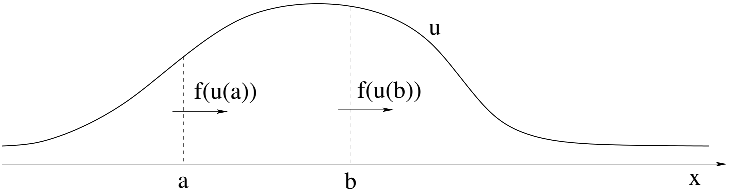
The Scalar Conservation Law

$$\mathbf{u_t} + \mathbf{f(u)_x} = \mathbf{0} \qquad \mathbf{u} = \text{conserved quantity}, \qquad \mathbf{f(u)} = \text{flux}$$

$$\frac{d}{dt}\int_a^b u(t,x)\,dx = \int_a^b u_t(t,x)\,dx = -\int_a^b f\big(u(t,x)\big)_x\,dx$$

$$= f\big(u(t,a)\big) - f\big(u(t,b)\big) = [\text{inflow at } a] - [\text{outflow at } b]$$



Weak solutions

conservation equation: $\mathbf{u_t + f(u)_x \;=\; 0}$

quasilinear form: $\mathbf{u_t + f'(u)u_x \;=\; 0}$

Conservation equation remains meaningful for $u = u(t,x)$ discontinuous, in distributional sense:

$$\int\!\!\int \left\{ u\phi_t + f(u)\phi_x \right\} dxdt \;=\; 0 \qquad \text{for all} \quad \phi \in \mathcal{C}^1_c$$

Need only : $u, f(u)$ locally integrable

Convergence of weak solutions

$$\mathbf{u_t} + \mathbf{f(u)_x} \;=\; \mathbf{0}$$

Assume: u_n is a solution, for every $n \geq 1$,

$$u_n \to u, \qquad f(u_n) \to f(u) \qquad \text{in} \quad \mathbf{L}^1_{loc}$$

then

$$\iint \{u\phi_t + f(u)\phi_x\}\, dxdt \;=\; \lim_{n\to\infty} \iint \{u_n\phi_t + f(u_n)\phi_x\}\, dxdt \;=\; 0$$

for all $\quad \phi \in \mathcal{C}^1_c$

(no need to check convergence of derivatives)

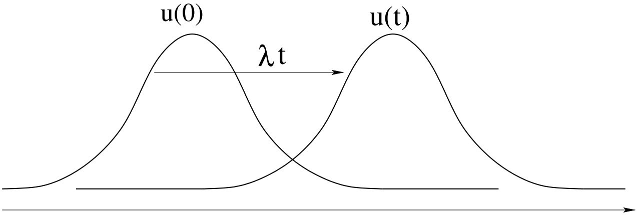
Scalar Equation with Linear Flux

$$u_t + f(u)_x = 0 \qquad\qquad f(u) = \lambda u$$

$$u_t + \lambda u_x = 0 \qquad\qquad u(0,x) = \phi(x)$$

Explicit solution: $u(t,x) = \phi(x - \lambda t)$

traveling wave with speed $f'(u) = \lambda$



The method of characteristics

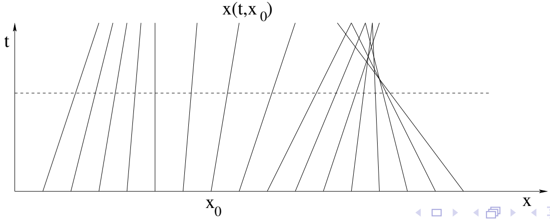
$$u_t + f'(u)u_x \ = \ 0 \qquad\qquad u(0,x) = \phi(x)$$

For each x_0, consider the straight line

$$t \ \mapsto \ x(t,x_0) \ = \ x_0 + tf'(\phi(x_0))$$

Set $u = \phi(x_0)$ along this line, so that $\dot{x}(t) = f'(u(t,x(t)))$. As long as characteristics do not cross, this yields a solution:

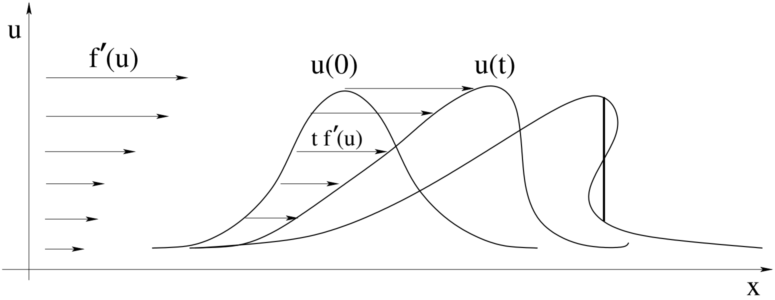
$$0 \ = \ \frac{d}{dt}u(t,x(t)) \ = \ u_t + \dot{x}u_x \ = \ u_t + f'(u)u_x$$



Loss of Regularity

$$u_t + f'(u)u_x \; = \; 0$$

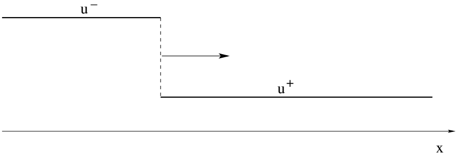
Assume: characteristic speed $f'(u)$ is not constant



Global solutions only in a space of discontinuous functions

$$u(t, \cdot) \in BV$$

Shocks



$$\mathbf{u_t + f(u)_x \ = \ 0}$$

$$u(t,x) \ = \ \begin{cases} u^- & \text{if} \quad x < \lambda t \\ u^+ & \text{if} \quad x > \lambda t \end{cases}$$

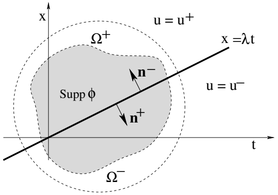
is a weak solution if and only if

$$\lambda \cdot [u^+ - u^-] \ = \ f(u^+) - f(u^-)$$ **Rankine - Hugoniot equations**

[speed of the shock] $\times$ [jump in the state] $=$ [jump in the flux]

Derivation of the Rankine - Hugoniot equation

$$\iint \Big\{ u\phi_t + f(u)\phi_x \Big\}\, dxdt \;=\; 0 \qquad \text{for all} \quad \phi \in \mathcal{C}^1_c$$



$$\mathbf{v} \;\doteq\; \Big(u\phi\,,\;\; f(u)\phi \Big)$$

$$0 \;=\; \iint_{\Omega^+\cup\Omega^-} \text{div }\; \mathbf{v}\, dxdt \;=\; \int_{\partial\Omega^+} \mathbf{n}^+ \cdot \mathbf{v}\, ds + \int_{\partial\Omega^-} \mathbf{n}^- \cdot \mathbf{v}\, ds$$

$$= \int \big[\lambda u^+ - f(u^+)\big]\, \phi(t,\lambda t)\, dt + \int \big[-\lambda u^- + f(u^-)\big]\, \phi(t,\lambda t)\, dt$$

$$= \int \Big[\lambda(u^+ - u^-) - \big(f(u^+) - f(u^-)\big)\Big]\, \phi(t,\lambda t)\, dt$$

Geometric interpretation

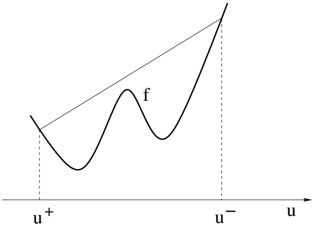
$$\lambda\,(u^+ - u^-) \;=\; f(u^+) - f(u^-) \;=\; \int_0^1 f'\big(\theta u^+ + (1-\theta)u^-\big)\cdot(u^+ - u^-)\;d\theta$$

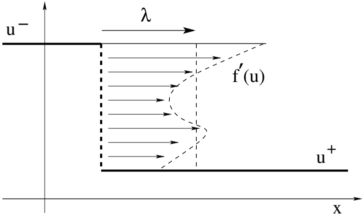
The Rankine-Hugoniot conditions hold if and only if the speed of the shock is

$$\lambda \;=\; \frac{f(u^+) - f(u^-)}{u^+ - u^-} \;=\; \int_0^1 f'\big(\theta u^+ + (1-\theta)u^-\big)\;d\theta$$

$$= \;\text{[average characteristic speed]}$$

scalar conservation law: $\qquad u_t + f(u)_x = 0$





$$\lambda \;=\; \frac{f(u^+) - f(u^-)}{u^+ - u^-} \;=\; \frac{1}{u^+ - u^-}\int_{u^-}^{u^+} f'(s)\, ds$$

[speed of the shock] $=$ [slope of secant line through u^-, u^+ on the graph of f]

$=$ [average of the characteristic speeds between u^- and u^+]

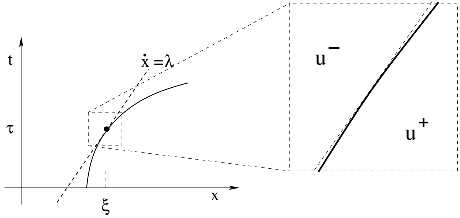
Points of approximate jump

The function $u = u(t,x)$ has an **approximate jump** at a point (τ, ξ) if there exists states $u^- \neq u^+$ and a speed λ such that, calling

$$U(t,x) \doteq \begin{cases} u^- & \text{if} \quad x < \lambda t, \\ u^+ & \text{if} \quad x > \lambda t, \end{cases}$$

there holds

$$\lim_{\rho \to 0+} \frac{1}{\rho^2} \int_{\tau-\rho}^{\tau+\rho} \int_{\xi-\rho}^{\xi+\rho} \Big| u(t,x) - U(t-\tau,\ x-\xi) \Big| \, dxdt = 0$$



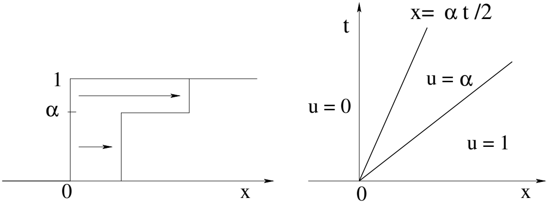
Theorem. *If u is a weak solution to a conservation law then the Rankine-Hugoniot equations hold at each point of approximate jump.*

Example: a Cauchy problem for Burgers' equation

$$u_t + (u^2/2)_x = 0 \qquad\qquad u(0,x) = \begin{cases} 1 & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$$

Each $\alpha \in [0,1]$ yields a weak solution

$$u_\alpha(t,x) = \begin{cases} 0 & \text{if} \quad x < \alpha t/2 \\ \alpha & \text{if} \quad \alpha t/2 \leq x < (1+\alpha)t/2 \\ 1 & \text{if} \quad x \geq (1+\alpha)t/2 \end{cases}$$



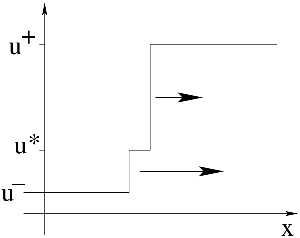
Stability conditions for shocks

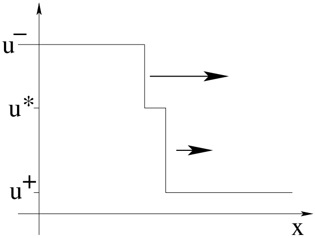
Perturb the shock with left and right states u^-, u^+ by inserting an intermediate state $u^* \in [u^-, u^+]$

Initial shock is stable $\Longleftrightarrow$

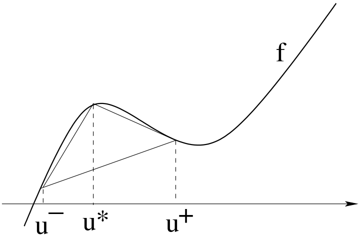
[speed of jump behind] $\geq$ [speed of jump ahead]

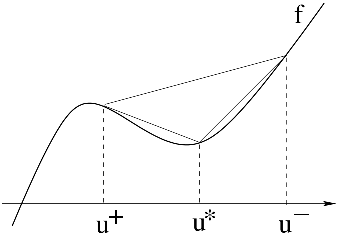
$$\frac{f(u^*) - f(u^-)}{u^* - u^-} \geq \frac{f(u^+) - f(u^*)}{u^+ - u^*}$$





speed of a shock = slope of a secant line to the graph of f

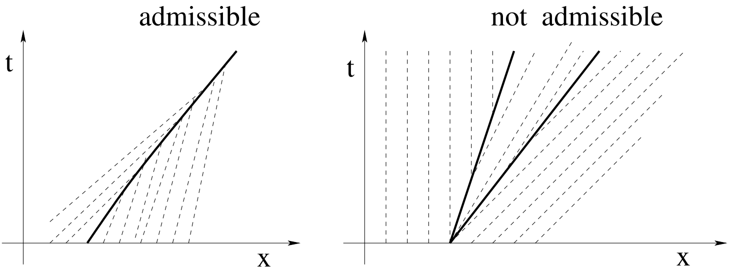




Stability conditions:

- when $u^- < u^+$ the graph of f should remain above the secant line
- when $u^- > u^+$, the graph of f should remain below the secant line

The Lax admissibility condition



A shock connecting the states u^-, u^+, travelling with speed $\lambda = \frac{f(u^+)-f(u^-)}{u^+-u^-}$ is *admissible* if

$$f'(u^-) \;\geq\; \lambda \;\geq\; f'(u^+)$$

i.e. characteristics do not move out from the shock from either side

Existence of solutions

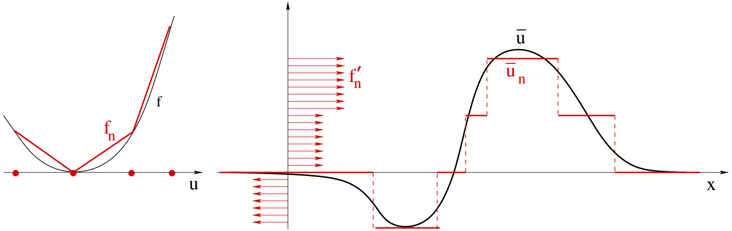
Cauchy problem: $\quad u_t + f(u)_x = 0\,, \qquad u(0,x) = \bar{u}(x)$

Polygonal approximations of the flux function (Dafermos, 1972)

Choose a piecewise affine function f_n such that

$$f_n(u) = f(u) \qquad u = j \cdot 2^{-n}\,, \qquad j \in \mathbb{Z}$$

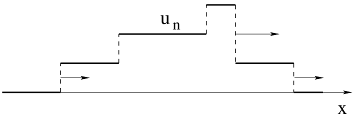
Approximate the initial data with a function $\bar{u}_n : \mathbb{R} \mapsto 2^{-n} \cdot \mathbb{Z}$

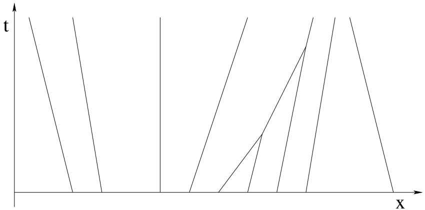


Front tracking approximations

piecewise constant approximate solutions: $u_n(t,x)$

$$(u_n)_t + f_n(u_n)_x = 0 \qquad\qquad u_n(0,x) = \bar{u}_n(x)$$





$$Tot.Var.(u_n(t,\cdot)) \le Tot.Var.(\bar{u}_n) \le Tot.Var.(\bar{u})$$

$\Longrightarrow$ as $n \to \infty$, a subsequence converges in $\mathbf{L}^1_{loc}([0,T] \times \mathbb{R})$ to a weak solution $u = u(t,x)$

A contractive semigroup of entropy weak solutions

$$u_t + f(u)_x \;=\; 0$$

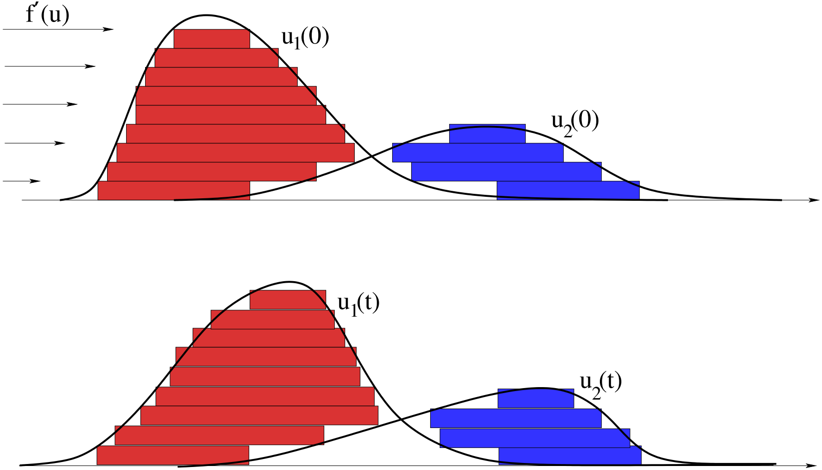
Two initial data in $\mathbf{L}^1(\mathbb{R})$: $\qquad u_1(0,x) = \bar{u}_1(x), \quad u_2(0,x) = \bar{u}_2(x)$

$\mathbf{L}^1$ - distance between solutions does not increase in time:

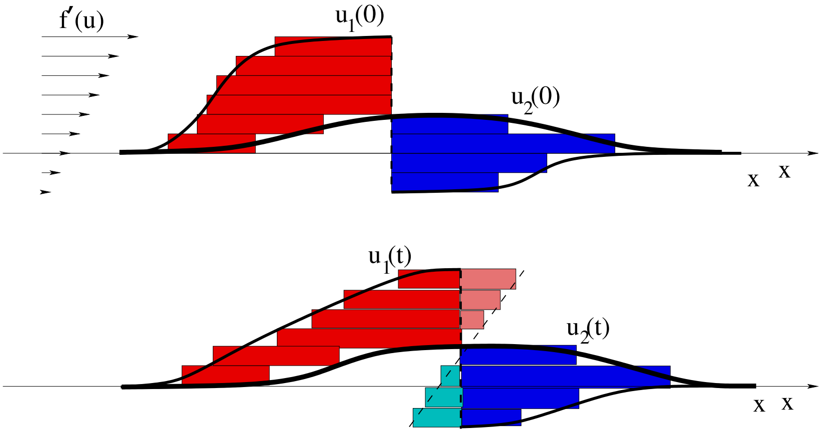
$$\|u_1(t,\cdot) - u_2(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R})} \;\leq\; \|\bar{u}_1 - \bar{u}_2\|_{\mathbf{L}^1(\mathbb{R})}$$

(not true for the $\mathbf{L}^p$ distance, $p > 1$)

The $\mathbf{L}^1$ distance between continuous solutions remains constant



The $\mathbf{L}^1$ distance decreases when a shock in one solution crosses the graph of the other solution



A related Hamilton-Jacobi equation

$$u_t + f(u)_x = 0 \qquad\qquad u(0,x) = \bar u(x)$$

$$U(t,x) = \int_{-\infty}^{x} u(t,y)\,dy$$

$$U_t + f(U_x) = 0 \qquad\qquad U(0,x) = \overline{U}(x) = \int_{-\infty}^{x} \bar u(y)\,dy$$

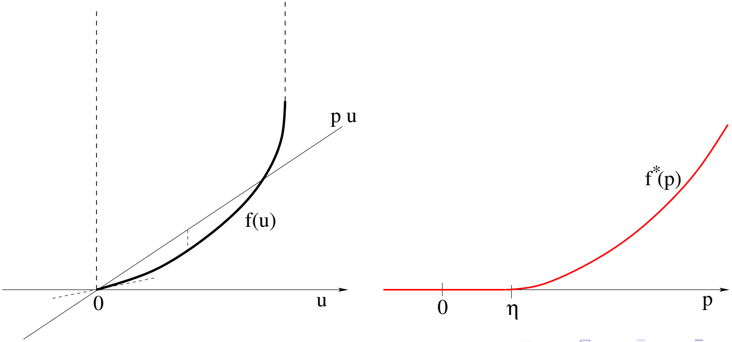
f convex $\qquad \Longrightarrow$

$U = U(t,x)$ is the value function for an optimization problem

Legendre transform

$$u \mapsto f(u) \in \mathbb{R} \cup \{+\infty\} \qquad \text{convex}$$

$$f^*(p) \doteq \max_u \{pu - f(u)\}$$

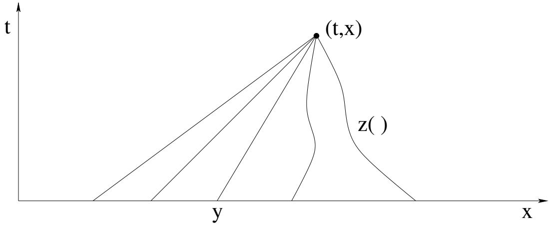


A representation formula

$$U_t + f(U_x) \;=\; 0 \qquad\qquad U(0,x) \;=\; \overline{U}(x)$$

$$U(t,x) \;=\; \inf_{z(\cdot)} \left\{ \int_0^t f^*(\dot z(s))\,ds + \overline{U}(z(0))\,; \qquad z(t) = x \right\}$$

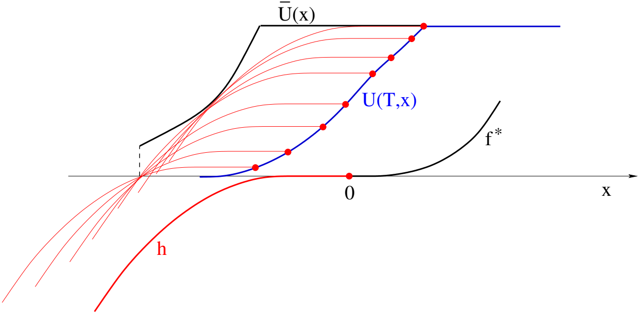
$$=\; \min_{y\in\mathbb{R}} \left\{ t\, f^*\left(\frac{x-y}{t}\right) + \overline{U}(y) \right\}$$



A geometric construction

$$U_t + f(U_x) \;=\; 0 \qquad\qquad U(0,x) \;=\; \overline{U}(x)$$

define $\quad h(s) \doteq -T\, f^*\left(\dfrac{-s}{T}\right)$



$$U(T,x) \;=\; \inf_y \left\{ \overline{U}(y) - h(y-x) \right\}$$

The Lax formula

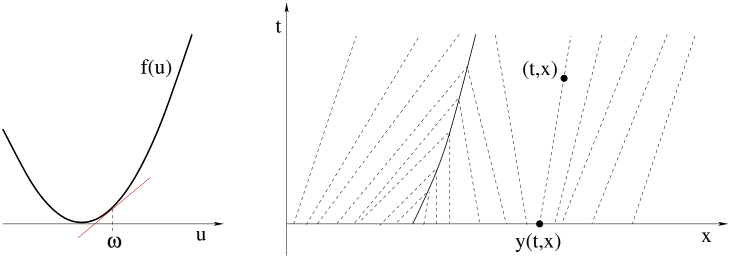
Cauchy problem: $$\begin{cases} u_t + f(u)_x &= 0\,, \\ u(0,x) &= \bar u(x) \end{cases}$$

For each $t > 0$, and all but at most countably many values of $x \in \mathbb{R}$, there exists a unique $y(t, x)$ s.t.

$$y(t,\, x) \;=\; \arg\min_{y\in\mathbb{R}} \left\{ t\, f^*\!\left(\frac{x-y}{t}\right) + \int_{-\infty}^{y} \bar u(s)\, ds \right\}$$

the solution to the Cauchy problem is

$$u(t,\, x) \;=\; (f')^{-1}\!\left(\frac{x - y(t,\, x)}{t}\right) \tag{1}$$



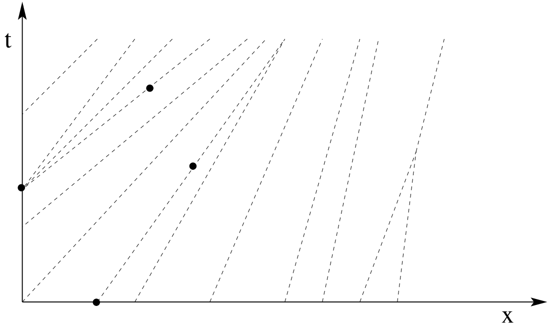
$$y(t,\, x) \;=\; \arg\min_{y\in\mathbb{R}} \left\{ t\, f^*\left(\frac{x-y}{t}\right) + \int_{-\infty}^{y} \bar u(s)\, ds \right\}$$

define the characteristic speed $\quad \xi \doteq \dfrac{x - y(t,x)}{t}$

if $\;f'(\omega) = \xi \qquad$ then $\qquad u(t,x) = \omega$

Initial-Boundary value problem

$$u_t + f(u)_x \ = \ 0 \qquad\qquad \begin{cases} u(0,x) & = \bar{u}(x) \qquad x > 0 \\ u(t,0) & = b(t) \qquad t > 0 \end{cases}$$



P. Le Floch, Explicit formula for scalar non-linear conservation laws with boundary condition, *Math. Models Appl. Sci.* (1988)

Systems of Conservation Laws

$$\begin{cases} \dfrac{\partial}{\partial t} u_1 + \dfrac{\partial}{\partial x} f_1(u_1, \ldots, u_n) & = & 0, \\ & \cdot \quad \cdot \quad \cdot & \\ \dfrac{\partial}{\partial t} u_n + \dfrac{\partial}{\partial x} f_n(u_1, \ldots, u_n) & = & 0 \end{cases}$$

$$u_t + f(u)_x = 0$$

$u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ conserved quantities

$f = (f_1, \ldots, f_n) : \mathbb{R}^n \mapsto \mathbb{R}^n$ fluxes

Hyperbolic Systems

$$u_t + f(u)_x = 0 \qquad\qquad u = u(t,x) \in \mathbb{R}^n$$

$$u_t + A(u)u_x = 0 \qquad\qquad A(u) = Df(u)$$

The system is **strictly hyperbolic** if each $n \times n$ matrix $A(u)$ has real distinct eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u)$$

right eigenvectors	$r_1(u), \ldots, r_n(u)$	(column vectors)
left eigenvectors	$l_1(u), \ldots, l_n(u)$	(row vectors)

$$Ar_i = \lambda_i r_i \qquad\qquad l_i A = \lambda_i l_i$$

Choose bases so that $\quad l_i \cdot r_j = \begin{cases} 1 & \textit{if} \quad i = j \\ 0 & \textit{if} \quad i \neq j \end{cases}$

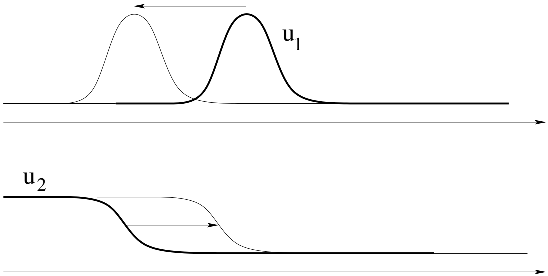
A linear hyperbolic system

$$u_t + Au_x = 0 \qquad\qquad u(0,x) = \phi(x)$$

$$\lambda_1 < \cdots < \lambda_n \quad \text{eigenvalues} \qquad\qquad r_1, \ldots, r_n \quad \text{eigenvectors}$$

Explicit solution: **linear superposition of travelling waves**

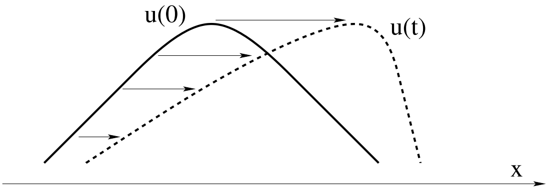
$$u(t,x) = \sum_i \phi_i(x - \lambda_i t) r_i \qquad\qquad \phi_i(s) = l_i \cdot \phi(s)$$



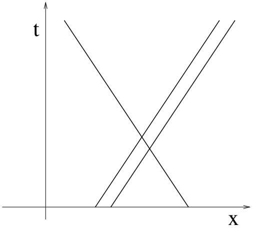
Nonlinear effects - 1

$$u_t + A(u)u_x = 0$$

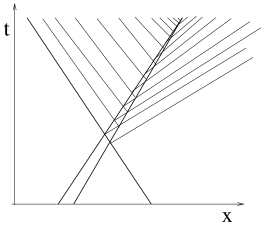
eigenvalues depend on u $\Longrightarrow$ waves change shape



eigenvectors depend on u $\Longrightarrow$ nontrivial wave interactions



linear



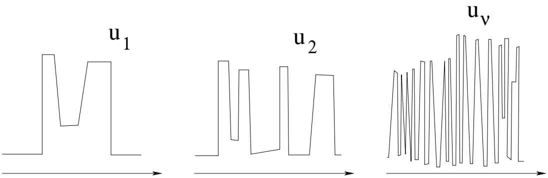
nonlinear

$$u_t + f(u)_x = 0 \qquad\qquad u(0,x) = \bar{u}(x)$$

- Construct a sequence of approximate solutions u_m

- Show that (a subsequence) converges: $u_m \to u$ in $\mathbf{L}^1_{loc}$

$$\Longrightarrow \quad u \text{ is a weak solution}$$



Need: a-priori bound on the total variation (J. Glimm, 1965)

Building block: the Riemann Problem

$$u_t + f(u)_x = 0 \qquad\qquad u(0,x) = \begin{cases} u^- & \text{if} \quad x < 0 \\ u^+ & \text{if} \quad x > 0 \end{cases}$$

B. Riemann 1860: 2×2 system of isentropic gas dynamics

P. Lax 1957: $n \times n$ systems (+ special assumptions)

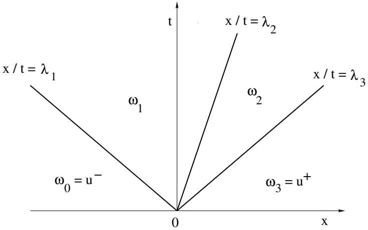
T. P. Liu 1975 $n \times n$ systems (generic case)

S. Bianchini 2003 (vanishing viscosity limit for general hyperbolic systems, possibly non-conservative)

invariant w.r.t. symmetry: $\quad u^\theta(t,x) \doteq u(\theta t, \theta x) \qquad \theta > 0$

Riemann Problem for Linear Systems

$$u_t + Au_x = 0 \qquad\qquad u(0,x) = \begin{cases} u^- & \text{if} \quad x<0 \\ u^+ & \text{if} \quad x>0 \end{cases}$$

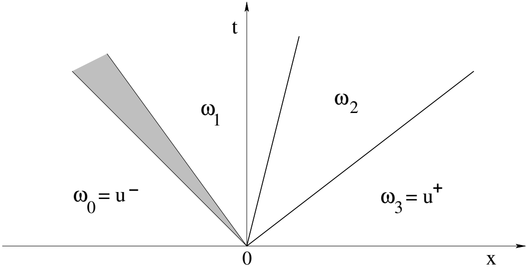


$$u^+ - u^- = \sum_{j=1}^{n} c_j r_j \qquad\qquad \text{(sum of eigenvectors of } A\text{)}$$

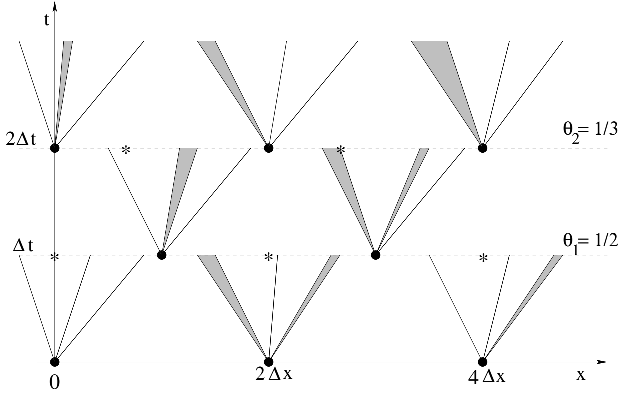
$$\text{intermediate states :} \quad \omega_i \doteq u^- + \sum_{j\le i} c_j r_j$$

i-th jump: $\omega_i - \omega_{i-1} = c_i r_i$ travels with speed λ_i

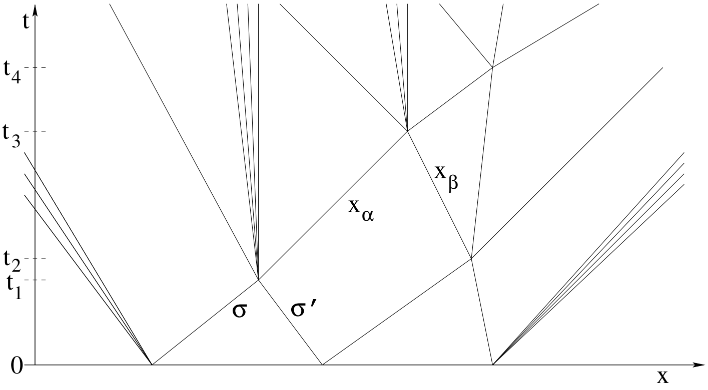
General solution of the Riemann problem: concatenation of elementary waves



Glimm scheme: piecing together solutions of Riemann problems on a fixed grid in the t-x plane



Front tracking scheme: piecing together piecewise constant solutions of Riemann problems at points where fronts interact



Existence of solutions

$$u_t + f(u)_x = 0, \qquad u(0,x) = \bar{u}(x)$$

Theorem (Glimm 1965).

Assume:

- *system is strictly hyperbolic (+ some technical assumptions)*

Then there exists $\delta > 0$ *such that, for every initial condition* $\bar{u} \in \mathbf{L}^1(\mathbb{R};\ \mathbb{R}^n)$ *with*

$$Tot.Var.(\bar{u}) \leq \delta,$$

the Cauchy problem has an entropy admissible weak solution $u = u(t,x)$ *defined for all* $t \geq 0$.

Uniqueness and continuous dependence on the initial data

$$u_t + f(u)_x \ = \ 0 \qquad\qquad u(0,x) = \bar u(x)$$

Theorem (A.B.- R.Colombo, B.Piccoli, T.P.Liu, T.Yang, 1994-1998).

For every initial data $\bar u$ with small total variation, the front tracking approximations converge to a unique limit solution $u : [0,\infty[\ \mapsto \mathbf{L}^1(\mathbb{R})$.

The flow map $(\bar u, t) \mapsto u(t,\cdot) \doteq S_t\bar u$ is a uniformly Lipschitz semigroup:

$$S_0\bar u = \bar u, \qquad S_s(S_t\bar u) = S_{s+t}\bar u$$

$$\big\|S_t\bar u - S_s\bar v\big\|_{\mathbf{L}^1} \ \leq \ L\cdot\big(\|\bar u - \bar v\|_{\mathbf{L}^1} + |t-s|\big) \qquad \text{for all} \ \ \bar u, \bar v, \ \ s,t\geq 0$$

Theorem (A.B.- P. LeFloch, M.Lewicka, P.Goatin, 1996-1998).

Any entropy weak solution to the Cauchy problem coincides with the limit of front tracking approximations, hence it is unique

Vanishing viscosity approximations

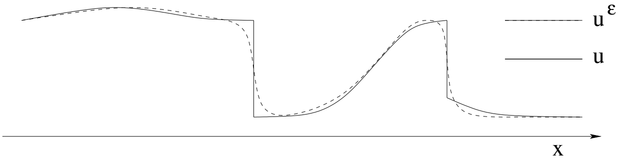
Claim: weak solutions of the hyperbolic system

$$u_t + f(u)_x = 0$$

can be obtained as limits of solutions to the parabolic system

$$u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon\, u^\varepsilon_{xx}$$

letting the viscosity $\varepsilon \to 0+$



Theorem (S. Bianchini, A. Bressan, *Annals of Math.* 2005)

Consider a strictly hyperbolic system with viscosity

$$u_t + A(u)u_x = \varepsilon\, u_{xx} \qquad\qquad u(0,x) = \bar u(x)\,. \tag{CP}$$

If Tot.Var.$\{\bar u\}$ is sufficiently small, then (CP) admits a unique solution $u^\varepsilon(t,\cdot) = S^\varepsilon_t \bar u$, defined for all $t \geq 0$. Moreover

$$\text{Tot.Var.}\bigl\{S^\varepsilon_t \bar u\bigr\} \;\leq\; C\,\text{Tot.Var.}\{\bar u\}\,, \qquad\qquad (\textbf{BV bounds})$$

$$\Bigl\|S^\varepsilon_t \bar u - S^\varepsilon_t \bar v\Bigr\|_{\mathbf{L}^1} \;\leq\; L\,\|\bar u - \bar v\|_{\mathbf{L}^1} \qquad\qquad (\mathbf{L^1}\ \textbf{stability})$$

(Convergence) If $A(u) = Df(u)$, then as $\varepsilon \to 0$, the viscous solutions u^ε converge to the unique entropy weak solution of the system of conservation laws

$$u_t + f(u)_x \;=\; 0$$

Main open problems

- Global existence of solutions to hyperbolic systems
  for initial data $\bar{u}$ with **large total variation**

- Existence of entropy weak solutions
  for systems in **several space dimensions**