The Scalar Conservation Law

\[ u_t + f(u)_x = 0 \]

\( u = \) conserved quantity, \( f(u) = \) flux

\[
\frac{d}{dt} \int_a^b u(t,x) \, dx = \int_a^b u_t(t,x) \, dx = - \int_a^b f(u(t,x))_x \, dx
\]

\[ = f(u(t,a)) - f(u(t,b)) = [\text{inflow at } a] - [\text{outflow at } b] \]

Diagram: A graph showing the function \( u \) with points \( a \) and \( b \), and the values \( f(u(a)) \) and \( f(u(b)) \). The function \( f(u) \) is also shown as the flux at these points.
Weak solutions

**Conservation equation:** \( u_t + f(u)_x = 0 \)

**Quasilinear form:** \( u_t + f'(u)u_x = 0 \)

Conservation equation remains meaningful for \( u = u(t,x) \) discontinuous, in distributional sense:

\[
\int \int \left\{ u\phi_t + f(u)\phi_x \right\} \, dx\,dt = 0 \quad \text{for all} \quad \phi \in C^1_c
\]

Need only: \( u, f(u) \) locally integrable
Convergence of weak solutions

\[ u_t + f(u)_x = 0 \]

Assume: \( u_n \) is a solution, for every \( n \geq 1 \),

\[ u_n \to u, \quad f(u_n) \to f(u) \quad \text{in} \quad L^1_{loc} \]

then

\[
\int \int \left\{ u \phi_t + f(u) \phi_x \right\} \, dxdt = \lim_{n \to \infty} \int \int \left\{ u_n \phi_t + f(u_n) \phi_x \right\} \, dxdt = 0
\]

for all \( \phi \in C^1_c \)

(no need to check convergence of derivatives)
Scalar Equation with Linear Flux

\[ u_t + f(u)_x = 0 \quad f(u) = \lambda u \]

\[ u_t + \lambda u_x = 0 \quad u(0, x) = \phi(x) \]

Explicit solution: \[ u(t, x) = \phi(x - \lambda t) \]

**traveling wave** with speed \( f'(u) = \lambda \)
The method of characteristics

\[ u_t + f'(u)u_x = 0 \quad u(0, x) = \phi(x) \]

For each \( x_0 \), consider the straight line

\[ t \mapsto x(t, x_0) = x_0 + tf'(\phi(x_0)) \]

Set \( u = \phi(x_0) \) along this line, so that \( \dot{x}(t) = f'(u(t, x(t))) \). As long as characteristics do not cross, this yields a solution:

\[ 0 = \frac{d}{dt} u(t, x(t)) = u_t + \dot{x} u_x = u_t + f'(u)u_x \]
Loss of Regularity

\[ u_t + f'(u)u_x = 0 \]

Assume: characteristic speed \( f'(u) \) is not constant

Global solutions only in a space of discontinuous functions

\[ u(t, \cdot) \in BV \]
\[
\begin{align*}
    u_t + f(u)_x &= 0 \\
    u(t, x) &= \begin{cases} 
        u^- & \text{if } x < \lambda t \\
        u^+ & \text{if } x > \lambda t
    \end{cases}
\end{align*}
\]

is a weak solution if and only if

\[
\lambda \cdot [u^+ - u^-] = f(u^+) - f(u^-) \quad \text{Rankine - Hugoniot equations}
\]

\[
\text{[speed of the shock] } \times \text{[jump in the state]} = \text{[jump in the flux]}
\]
Derivation of the Rankine - Hugoniot equation

\[ \int \int \left\{ u \phi_t + f(u) \phi_x \right\} \, dx \, dt = 0 \quad \text{for all} \quad \phi \in C^1_c \]

\[ \mathbf{v} = \left( u \phi, \ f(u) \phi \right) \]

\[ 0 = \int \int_{\Omega^+ \cup \Omega^-} \text{div} \ \mathbf{v} \, dx \, dt = \int_{\partial \Omega^+} \mathbf{n}^+ \cdot \mathbf{v} \, ds + \int_{\partial \Omega^-} \mathbf{n}^- \cdot \mathbf{v} \, ds \]

\[ = \int \left[ \lambda u^+ - f(u^+) \right] \phi(t, \lambda t) \, dt + \int \left[ - \lambda u^- + f(u^-) \right] \phi(t, \lambda t) \, dt \]

\[ = \int \left[ \lambda (u^+ - u^-) - (f(u^+) - f(u^-)) \right] \phi(t, \lambda t) \, dt \]
Geometric interpretation

\[ \lambda (u^+ - u^-) = f(u^+) - f(u^-) = \int_0^1 f'(\theta u^+ + (1-\theta)u^-) \cdot (u^+ - u^-) \, d\theta \]

The Rankine-Hugoniot conditions hold if and only if the speed of the shock is

\[ \lambda = \frac{f(u^+) - f(u^-)}{u^+ - u^-} = \int_0^1 f'(\theta u^+ + (1-\theta)u^-) \, d\theta \]

= [average characteristic speed]
scalar conservation law: \[ u_t + f(u)_x = 0 \]

\[ \lambda = \frac{f(u^+) - f(u^-)}{u^+ - u^-} = \frac{1}{u^+ - u^-} \int_{u^-}^{u^+} f'(s) \, ds \]

[speed of the shock] = [slope of secant line through \( u^- \), \( u^+ \) on the graph of \( f \)]

= [average of the characteristic speeds between \( u^- \) and \( u^+ \)]
Points of approximate jump

The function $u = u(t, x)$ has an **approximate jump** at a point $(\tau, \xi)$ if there exists states $u^- \neq u^+$ and a speed $\lambda$ such that, calling

$$U(t, x) \equiv \begin{cases} u^- & \text{if } x < \lambda t, \\ u^+ & \text{if } x > \lambda t, \end{cases}$$

there holds

$$\lim_{\rho \to 0^+} \frac{1}{\rho^2} \int_{\tau-\rho}^{\tau+\rho} \int_{\xi-\rho}^{\xi+\rho} \left| u(t, x) - U(t - \tau, x - \xi) \right| \, dx \, dt = 0$$

**Theorem.** If $u$ is a weak solution to a conservation law then the Rankine-Hugoniot equations hold at each point of approximate jump.
Weak solutions can be non-unique

Example: a Cauchy problem for Burgers’ equation

\[ u_t + (u^2/2)_x = 0 \quad u(0, x) = \begin{cases} 
1 & \text{if } x \geq 0 \\
0 & \text{if } x < 0 
\end{cases} \]

Each \( \alpha \in [0, 1] \) yields a weak solution

\[ u_\alpha(t, x) = \begin{cases} 
0 & \text{if } x < \alpha t/2 \\
\alpha & \text{if } \alpha t/2 \leq x < (1 + \alpha)t/2 \\
1 & \text{if } x \geq (1 + \alpha)t/2 
\end{cases} \]
Stability conditions for shocks

Perturb the shock with left and right states $u^-, u^+$ by inserting an intermediate state $u^* \in [u^-, u^+]$

Initial shock is stable $\iff$

\[ \text{[speed of jump behind]} \geq \text{[speed of jump ahead]} \]

\[ \frac{f(u^*) - f(u^-)}{u^* - u^-} \geq \frac{f(u^+) - f(u^*)}{u^+ - u^*} \]
speed of a shock = slope of a secant line to the graph of $f$

Stability conditions:

- when $u^- < u^+$ the graph of $f$ should remain above the secant line
- when $u^- > u^+$, the graph of $f$ should remain below the secant line
The Lax admissibility condition

A shock connecting the states $u^-, u^+$, travelling with speed \( \lambda = \frac{f(u^+)-f(u^-)}{u^+-u^-} \) is **admissible** if

\[
    f'(u^-) \geq \lambda \geq f'(u^+)
\]

i.e. characteristics do not move out from the shock from either side.
Existence of solutions

Cauchy problem: \( u_t + f(u)_x = 0 \), \( u(0, x) = \bar{u}(x) \)

Polygonal approximations of the flux function (Dafermos, 1972)

Choose a piecewise affine function \( f_n \) such that

\[
    f_n(u) = f(u) \quad u = j \cdot 2^{-n} , \quad j \in \mathbb{Z}
\]

Approximate the initial data with a function \( \bar{u}_n : \mathbb{R} \mapsto 2^{-n} \cdot \mathbb{Z} \)
piecewise constant approximate solutions: $u_n(t, x)$

\[
(u_n)_t + f_n(u_n)_x = 0 \quad \text{and} \quad u_n(0, x) = \bar{u}_n(x)
\]

\[
\text{Tot. Var.}(u_n(t, \cdot)) \leq \text{Tot. Var.}(\bar{u}_n) \leq \text{Tot. Var.}(\bar{u})
\]

$\implies$ as $n \to \infty$, a subsequence converges in $L^1_{\text{loc}}([0, T] \times \mathbb{R})$ to a weak solution $u = u(t, x)$
A contractive semigroup of entropy weak solutions

\[ u_t + f(u)_x = 0 \]

Two initial data in \( L^1(\mathbb{R}) \): \[ u_1(0,x) = \bar{u}_1(x), \quad u_2(0,x) = \bar{u}_2(x) \]

\( L^1 \) - distance between solutions does not increase in time:

\[ \| u_1(t,\cdot) - u_2(t,\cdot) \|_{L^1(\mathbb{R})} \leq \| \bar{u}_1 - \bar{u}_2 \|_{L^1(\mathbb{R})} \]

(not true for the \( L^p \) distance, \( p > 1 \))
The $L^1$ distance between continuous solutions remains constant.
The $L^1$ distance decreases when a shock in one solution crosses the graph of the other solution.
A related Hamilton-Jacobi equation

\[ u_t + f(u)_x = 0 \quad u(0, x) = \bar{u}(x) \]

\[ U(t, x) = \int_{-\infty}^{x} u(t, y) \, dy \]

\[ U_t + f(U_x) = 0 \quad U(0, x) = \bar{U}(x) = \int_{-\infty}^{x} \bar{u}(y) \, dy \]

\( f \) convex \quad \implies \quad U = U(t, x) \text{ is the value function for an optimization problem}
Legendre transform

\[ u \mapsto f(u) \in \mathbb{R} \cup \{+\infty\} \quad \text{convex} \]

\[ f^*(p) \overset{\text{def}}{=} \max_u \{pu - f(u)\} \]

\[ f^*(p) \]

\[ f(u) \]

\[ (0, 0) \]

\[ (\eta, 0) \]

\[ (0, \infty) \]

\[ (\infty, 0) \]
A representation formula

\[ U_t + f(U_x) = 0 \quad U(0,x) = \overline{U}(x) \]

\[ U(t, x) = \inf_{z(\cdot)} \left\{ \int_0^t f^*(\dot{z}(s)) \, ds + \overline{U}(z(0)) ; \quad z(t) = x \right\} \]

\[ = \min_{y \in \mathbb{R}} \left\{ t f^* \left( \frac{x - y}{t} \right) + \overline{U}(y) \right\} \]

\[ \text{Alberto Bressan (Penn State)} \]
A geometric construction

\[ U_t + f(U_x) = 0 \quad \text{and} \quad U(0, x) = \overline{U}(x) \]

define \( h(s) \equiv -T f^*(\frac{s}{T}) \)

\[ U(T, x) = \inf_y \left\{ \overline{U}(y) - h(y - x) \right\} \]
The Lax formula

Cauchy problem: \[
\begin{align*}
&\quad u_t + f(u)_x = 0, \\
&\quad u(0, x) = \bar{u}(x)
\end{align*}
\]

For each \( t > 0 \), and all but at most countably many values of \( x \in \mathbb{R} \), there exists a unique \( y(t, x) \) s.t.

\[
y(t, x) = \arg \min_{y \in \mathbb{R}} \left\{ t f^* \left( \frac{x - y}{t} \right) + \int_{-\infty}^{y} \bar{u}(s) \, ds \right\}
\]

the solution to the Cauchy problem is

\[
u(t, x) = (f')^{-1} \left( \frac{x - y(t, x)}{t} \right) \quad (1)
\]
\[ y(t, x) = \arg\min_{y \in \mathbb{R}} \left\{ t f^* \left( \frac{x - y}{t} \right) + \int_{-\infty}^{y} \bar{u}(s) \, ds \right\} \]

define the characteristic speed \( \xi = \frac{x - y(t, x)}{t} \)

if \( f'(\omega) = \xi \) then \( u(t, x) = \omega \)
Initial-Boundary value problem

\[ u_t + f(u)_x = 0 \]

\( \begin{cases} 
    u(0, x) = \bar{u}(x) & x > 0 \\
    u(t, 0) = b(t) & t > 0 
\end{cases} \)

Systems of Conservation Laws

\[
\begin{align*}
\frac{\partial}{\partial t} u_1 + \frac{\partial}{\partial x} f_1(u_1, \ldots, u_n) &= 0, \\
\vdots \\
\frac{\partial}{\partial t} u_n + \frac{\partial}{\partial x} f_n(u_1, \ldots, u_n) &= 0
\end{align*}
\]

\[u_t + f(u)_x = 0\]

\[u = (u_1, \ldots, u_n) \in \mathbb{R}^n\] conserved quantities

\[f = (f_1, \ldots, f_n) : \mathbb{R}^n \rightarrow \mathbb{R}^n\] fluxes
Hyperbolic Systems

\[ u_t + f(u)_x = 0 \quad \text{and} \quad u = u(t, x) \in \mathbb{R}^n \]

\[ u_t + A(u)u_x = 0 \quad \text{and} \quad A(u) = Df(u) \]

The system is **strictly hyperbolic** if each \( n \times n \) matrix \( A(u) \) has real distinct eigenvalues

\[ \lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u) \]

right eigenvectors \( r_1(u), \ldots, r_n(u) \) (column vectors)

left eigenvectors \( l_1(u), \ldots, l_n(u) \) (row vectors)

\[ Ar_i = \lambda_i r_i \quad \text{and} \quad l_i A = \lambda_i l_i \]

Choose bases so that \( l_i \cdot r_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \)
A linear hyperbolic system

\[ u_t + Au_x = 0 \quad \text{and} \quad u(0, x) = \phi(x) \]

\[ \lambda_1 < \cdots < \lambda_n \quad \text{eigenvalues} \quad r_1, \ldots, r_n \quad \text{eigenvectors} \]

Explicit solution: \textit{linear superposition of travelling waves}

\[ u(t, x) = \sum_i \phi_i(x - \lambda_i t) r_i \quad \phi_i(s) = l_i \cdot \phi(s) \]
Nonlinear effects - 1

\[ u_t + A(u)u_x = 0 \]

eigenvalues depend on \( u \) \( \implies \) waves change shape

Alberto Bressan (Penn State)
eigenvectors depend on $u$ \implies$ nontrivial wave interactions
Global solutions to the Cauchy problem

\[ u_t + f(u)_x = 0 \quad u(0, x) = \bar{u}(x) \]

- Construct a sequence of approximate solutions \( u_m \)

- Show that (a subsequence) converges: \( u_m \rightarrow u \) in \( L^1_{loc} \)

\[ \implies u \text{ is a weak solution} \]

Need: a-priori bound on the total variation (J. Glimm, 1965)
Building block: the Riemann Problem

\[ u_t + f(u)_x = 0 \]
\[ u(0,x) = \begin{cases} 
  u^- & \text{if } x < 0 \\
  u^+ & \text{if } x > 0 
\end{cases} \]

**B. Riemann 1860:** 2 × 2 system of isentropic gas dynamics

**P. Lax 1957:** \( n \times n \) systems (+ special assumptions)

**T. P. Liu 1975** \( n \times n \) systems (generic case)

**S. Bianchini 2003** (vanishing viscosity limit for general hyperbolic systems, possibly non-conservative)

Invariant w.r.t. symmetry:

\[ u^\theta(t,x) = u(\theta t, \theta x) \quad \theta > 0 \]
Riemann Problem for Linear Systems

\[ u_t + Au_x = 0 \]
\[ u(0, x) = \begin{cases} 
  u^- & \text{if } x < 0 \\
  u^+ & \text{if } x > 0 
\end{cases} \]

Intermediate states:

\[ \omega_i = u^- + \sum_{j \leq i} c_j r_j \]

\[ i\text{-th jump: } \omega_i - \omega_{i-1} = c_i r_i \text{ travels with speed } \lambda_i \]
General solution of the Riemann problem: concatenation of elementary waves

\[ \omega_0 = u^- \quad \omega_1 \quad \omega_2 \quad \omega_3 = u^+ \]

Alberto Bressan (Penn State)
Construction of a sequence of approximate solutions

**Glimm scheme:** piecing together solutions of Riemann problems on a fixed grid in the $t$-$x$ plane
**Front tracking scheme:** piecing together piecewise constant solutions of Riemann problems at points where fronts interact.
Existence of solutions

\[ u_t + f(u)_x = 0, \quad u(0, x) = \bar{u}(x) \]

Theorem (Glimm 1965).

Assume:
• system is strictly hyperbolic (+ some technical assumptions)

Then there exists \( \delta > 0 \) such that, for every initial condition \( \bar{u} \in L^1(\mathbb{R}; \mathbb{R}^n) \) with

\[ \text{Tot.Var.}(\bar{u}) \leq \delta, \]

the Cauchy problem has an entropy admissible weak solution \( u = u(t, x) \) defined for all \( t \geq 0 \).
Uniqueness and continuous dependence on the initial data

\[ u_t + f(u)_x = 0 \quad u(0, x) = \bar{u}(x) \]


For every initial data \( \bar{u} \) with small total variation, the front tracking approximations converge to a unique limit solution \( u : [0, \infty[ \mapsto L^1(\mathbb{R}) \).

The flow map \( (\bar{u}, t) \mapsto u(t, \cdot) = S_t \bar{u} \) is a uniformly Lipschitz semigroup:

\[ S_0 \bar{u} = \bar{u}, \quad S_s(S_t \bar{u}) = S_{s+t} \bar{u} \]

\[ \| S_t \bar{u} - S_s \bar{v} \|_{L^1} \leq L \cdot (\| \bar{u} - \bar{v} \|_{L^1} + |t - s|) \quad \text{for all} \quad \bar{u}, \bar{v}, \quad s, t \geq 0 \]


Any entropy weak solution to the Cauchy problem coincides with the limit of front tracking approximations, hence it is unique.
Claim: weak solutions of the hyperbolic system

\[ u_t + f(u)_x = 0 \]

can be obtained as limits of solutions to the parabolic system

\[ u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon u^\varepsilon_{xx} \]

letting the viscosity \( \varepsilon \to 0^+ \)
Consider a strictly hyperbolic system with viscosity

\[ u_t + A(u)u_x = \varepsilon u_{xx} \quad \text{and} \quad u(0, x) = \bar{u}(x). \] (CP)

If Tot.Var.\(\{\bar{u}\}\) is sufficiently small, then (CP) admits a unique solution \(u^\varepsilon(t, \cdot) = S_t^\varepsilon \bar{u}\), defined for all \(t \geq 0\). Moreover

\[ \text{Tot.Var.}\{S_t^\varepsilon \bar{u}\} \leq C \text{Tot.Var.}\{\bar{u}\}, \] (BV bounds)

\[ \|S_t^\varepsilon \bar{u} - S_t^\varepsilon \bar{v}\|_{L^1} \leq L \|\bar{u} - \bar{v}\|_{L^1} \] (L^1 stability)

(Convergence) If \(A(u) = Df(u)\), then as \(\varepsilon \to 0\), the viscous solutions \(u^\varepsilon\) converge to the unique entropy weak solution of the system of conservation laws

\[ u_t + f(u)_x = 0 \]
Main open problems

- Global existence of solutions to hyperbolic systems for initial data $\bar{u}$ with **large total variation**

- Existence of entropy weak solutions for systems in **several space dimensions**