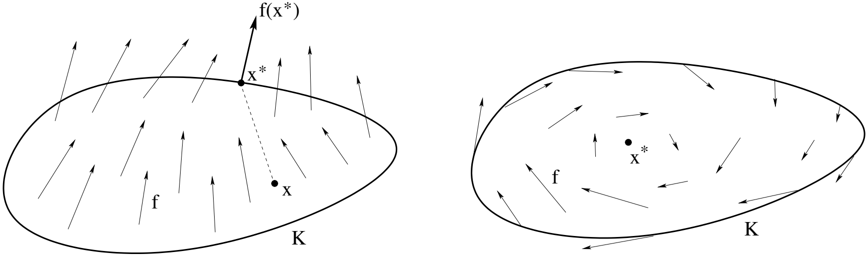
A variational inequality

$K \subset \mathbb{R}^n$ closed, bounded convex set, $\quad f : K \mapsto \mathbb{R}^n$ continuous

Then there exists $x^* \in K$ such that

$$\langle x - x^*,\ f(x^*)\rangle \ \leq\ 0 \qquad \text{for all } x \in K$$

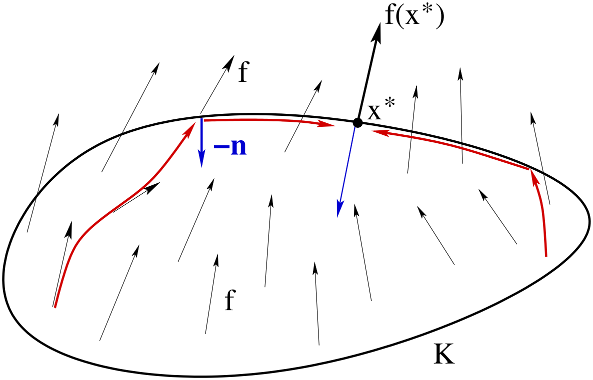
Either $f(x^*) = 0$, or $f(x^*)$ is an outer normal vector to K at x^*



If $f(x)$ is tangent, or points inward at every boundary point of K, then $f(x^*) = 0$

A constrained evolution

Trajectories of $\dot{x} = f(x)$ are constrained to remain in K by a frictionless barrier

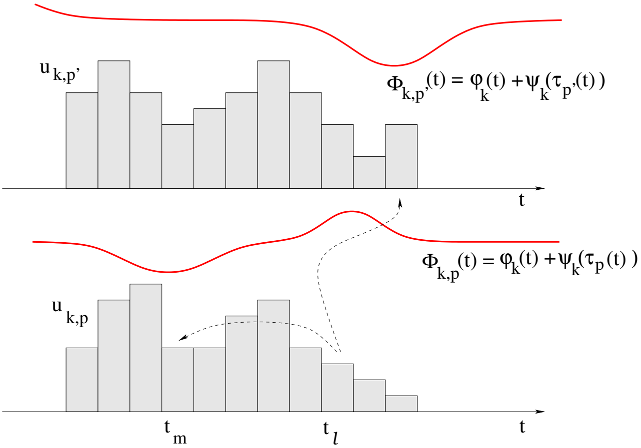


There exists a point $x^* \in K$ that does not move.

Finite dimensional approximations

On a family $\mathcal{K}$ of admissible piecewise constant departure rates $u = (u_{k,p})$, define an evolution equation

$$\frac{d}{d\theta} u = \Psi(u)$$



Existence of a Nash equilibrium on a network

The map $\Psi : \mathcal{K} \mapsto \mathbb{R}^N$ is continuous and inward-pointing
hence it admits a zero: $\Psi(\bar{u}) = 0$

The departure rates $\bar{u} = (\bar{u}_{k,p})$ represent a Galerkin approximation to a Nash equilibrium

Letting the discretization step Δt approach zero, taking subsequences:

departure rates:	$\bar{u}^{\nu}_{k,p}(\cdot) \rightharpoonup \bar{u}_{k,p}(\cdot)$	weakly
arrival times:	$\tau^{\nu}_{p}(\cdot) \to \tau_{p}(\cdot)$	uniformly

The departure rates $\bar{u}_{k,p}(\cdot)$ provide a Nash equilibrium

Work in progress

- More general conditions at junctions (K. Han, B. Piccoli)
- Necessary conditions for globally optimal solutions on networks
  No queues ? No shocks ?

Stability of Nash equilibrium ?

To justify the practical relevance of a Nash equilibrium, we need to

- analyze a suitable dynamic model
- check whether the rate of departures asymptotically converges to the Nash equilibrium

Assume: drivers can change their departure time on a day-to-day basis, in order to decrease their own cost (one group of drivers, one single road)

Introduce an additional variable θ counting the number of days on the calendar.

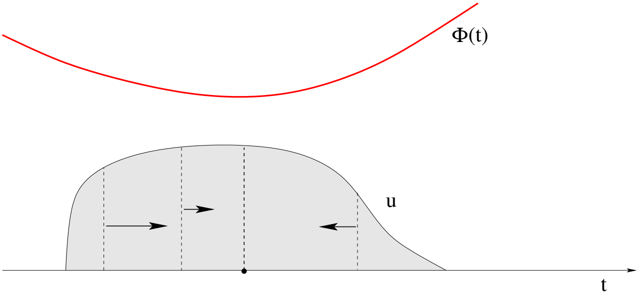
$$\bar{u}(t,\theta) \doteq \text{rate of departures at time } t, \text{ on day } \theta$$

$$\Phi(t,\theta) \doteq \text{cost to a driver starting at time } t, \text{ on day } \theta$$

A conservation law with non-local flux

Model 1: drivers gradually change their departure time, drifting toward times where the cost is smaller.
If the rate of change is proportional to the gradient of the cost, this leads to the conservation law

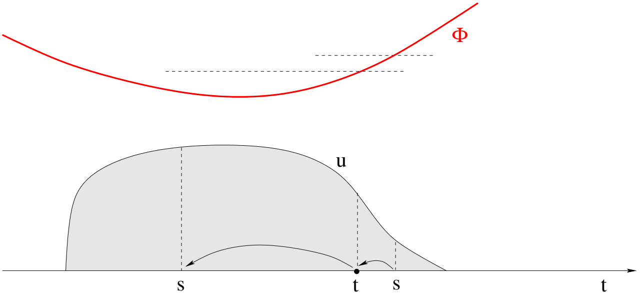
$$\bar{u}_\theta + [\Phi_t \, \bar{u}]_t \;=\; 0$$



An integral evolution equation

Model 2: drivers jump to different departure times having a lower cost.
If the rate of change is proportional to the difference between the costs, this yields

$$\frac{d}{d\theta}\bar{u}(t) \;=\; \int \bar{u}(s)\Big[\Phi(s)-\Phi(t)\Big]_+ ds - \int \bar{u}(t)\Big[\Phi(t)-\Phi(s)\Big]_+ ds$$



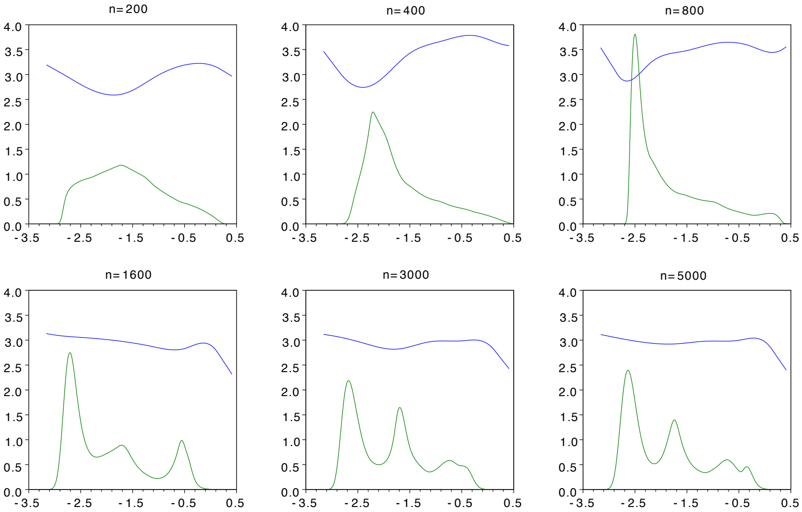
Numerical experiments (Wen Shen, 2011)

Question: as $\theta \to \infty$, does the departure rate $\overline{u}(t, \theta)$ approach the unique Nash equilibrium?

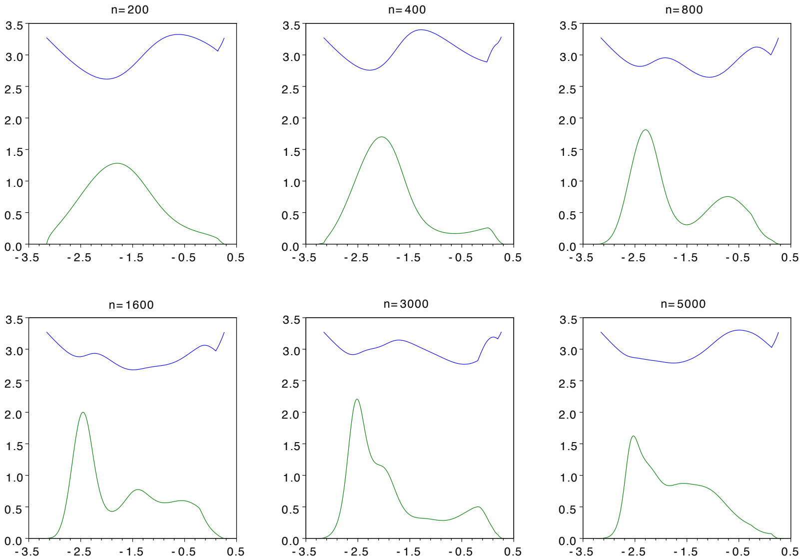
Flux function: $f(\rho) = \rho(2 - \rho)$

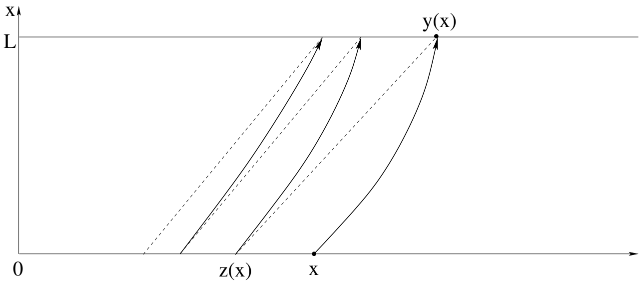
Departure and arrival costs: $\varphi(t) = -t, \qquad \psi(t) = e^t$

Numerical simulation: Model 1



Numerical simulation: Model 2





main difficulty: non-local dependence

linearized equation: $$\frac{d}{d\theta}Y(x) = \Big[\alpha(x)\Big(\beta(x)Y(x) - Y(z(x))\Big)\Big]_x$$